Matching Domination of Cartesian Product of Two Graphs

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Abstract—The paper concentrates on the theory of domination in graphs. In this paper we define a new parameter on domination called matching domination set, matching domination number and we have investigated some properties on matching domination of cartesian product of two graphs. The following are the results:

- **If u_{i1} and u_{j2} then**
  $\deg_{G_1(C)G_2}(u_i, v_j) = \deg G_1(u_i).\deg G_2(v_j)$
- **If G_1, G_2 are simple finite graphs without isolated vertices. $G_1(C)G_2$ is a simple finite graph without isolated vertices.**
- **The cartisean product graph of two simple graphs is not a complete graph.**
- **If G_1 and G_2 are bipartite graphs them $G_1(C)G_2$ is a bipartite graph.**
- **If G_1 and G_2 are any two graphs without isolated vertices then**

  $\gamma_m[G_1(C)G_2] \leq |V_1|.\gamma_{md}(G_2); \gamma md(G_1).|V_2|$

Keywords - *Cartesian product of graphs, Domination Set, Domination number, Complete graphs, Isolated vertices, degree, regular graphs, bipartite graphs.*

I. INTRODUCTION

The study on dominating sets was initiated as a problem in the game of chess in 1850. It is about the placement of the minimum number of Queens/rooks/horses, in the game of chess so as to cover every square in the chess board. However a precise notion of a dominating set is said to be given by Konig [12], Berge [13] and Ore [7], Vizing [14] were the first to derive some interesting results on dominating sets. Since then a number of graph theorists Konig [15], Ore [7], Bauer Harary [16], Laskar [5], Berge [13], Cockayne [l7], Hedetniemi [10], Alavi[18], Allan [19], Chartrand [18], Kulli [3], Sampathkumar [3], Walikar[20], Arumugam [21], Acharya [22], Neeralgi [23], Nagaraja Rao [15] and many others have done very interesting and significant work in the domination numbers and other related topics. Cockayne [17] and Hedetniemi [10] gave an exhaustive survey of research on the theory of dominating sets in 1975 and it was updated in 1978 by Cockayne [17]. A survey on the topics on domination was also done by Hedetniemi and Laskar recently.A domination number is defined to be the minimum cardinality of all dominating sets in the graph G and a set $S \subseteq V$ is said to be a dominating set in a graph, if every vertex in V/S is adjacent to some vertex in S.

In this paper, we have defined two new domination parameters viz., matching domination set and matching domination number.

The matching domination is defined as follows:

Let $G :< V, E >$ be a finite graph without isolated vertices.

Let $S \subseteq V$. A dominating set S or G is called a matching dominating set if the induced subgraph $< S >$ admits a perfect matching. The cardinality of a minimum matching dominating set is called the matching domination number.

We have obtained the matching domination of the product of two graphs G_1 and G_2 in cartesian product graphs and obtained an expression for this number in terms of matching domination number of $G_1$and G_2. While obtaining these results, we have obtained several other interesting results on matching domination on cartesian product of two graphs .

II. CARTESIAN PRODUCT OF GRAPHS

A. Definition 2.1

lf G_1, G_2 are two simple graphs with their vertex sets $V_1 : \{u_1,u_2, ...\}$ and $V_2 : \{v_1, v_2 ....\}$ respectively then the Cartesian product of these two graphs is defined to be a graph with its vertex set as $V_1xV_2 : \{w_1, w_2,.....\}$ and if $w_1 = (u_1,v_1),w_2 = (u_2,v_2)$ then w_1,w_2 is an edge in this product graph if and only if either $u_1 = u_2$ and $v_1, v_2 \epsilon E(G_2)$ or $u_1, u_2 \epsilon E(G_1)$ and $v_1 = v_2$. This product graph is denoted by $G_1(C)G_2$. Illustration follows

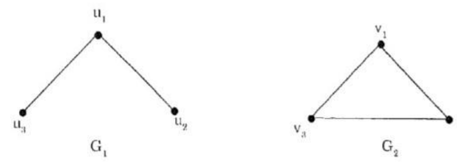

Fig. 1. .

