Generalized I-Convergent Difference Double Sequence Spaces Defined By A Moduli Sequence

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Abstract:-In this article we introduce the sequence space $c_0^I(F, \Delta^n)$ and $\ell_\infty^I(F, \Delta^n)$ for the sequence of F = (f_k) and given some inclusion relations.

Keywords:-Ideal Filter, Sequence of Moduli, Difference Sequence Space, F-Convergent Sequence Space.

I. INTRODUCTION

Let ω, ℓ_∞, c_0 be the set of all sequences of complex numbers, the linear spaces of bounded, convergent and null sequences x = (x_k) with complex terms, respectively, normed by

$$\|x\|_\infty = \text{where } K \in N = 1, 2, 3 \ldots..$$

The idea of difference sequence spaces was introduced by H. Kizmaz [10]. In 1981, Kizmaz defined the sequence spaces as follows;

$$\ell_\infty(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in \ell_\infty\},$$
$$c(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c\},$$
$$c_0(\Delta) = \{x = (x_k) \in \omega : (\Delta x_k) \in c_0\},$$

Where

$$\Delta_x = (x_k - x_{k+1}) \quad \text{and} \quad \Delta_x^0 = (x_k),$$

There are Banach spaces with the norm

$$\| x \|_\Delta = |x_1| + \| \Delta_x \|_\infty .$$

Later Colak and Et [2] defined the sequence spaces:

$$\ell_\infty(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in \ell_\infty\},$$

$$c(\Delta^n) = \{x = (x_k) \in \omega(\Delta^n x_k) \in c\},$$

$$c_0(\Delta^n) = \{x = (x_k) \in \omega : (\Delta^n x_k) \in c_0\},$$

Where

$n \in N$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^n x = (\Delta^n x_k) = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1})\}$

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^n \begin{bmatrix} n \\ v \end{bmatrix} x_{k+v},$$

And so that these are Banach space with the norm

$$\| x \|_\Delta = \sum_{i=1}^{n} |x_i| + \| \Delta^n x \|_\infty .$$

The idea of modulus was defined by Nakano [15] in 1953. A function
$f : [0, \infty) \to [0, \infty)$ is called a modulus if

(i) $f(t) = 0$ if and only if t = 0,
(ii) $f(t+u) \le f(t)+f(u)$, for all $t,u \ge \theta$,
(iii) f is increasing and
(iv) f is continuous from the right at 0.

Let X be a sequence spaces. Then the sequence spaces X(f) is defined as

$$X(f) = \{x = (x_k) : (f(|x_k|)) \in X\}$$

For a modulus f. Maddox and Ruckle [14,16]

Kolak [11, 12] gave an extension of X(f) by considering a sequence of moduli F = (f_k) that is

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

After then Gaur and Mursaleen [9] defined the following sequence spaces

$$\ell_\infty(F, \Delta) = \{x = (\Delta x_k) \in \ell_\infty(F)\},$$
$$c_0(F, \Delta) = \{x = (x_k) : (\Delta x_k) \in c_0(F)\},$$

For a sequence of moduli $F = (f_k)$.

We defined the following sequence spaces :

$$\ell^2_\infty(F, \Delta^n_m) = \{x = (x_{i,j}) : (\Delta^n_m x_{i,j}) \in \ell^2_\infty(F)\},$$

$$c^2_0(F, \Delta^n_m) = \{x = (x_{i,j}) : (\Delta^n_m x_{i,j}) \in c^2_0(F)\},$$

Where

$$\Delta^n_m x_{ij} = \sum_{u=0}^{m} \sum_{v=0}^{n} (-1)^{u+v} \binom{n}{u}\binom{n}{v} x_{i+mu,\, k+mv}$$

for a sequence of moduli $F = (f_{ij})$ we will give the necessary and sufficient conditions for the inclusion relations between $X(\Delta^n_m)$ and sufficient conditions for the inclusion relations between $X(\Delta^n_m)$ and $Y(F, \Delta^n_m)$, where $X,Y = \ell^2_\infty$ or c^2_0. Sequences of moduli have been studied by C.A. Bektas and R. Colak [1] and many other authors.

The notion of statical convergence was introduced by H. Fast [6]. Later on it was studied by J.A. Fridy [7,8] from the sequence space point view and linked with the summability theory.

The notion of I-convergence is a generalization of the statical convergence. It was studied at initial stage by Kostyrko, Salat and Wilezynski [13]. Later on it was studied by Salat [19], Salat, Tripathy and Ziman [20], Demric [3].

