FUZZY Γ- IDEALS OF Γ- SEMI GROUPS

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Abstract— In this paper, we considered some properties and characterizations of fuzzy Γ - ideals such as fuzzy interior Γ - ideals and fuzzy bi Γ - ideals of Γ - semi groups and investigate some of their properties. We also have studied fuzzy quasi Γ - ideals and fuzzy left (right, two sided) Γ - ideals of Γ - semi groups.

Keywords— Γ -Semi Group, Fuzzy bi Γ - Ideal, Fuzzy Quasi Γ - Ideal, Fuzzy Interior Γ - Ideal.

I. INTRODUCTION AND PRELIMINARIES

The fundamental concept of a fuzzy set was introduced by L.A.Zadeh in 1965[4]. The concept of fuzzy ideals in semi groups was introduced by N.Kuroki in 1979[2]. N.Kuroki [3] introduced fuzzy left (right) ideals, fuzzy bi ideals and fuzzy interior ideals. Some basic concepts of fuzzy algebra such as fuzzy left (right) ideals and fuzzy bi ideals in a fuzzy semi group were introduced by Dib [7] in 1994. D.R.Prince Williams and K.B.Latha introduced fuzzy Γ - ideal and fuzzy bi Γ - ideal [1].

Definition 1.1: A mapping $\mu : S \to [0,1]$ is called fuzzy set of S and the compliment of a set μ, denoted by $\mu^{'}$, is the fuzzy subset in S defined by $\mu^{'} = 1 - \mu(x)$ $for\ all\ \ x \in S$. Let the level set of a fuzzy set μ of S is defined as $U(\mu,t) = \{x \in S / \mu(x) \geq t\}$. Note that Γ - semi group S can be considered as a fuzzy set of itself and we write $S = C_S$ i.e. $S(x) = 1$ for all $x \in S$.

Definition 1.2: Let $S = \{x, y, z,...\}$ and $\Gamma = \{\alpha, \beta, \gamma,...\}$ be two non-empty sets then S is called a Γ -semi group if it satisfies (i) $x\gamma y \in S$ (ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ $for\ x, y, z \in S\ \ and\ \ \alpha, \beta \in \Gamma$.

Definition 1.3: A fuzzy set μ of S is called a fuzzy sub Γ - semi group of S if $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S\ and\ \alpha \in \Gamma$.

Definition 1.4: A fuzzy set μ of S is called a fuzzy left (right) Γ - ideal of S if $\mu(x\alpha y) \geq \mu(y)$ ($\mu(x\alpha y) \geq \mu(x)$) for all $x, y \in S\ and\ \alpha \in \Gamma$.

Definition 1.5: A fuzzy set μ of S is called a fuzzy Γ - ideal of S if it is both fuzzy left Γ - ideal and fuzzy right Γ - ideal of S.

Definition 1.6: A fuzzy sub Γ - semi group μ of S is called a fuzzy bi Γ - ideal of S if $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in S$ $and\ \alpha, \beta \in \Gamma$

Definition 1.7: A fuzzy sub Γ - semi group μ of S is called a fuzzy interior Γ - ideal of S if $\mu(x\alpha y\beta z) \geq \mu(y)$ for all $x, y, z \in S\ and\ \alpha, \beta \in \Gamma$.

Definition 1.8: Let μ_1 and μ_2 be two fuzzy sets of Γ - semi group S. Then $\mu_1 \cap \mu_2\ and\ \mu_1 \cup \mu_2$ are defined by $(\mu_1 \cap \mu_2)(a) = \min\{\mu_1(a), \mu_2(a)\}$ and $(\mu_1 \cup \mu_2)(a) = \max\{\mu_1(a), \mu_2(a)\}$.

We denote $\wedge$ - minimum or infimum and $\vee$ - maximum or supremum then $(\mu_1 \cap \mu_2)(a) = \mu_1(a) \wedge \mu_2(a)$, $(\mu_1 \cup \mu_2)(a) = \mu_1(a) \vee \mu_2(a)$

Definition 1.9: Let μ_1 and μ_2 be any two fuzzy sets of a Γ - semi group S. Then their fuzzy product $\mu_1 \circ \mu_2$ is defined by

$$(\mu_1 \circ \mu_2)(a) = \begin{cases} \bigvee_{a = x\alpha y} \{\mu_1(x) \wedge \mu_2(y)\}\ if\ a = x\alpha y\ \ for\ x, y \in S, \alpha \in \Gamma \\ 0 \quad otherwise \end{cases}$$