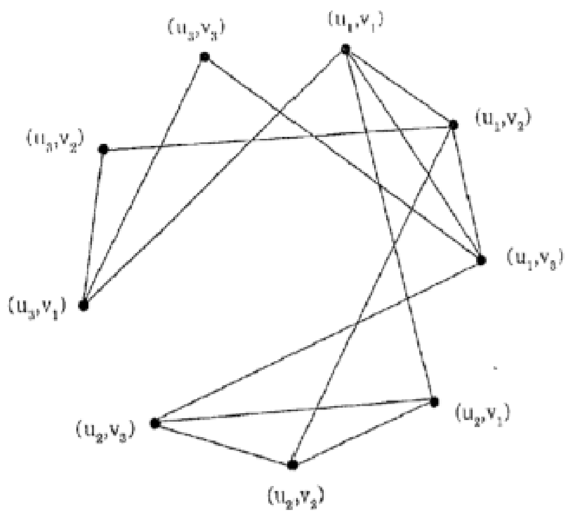

Fig. 2. .

It can be proved that in this product also that if G_1, G_2 are finite graphs without isolated vertices then $G_1(C)G_2$ is a finite graph without isolated vertices.

For this, we first prove the following theorem

B. Theorem 2.2

If u_{i1} and u_{j2} then

$$\deg_{G_1}(C)G2\,({}^{u}i,{}^{v}j) = \deg_{G_1}(u_i).\deg_{G_2}(v_j)$$

Proof :

By definition 2.1, (u_i, v_j) is adjacent with all the vertices in $u_i x N_{G2}(v_j)$ and $N_{G1}(u_i) x v_j$

Further, $|N_{G1}(u_i)| = \deg_{G1} u_i$ and $|N_{G2}(v_j)| = \deg_{G2} v_j$ We have the following result as an immediate consequence.

C. Theorem 2.3

If G_1, G_2 are simple finite graphs without isolated vertices. $G_1(C)G_2$ is a simple finite graph without isolated vertices.\

Proof :

Then $G_1(C)G_2$ is a simple finite graph follows from the definition 2.1.

Further G_1, G_2 being graphs without isolated vertices. $\deg_{G1}(u_i) = 0$ for any i, $\deg_{G2}(v_j) = 0$ for any j.

Hence from the Theorem 2.2, $\deg_{G1}(C)G2\,({}^{u}i,{}^{v}j) = 0$ for any i,j.

Thus $G_1(C)G_2$ does not have any isolated vertices.

It is interesting to see that the cartesian product of two simple graphs is not a comlete graph and if G_1, G_2 are bipartite then $G_1(C)G_2$ is also a bipartite graph.

D. Theorem 2.4

The cartisean product graph of two simple graphs is not a complete graph.

Proof :
If

$$V_1 = \{u_1,u_2,.......,u_m\}$$

and

$$V_2 = \{v_1,v_2,.......,v_n\}$$

then in $G_1(C)G_2$ the vertex (u_i, v_i) is not adjacent with (u_j, v_j) even if u_1 and u_j are adjacent and/or v_i and v_j are adjacent (By definition 2.1).

E. Theorem 2.5

If G_1 and G_2 are bipartite graphs them $G_1(C)G_2$ is a bipartite graph.

Proof :

Suppose G_1 is a bipartite graph with bipartite X_1, Y and G_2 a bipartite graph with bipartition (X_2, Y_2) where

$$X_1 = \{x_1,x_2, ...,x_r\}$$

$$Y_1 = \{y_1,y_2, ...,y_s\}$$

$$X_2 = \{u_1,u_2, ...,u_m\}$$

$$Y_2 = \{v_1,v_2, ...,v_n\}$$

We know that $X_{11} = V_1$ and $X_{22} = V_2$ and also

$$X_{11} = \phi = X_{22}.$$

Now

$$V_1 x V_2 = \{X_{11}\}x\{X_{22}\}$$

$$= \{X_1 x X_2\} \cup \{X_1 x Y_2\} \cup \{X_2 x Y_1\} \cup \{X_2 Y_2\}$$

This vertex set can partitioned as

$$X = \{X_1 x X_2\} \cup \{Y_1 x Y_2\}$$

and

$$Y = \{X_1 x Y_2\} \cup \{X_2 x Y_1\}$$

It can easily seen that no two vertices in X are adjacent in $G_1(C)G_2$ for if t_1, t_2 are two any vertices in X, then $t_1 t_2 \epsilon \{X_1 x X_2\} \cup \{Y_1 x Y_2\}$

$$\Longrightarrow t_1, t_2 \epsilon \{X_2 x X_2\} or \{Y_1 x Y_2\}$$

Case(i)
if

$$t_1 \epsilon \{X_1 x X_2\}, t_2 \epsilon \{X_1 x X_2\}$$

$$t_1 \epsilon \{X_1 x X_2\}$$

$$\Longrightarrow t_1 = (x_i, u_j) \text{ say and } t_2 = (x_k, u_l)$$

If $x_i = x_k; u_i, u_k \epsilon X_2$ and no two vertices of X_2 are adjacent by hypothesis; on the other hand if $u_j = u_l$ and if x_i, $x_k \epsilon X_1$ as no two vertices of X_1 are adjacent (by hypothesis), t_1, t_2 are not adjacent.