Let N be a non empty set. Then a family of sets $I \subseteq 2^N$ (power set of N) is said to be an ideal if I is additive i.e. $(A,B) \in I \Rightarrow (A \bigcup B) \in I$ and i.e. $A \in I$, $B \subseteq A \Rightarrow B \in I$. A non empty family of sets $\pounds(I) \subseteq 2^N$ is said to be filter on N if and only if $\Phi \notin \pounds(I)$ for A, B $\in \pounds(I)$ we have $(A \bigcap B) \in \pounds(I)$ and each $A \in \pounds(I)$ and $A \subseteq B$ implies $B \in \pounds(I)$.

An ideal $I \subseteq 2^N$ is called non trivial it $I \neq 2^N$. A non trivial $I \subseteq 2^N$ is called admissible if $\{(x) : x \in N\} \subseteq I$. A non trivial ideal is maximal ideal is maximal it there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. For each ideal I, there exist a filter £(I) corresponding to I, i.e. $\pounds(I) = \{K \subseteq N : K^c \in I\}$, where $K^c = N - K$.

Definition 1.1: A sequence $(x_{ij}) \in \omega$ is said to be I-convergent to a number L if for every $\varepsilon > 0$. $\{i,j \in N : |xij - L| \geq \varepsilon\} \in I$. In this case we write $I - \lim_{i+j\to\infty} x_{ij} = L$.

Definition 1.2: A sequence $(x_{ij}) \in \omega$ is said to be I-null if L = 0. In this case we write $I - \lim_{i+j\to\infty} x_{ij} = 0$.

Definition 1.3: A sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for every $\varepsilon > 0$, there exist a number $m = m(\varepsilon)$ such that $\{i,j \in N : |x_{ij} - x_{m,n}| \geq \varepsilon\} \in I$.

Definition 1.4: A sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exist $M > 0$ such that $\{K \in N : |x_{ij}| \geq M\}$.

We need the following Lemmas.

Lemma 1.5 : The condition $\sup_{ij} f_{ij}(t) < \infty$, $t > 0$ hold if and only if there is a point $t_0 > 0$ such that $\sup_{ij} f_{ij}(t_0) < \infty$ (see [1, 9]).

Lemma 1.6: The condition $\inf_{ij} f_{ij}(t) > 0$ hold if and only if there exist is a point $t_0 > 0$ such that $\inf_{ij} f_{ij}(t_0) > 0$ (see [1, 9]).

Lemma 1.7: Let $K \in \pounds(I)$ and $M \subseteq N$. If $M \neq I$, the $M \bigcap K \neq I$ (see [20]).

Lemma 1.8: If $I \subseteq 2^N$ and $M \subseteq N$. If $M \neq I$ then $M \bigcap K \neq I$ (see [13]).

II. MAIN RESULT

In this article we introduce the following classes of sequence cpaces.

$${}^2c^I_0(F, \Delta^n_m) = \{(x_{i,j}) \in \omega : I - \lim_{i+j\to\infty} f_{i,j}(|\Delta^n_m x_{i,j}|) = 0\} \in I,$$

$${}^2\ell^I_\infty(F, \Delta^n_m) = \{(x_{i,j}) \in \omega : I - \sup_{i,j} f_{i,j}(|\Delta^n_m x_{i,j}|) < \infty\} \in I.$$

Theorem 2.1: For a sequence $F = f_{i,j}$ of moduli, the following statements are equivalent :

(a) ${}^{2}\ell_{\infty}^{I}(\Delta_m^n) \subseteq {}^{2}\ell_{\infty}^{I}(F, \Delta_m^n)$,

(b) ${}^{2}c_0^{I}(\Delta_m^n) \subseteq {}^{2}c_0^{I}(F, \Delta_m^n)$,

(c) $\mathrm{Sup}_{i,j} f_{i,j}(t) < \infty$, $(t > 0)$.

Proof: (a) Implies (b) is obvious.