Definition 1.10: A fuzzy sub Γ - semi group μ of S is called a fuzzy bi Γ -ideal of a Γ - semi group S if $\mu(x\alpha y\beta z) \geq \mu(x) \wedge \mu(z)\}$ for all $x, y, z \in S\ and$ $\alpha, \beta \in \Gamma$

Definition 1.11: A fuzzy set μ of a Γ - semi group S is a fuzzy quasi Γ - ideal of S if $(\mu \circ S) \cap (S \circ \mu) \subseteq \mu$

Based on these preliminaries we prove some results on fuzzy Γ - ideals, fuzzy bi Γ - ideals, fuzzy interior Γ - ideals and fuzzy quasi Γ - ideals of Γ -semi group S.

II. MAIN RESULTS

Theorem 2.1.: Let S be a Γ -semi group. (i) If μ_1 and μ_2 are fuzzy sub $\Gamma -$semi groups of S, then $\mu_1 \cup \mu_2$ is a fuzzy sub $\Gamma-$semi group of S. (ii) If μ_1 and μ_2 are fuzzy Γ- ideals of S, then $\mu_1 \cup \mu_2$ is a fuzzy Γ- ideal of S.

Proof: (i) Let μ_1 and μ_2 be two fuzzy sub $\Gamma-$semi groups of S. Then we have $\mu_1(x\alpha y) \geq \{\mu_1(x) \wedge \mu_1(y)\}$ and $\mu_2(x\alpha y) \geq \{\mu_2(x) \wedge \mu_2(y)\}$

Consider

$$\begin{aligned}(\mu_1 \cup \mu_2)(x\alpha y) &= \{\mu_1(x\alpha y) \vee \mu_2(x\alpha y)\}\\ &\geq [\mu_1(x) \wedge \mu_1(y)] \vee [\mu_2(x) \wedge \mu_2(y)]\\ &= [\mu_1(x) \vee \mu_2(x)] \wedge [\mu_1(y) \vee \mu_2(y)]\\ &= (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)\end{aligned}$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy sub $\Gamma-$semi group of S.

(ii) Let μ_1 *and* μ_2 be fuzzy Γ- ideals of Γ-semi group S. Then we have $\mu_1(x\alpha y) \geq \mu_1(x)$, $\mu_1(x\alpha y) \geq \mu_1(y)$ and $\mu_2(x\alpha y) \geq \mu_2(x)$, $\mu_2(x\alpha y) \geq \mu_2(y)$.

Consider

$$\begin{aligned}(\mu_1 \cup \mu_2)(x\alpha y) &= \mu_1(x\alpha y) \vee \mu_2(x\alpha y)\\ &\geq \mu_1(x) \vee \mu_2(x)\\ &\geq (\mu_1 \cup \mu_2)(x)\end{aligned}$$

$$(\mu_1 \cup \mu_2)(x\alpha y) \geq (\mu_1 \cup \mu_2)(x)$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy left Γ- ideal of Γ-semi group S.

And

$$\begin{aligned}(\mu_1 \cup \mu_2)(x\alpha y) &= \mu_1(x\alpha y) \vee \mu_2(x\alpha y)\\ &\geq \mu_1(y) \vee \mu_2(y)\\ &\geq (\mu_1 \cup \mu_2)(y)\end{aligned}$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy right Γ- ideal of S.

Hence $\mu_1 \cup \mu_2$ is a fuzzy Γ- ideal of S.

Theorem 2.2.: Let S be a Γ-semi group. (i) If μ_1 and μ_2 are fuzzy sub $\Gamma-$semi groups, then $\mu_1 \cap \mu_2$ is a fuzzy sub $\Gamma-$semi group of S. (ii) If μ_1 and μ_2 are fuzzy Γ- ideals of S, then $\mu_1 \cap \mu_2$ is a fuzzy Γ- ideal of S.

Proof: Similar to the proof of Theorem **2.1.**

Theorem 2.3.: Let μ_1 and μ_2 be two fuzzy bi Γ- ideals of a Γ-semi group S. Then $\mu_1 \cap \mu_2$ is a fuzzy bi Γ- ideal of S.