case (ii)
if

$$t_1 \epsilon \{X_1 x X_2\}$$
$$t_2 \epsilon \{Y_1 x Y_2\}$$

From the definition 2.1, it is evident that t_1, t_2 are not adjacent since no x_i is equal to any y_k and so also no u_j is equal to any u_l as $X_1 \cap Y_1 = \phi$, further u_j and u_l are in two different partitions of G_2. So t_1, t_2 are not adjacent in this case also.
case (iii)

$$t_1 \epsilon \{Y_1 x Y_2\}$$
$$t_2 \epsilon \{X_1 x X_2\}$$

This case is similar to the case (ii) and thus t_1, t_2 are not adjacent in this case also.

case (iv)

$$t_1 \epsilon \{Y_1 x Y_2\}$$
$$t_2 \epsilon \{X_1 x X_2\}$$

This case is similar to case (i) discussed above and so t_1, t_2 are not adjacent in this case too.
Thus we have proved that no two vertices in X are adjacent. Similarly it can he proved that no two vertices in Y are adjacent.
Hence the Theorem.

III. MATCHING DOMINATION NUMBER

A. Definition 3.1

A set $S \subseteq V$ is said to be a dominating set in a graph G if every vertex in V/S is adjacent to some vertex in S and the domination number $'\gamma'$ of G is defined to be the minimum cardinality of all dominating sets in G.

We have introduced a new parameter called the matching domination set of a graph.

It is defined as follows:

B. Definition 3.2

A dominating set of a graph G is said to be matching dominating set if the induced subgraph < D > admits a perfect matching.

The cardinality of the smallest matching dominating set is called matching domination number and is denoted by γ_m
Illustration

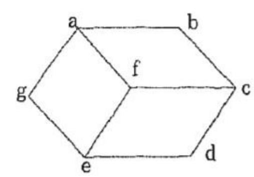

Fig. 3.

In this graph {a, b, c, f, e, g } is a matching domination set, since this is a dominating set and the induced subgraph {a, b, c, e, f, g } has perfect matching formed by the edges af, bc, eg, {a,b,e,f } is also matching dominating set. Similarly {a,b,c,g} is a matching dominating set where the induced subgraph of this set admits a perfect matching given by the edges be,ag.

However there are no matching dominating sets of lower cardinality and it follows that the matching domination number of the graph in figure 3 is 4. Thus a graph can have many matching dominating sets of minimal cardinality. We make the following observations as an immediate consequence.
(a) Not all dominating sets are matching domination sets. For example in figure 3, {a,c,e } is a dominating set but it is not a matching dominating set.
(b) The cardinality of matching dominating set is always even. The matching dominating set D of a graph requires the admission of a perfect matching by the induced subgraph < D >. Thus it is necessary that D has even number of vertices for admitting a perfect matching.

(c) Not all dominating sets with even number of vertices are matching dominating sets. For example in figure 3,{b,d,g,f } is a dominating set containing even number of vertices, but induced subgraph formed by these four vertices does not have a perfect matching.

(d) The necessary condition for a graph G to have matching dominating set is that G is a graph without isolated vertices. The matching domination number of the graph G (figure 3) is 4, where as the domination number is 2 ; {a,d } being a minimal dominating set. If G is a graph with isolated vertices then any dominating set should include these isolated vertices and consequently the induced subgraph of this set containing isolated vertices will not admit a perfect matching.

C. Theorem 3.3

If G_1 and G_2 are any two graphs without isolated vertices then $\gamma_m[G_1(C)G_2] \leq |V_1| .\gamma_{md}(G_2); \gamma_{md}(G_1). |V_2|$

Proof :

Let

$V(G_1) : \{u_1, u_2 ...., u_p\} = V_1$

$V(G_2) : \{v_1, v_2 ...., v_q\} = V_2$

Let D1 : $\{u_{d1} ; u_{d2} ;.....u_{d2r}\}$ be the matching dominating set of minimum cardinality of G,. Let the induced sub graph $< D_1 >$ of G1 have a perfect matching in $< D_1 >$ constitute by the edges $u_{d1} ; u_{d2} ; u_{d3} ; u_{d4} ; ....ud_{2r-1} ; ud_{2r}$.

Similarly let $D_2 : \{v_{d1} ; v_{d2} ; ......v_{d2r}\}$ be the matching dominating set of minimum cardinality of G2. Let the induced subgraph $< D_2 >$ of G_2 have a perfect matching in $< D_2 >$ constituted by the edges $v_{d1} ; v_{d2} ; v_{d3} ; v_{d4} ; v_{d2r-11} ; v_{d2r}$.

Then it can easily seen that the sets

$$
D : \begin{array}{l}
\{ (u_1; v_{d1}) (u_1; v_{d2}) : : : (u_1; v_{d2r}) \\
(u_2; v_{d1}) (u_2; v_{d2}) : : : (u_2; v_{d2r}) \\
... \\
... \\
... \\
... \\
(u_p; v_{d1}) (u_p; v_{d2}) : : : (u_p; v_{d2r})
\end{array}
$$

and the set

$$
D' : \begin{array}{l}
(u_{d1} ; v_1) (u_{d2} ; v_1) : : : (u_{d2r} ; v_1) \\
(u_{d1} ; v_2) (u_{d2} ; v_2) : : : (u_{d2r} ; v_2) \\
... \\
... \\
... \\
... \\
(u_{d1} ; v_p) (u_{d2} ; v_p) : : : (u_{d2r} ; v_p)
\end{array}
$$

will be both matching dominating sets of $G_1(C)G_2$. Hence, it follows that the minimal matching dominating set will be $\leq \min |D|, |D'|$.

$$|D| = p.2s = |v_1| .\gamma_m(G_2)$$

$$|D'| = 2r.q = \gamma_m(G_1). |v_2|$$

Hence

$$\gamma_m(G_1(C)G_2) \leq \min.\{2ps,2qr\}$$
$$\leq \min.\gamma_m(G_1). |V_2|, |V_1| \gamma_m(G_2)$$

Illustrate follows:

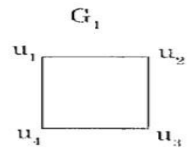

Fig. 4. matching domination set $\{u_1, u_2\}$, p = 4, γm(G1) = 2

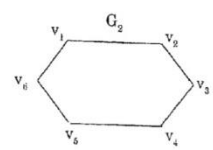

Fig. 5. matching domination set $\{v_1, v_2; v_4, v_5\}$, q = 6, γm(G1) = 4

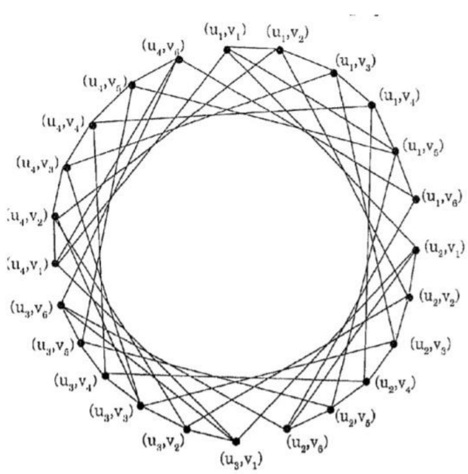

Fig. 6. $G_1(C)G_2$

matching domination set

$D = \{u_1,u_2\} x |V_2|$

$= \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_1, v_5)(u_1, v_6),$

$(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5), (u_2, v_6)\}$

$|D| = 12$

matching domination set

$$D' = | V_1 | x\{v_1, v_2, v_4, v_5\}$$

$= \{(u_1, v_1), (u_1, v_2), (u_1, v_4), (u_1, v_5)$

$\{(u_2, v_1), (u_2, v_2), (u_2, v_4), (u_2, v_5)$

$\{(u_3, v_1), (u_3, v_2), (u_3, v_4), (u_3, v_5)$

$\{(u_4, v_1), (u_4, v_2), (u_4, v_4), (u_4, v_5)$

$$| D' | = 16$$

However

$\{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_1, v_5), (u_1, v_6), (u_3, v_1), (u_3, v_2), (u_3, v_4), (u_3, v_5)\}$ is a dominating set of $G_1(C)G_2$ with cardinality 10.

Hence,

$$\gamma_m\{G_1(C)G_2\} \leq \min\{D, D'\}$$

IV. Conclusion

The Theory of domination has been the nucleus research activity in graph theory in recent times. This is largely due to a variety of new parameters that can developed from the basic definition of domination. The study of Cartesian product graphs,the matching domination of Cartesian product graphs has been providing us sufficient stimulation for obtaining some in-depth knowledge of the various properties of the graphs. It is hoped that encouragement provided by this study of these product graphs will be a good straight point for further research.

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