Proof: Let μ_1 and μ_2 be two fuzzy bi Γ- ideals of a Γ-semi group S. Then we have $\mu_1(x\alpha y\beta z) \geq \mu_1(x) \wedge \mu_1(z)$, and $\mu_2(x\alpha y\beta z) \geq \mu_2(x) \wedge \mu_2(z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

To prove that $\mu_1 \cap \mu_2$ is a fuzzy bi Γ- ideal of S, we shall to prove that it is fuzzy sub Γ-semi group. Consider

$$\begin{aligned}(\mu_1 \cap \mu_2)(x\alpha y) &= \mu_1(x\alpha y) \wedge \mu_2(x\alpha y)\\ &\geq [\mu_1(x) \wedge \mu_1(y)] \wedge [\mu_2(x) \wedge \mu_2(y)\\ &= [\mu_1(x) \wedge \mu_2(x)] \wedge [\mu_1(y) \wedge \mu_2(y)]\\ &= (\mu_1 \cap \mu_2)(x) \wedge (\mu_1 \cap \mu_2)(y)\end{aligned}$$

$$(\mu_1 \cap \mu_2)(x\alpha y) \geq (\mu_1 \cap \mu_2)(x) \wedge (\mu_1 \cap \mu_2)(y)$$

$\therefore$ $\mu_1 \cap \mu_2$ is a fuzzy sub $\Gamma-$semi group of S.

And

$$\begin{aligned}(\mu_1 \cap \mu_2)(x\alpha y\beta z) &= \mu_1(x\alpha y\beta z) \wedge \mu_2(x\alpha y\beta z)\\ &\geq [\mu_1(x) \wedge \mu_1(z)] \wedge [\mu_2(x) \wedge \mu_2(z)]\\ &= [\mu_1(x) \wedge \mu_2(x)] \wedge [\mu_1(z) \wedge \mu_2(z)]\\ &= (\mu_1 \cap \mu_2)(x) \wedge (\mu_1 \cap \mu_2)(z)\end{aligned}$$

$$(\mu_1 \cap \mu_2)(x\alpha y\beta z) \geq (\mu_1 \cap \mu_2)(x) \wedge (\mu_1 \cap \mu_2)(x)$$

$\therefore$ $\mu_1 \cap \mu_2$ is a fuzzy bi Γ- ideal of S.

Theorem 2.4.: Let μ_1 and μ_2 be two fuzzy bi Γ- ideals of a Γ-semi group of S. Then $\mu_1 \cup \mu_2$ is a fuzzy bi Γ- ideal of S.

Proof: Let μ_1 and μ_2 be two fuzzy bi Γ- ideals of a Γ-semi group S. Then for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. To prove that $\mu_1 \cup \mu_2$ is a fuzzy bi Γ- ideal of S, we shall prove that it is fuzzy sub Γ-semi group.

Consider

$$\begin{aligned}(\mu_1 \cup \mu_2)(x\alpha y) &= \mu_1(x\alpha y) \vee \mu_2(x\alpha y)\\ &\geq [\mu_1(x) \wedge \mu_1(y)] \vee [\mu_2(x) \wedge \mu_2(y)]\\ &= [\mu_1(x) \vee \mu_2(x)] \wedge [\mu_1(y) \vee \mu_2(y)]\\ &= (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)\end{aligned}$$

$$(\mu_1 \cup \mu_2)(x\alpha y) \geq (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(y)$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy sub $\Gamma-$semi group of S.

Again

$$\begin{aligned}(\mu_1 \cup \mu_2)(x\alpha y\beta z) &= \mu_1(x\alpha y\beta z) \vee \mu_2(x\alpha y\beta z)\\ &\geq [\mu_1(x) \wedge \mu_1(z)] \vee [\mu_2(x) \wedge \mu_2(z)]\\ &= [\mu_1(x) \vee \mu_2(x)] \wedge [\mu_1(z) \vee \mu_2(z)]\\ &= (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(z)\end{aligned}$$

$$(\mu_1 \cup \mu_2)(x\alpha y\beta z) \geq (\mu_1 \cup \mu_2)(x) \wedge (\mu_1 \cup \mu_2)(z)$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy bi Γ- ideal of S.

Theorem 2.5: A fuzzy set μ of a Γ-semi group S is a fuzzy bi Γ-ideal of S if and only $\mu \circ S \circ \mu \subseteq \mu$.

Proof: Let μ be a fuzzy set of a Γ-semi group S and let μ be a fuzzy bi Γ-ideal of S. Then we have $\mu\ (x\alpha y\beta z) \geq \{\mu\ (x) \wedge \mu\ (z)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Suppose $a = x\alpha\, y\beta\, z$ where $b = x\alpha y$. Then

Consider

$$\begin{aligned}(\mu \circ S \circ \mu)(a) &= \bigvee_{a=b\beta z} [(\mu \circ S)(b) \wedge \mu(z)]\\ &= \bigvee_{a=b\beta z} \{\bigvee_{b=x\alpha y} [\mu(x) \wedge S(y)] \wedge \mu(z)\}\\ &= \bigvee_{a=b\beta z} \{\bigvee_{b=x\alpha y} [\mu(x) \wedge 1] \wedge \mu(z)\}\end{aligned}$$

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$$= \bigvee_{a=x\alpha y\beta z}[\mu(x) \wedge \mu(z)]$$
$$\leq \bigvee_{a=x\alpha y\beta z} \mu(x\alpha y\beta z)$$
$$= \mu(a)$$
$$(\mu \circ S \circ \mu)(a) \leq \mu(a)$$
$$\therefore \ \mu \circ S \circ \mu \subseteq \mu$$

If $a \neq x\alpha y\beta z$, then $(\mu \circ S \circ \mu)(a) = 0 \leq \mu(a)$

$$\therefore \ \mu \circ S \circ \mu \subseteq \mu$$

Conversely, assume that $\mu \circ S \circ \mu \subseteq \mu$. Then for all $a \in S$ $\mu(a) \geq (\mu \circ S \circ \mu)(a)$

$$= \bigvee_{a=b\beta z}[(\mu \circ S)(b) \wedge \mu(z)]$$
$$= \bigvee_{a=b\beta z}\left\{\bigvee_{b=x\alpha y}[\mu(x) \wedge S(y)] \wedge \mu(z)\right\}$$
$$= \bigvee_{a=b\beta z}\{\bigvee_{b=x\alpha y}[\mu(x) \wedge 1] \wedge \mu(z)\}$$
$$= \bigvee_{a=x\alpha y\beta z}[\mu(x) \wedge \mu(z)]$$
$$\therefore \ \mu(x\alpha y\beta z) \geq [\mu(x) \wedge \mu(z)]$$

Hence μ is a fuzzy bi Γ-ideal of S.

Theorem 2.6: Let μ_1 be a fuzzy right Γ- ideal of Γ-semi group S and μ_2 be a fuzzy left Γ-ideal of S. Then $\mu_1 \cup \mu_2$ is a fuzzy quasi Γ- ideal of S.

Proof: Let μ_1 be a fuzzy right Γ- ideal and μ_2 be a fuzzy left Γ- ideal of S. Then we have

$\mu_1(x\alpha y) \geq \mu_1(x)$ and $\mu_2(x\alpha y) \geq \mu_2(y)$.

consider

$$((\mu_1 \cup \mu_2) \circ S)(a) = \bigvee_{a=x\alpha y}[(\mu_1 \cup \mu_2)(x) \wedge S(y)]$$
$$= \bigvee_{a=x\alpha y}[(\mu_1(x) \vee \mu_2(x) \wedge 1]$$
$$= \bigvee_{a=x\alpha y}[(\mu_1(x) \vee \mu_2(x)]$$
$$\leq \bigvee_{a=x\alpha y}[(\mu_1(x\alpha y) \vee \mu_2(x\alpha y)]$$
$$= (\mu_1(a) \vee \mu_2(a)$$
$$= (\mu_1 \cup \mu_2)(a)$$
$$((\mu_1 \cup \mu_2) \circ S)(a) \leq (\mu_1 \cup \mu_2)(a)$$
$$\therefore \ (\mu_1 \cup \mu_2) \circ S \subseteq \mu_1 \cup \mu_2$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy left Γ- ideal of S.

Similarly, we can prove that $\mu_1 \cup \mu_2$ is a fuzzy right Γ- ideal of S. i.e. $S \circ (\mu_1 \cup \mu_2) \subseteq \mu_1 \cup \mu_2$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy Γ- ideal of S.

To prove that $\mu_1 \cup \mu_2$ is a fuzzy quasi Γ- ideal of S, consider

$$[(\mu_1 \cup \mu_2) \circ S] \cap [S \circ (\mu_1 \cup \mu_2)] \subseteq (\mu_1 \cup \mu_2) \cap (\mu_1 \cup \mu_2)$$
$$\subseteq (\mu_1 \cup \mu_2)$$
$$[(\mu_1 \cup \mu_2) \circ S] \cap [S \circ (\mu_1 \cup \mu_2)] \subseteq (\mu_1 \cup \mu_2)$$

$\therefore$ $\mu_1 \cup \mu_2$ is a fuzzy quasi Γ- ideal of S.

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