(b) implies (c). Let ${}^{2}c_0^{I}(\Delta_m^n) \subseteq {}^{2}c_0^{I}(F, \Delta_m^n)$. Suppose that (c) is not true. Then by Lemma (1.5)

$$\sup_{i,j} f_{i,j}(t) = \infty, \text{ for all } t > 0,$$

And therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}\left(\frac{1}{i}\right) > i, \quad \text{for each } i = 1, 2, 3 \ldots.. \qquad (1)$$

Define $x = (x_{i,j})$ as follows

$$x_{ij} = \begin{cases} \frac{1}{i} & \text{if } i, j = k_i \;\; i = 1,2,3 \ldots..; \\ 0 & \text{otherewise}. \end{cases}$$

Then $x \in {}^{2}c_0^{I}(\Delta_m^n)$ but by (1), $x \notin \ell_{\infty}^{I}(F, \Delta_m^n)$ which contradicts (b). Hence (c) must hold, (c) implies (a). Let (c) be satisfied and $x \in {}^{2}\ell_{\infty}^{I}(F, \Delta_m^n)$. If we suppose that $x \notin {}^{2}\ell_{\infty}^{I}(F, \Delta_m^n)$ then

$$\sup_{i,j} f_{i,j}(|\Delta_m^n x_{i,j}|) = \infty \text{ for } x \in {}^{2}\ell_{\infty}^{I}$$

If we take $t = |\Delta_m^n x|$ then $\sup_{i,j} f_{i,j}(t) = \infty$ which contradicts (c).

Hence ${}^{2}\ell_{\infty}^{I}(\Delta_m^n) \subseteq {}^{2}\ell_{\infty}^{I}(F, \Delta_m^n)$.

Theorem 2.2: For a sequence $F = f_{i,j}$ is a sequence of moduli, the following statements are equivalent :

(a) ${}^{2}c_0^{I}(F, \Delta_m^n) \subseteq {}^{2}c_0^{I}(\Delta_m^n)$,

(b) ${}^{2}c_0^{I}(F, \Delta_m^n) \subseteq {}^{2}\ell_{\infty}^{I}(\Delta_m^n)$,

(c) $\mathrm{Inf}_{i,j} f_{i,j}(t) > 0$, $(t > 0)$.

Proof: (a) implies (b) is obvious.

(b) implies (c). Let ${}^{2}c_0^{I}(F, \Delta_m^n) \subseteq {}^{2}\ell_{\infty}^{I}(\Delta_m^n)$. Suppose that (c) is not true. Then by Lemma (1.6)

$$\mathrm{Inf}_{i,j} f_{i,j}(t) = 0, \;\; (t > 0)$$

And therefore there is a sequence (k_i) of positive integers such that

$$f_{k_i}(i^2) < \frac{1}{i} \text{ for each } I = 1, 2, 3 \ldots.. \qquad (2)$$

Define $x = (x_{i,j})$ as follows

$$x_{i,j} = \begin{cases} i^2, & \text{if } k = k_i \;\; i = 1, 2, 3\ldots..; \\ 0 & \text{otherwise} \end{cases}$$

By (2) $x \in {}^{2}c_0^{I}(F, \Delta_m^n)$ but $x \notin \ell_{\infty}^{I}(\Delta^n)$ which contradicts (b). Hence (c) must hold. (c) implies (a). Let (c) be satisfied and $x \in {}^{2}c_0^{I}(F, \Delta_m^n)$ that is

$$I - \lim_{i+j\to\infty} f_{i,j}(|\Delta_m^n x_{i,j}|) = 0$$

Suppose that $x \notin {}^{2}c_0^{I}(\Delta_m^n)$. Then for some number $\varepsilon_0 > 0$ and positive integer k_0 we have $|\Delta_m^n x_{i,j}| \leq c_0$ for $ki,j > k_0$. Therefore $f_{ij}(c_0) \geq f_{ij}(|\Delta_m^n x_{i,j}|)$ for $I,j > k_0$ and hence $\lim_{i+j\to\infty} f_{i,j}(c_0) > 0$,

Which contradicts our assumption that $x \notin {}^{2}c_0^{I}(\Delta_m^n)$.

$$\text{Thus } {}^{2}c_0^{I}(F, \Delta_m^n) \subseteq {}^{2}c_0^{I}(\Delta_m^n).$$

Theorem 2.3: The inclusion ${}^{2}\ell_{\infty}^{I}(F, \Delta_m^n) \subseteq {}^{2}c_0^{I}(\Delta_m^n)$ holds in and only if

$$\lim_{i+j\to\infty} f_{i,j}(t) = \infty \text{ for } t > 0 \quad \ldots\ldots\ldots \; (3)$$

Proof: Let ${}^{2}\ell_{\infty}^{I}(F, \Delta_m^n) \subseteq {}^{2}c_0^{I}(\Delta_m^n)$

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such that $\lim_{i+j\to\infty} f_{ij}(t) = \infty$ for $t > 0$ does not hold. Then there is a number $t_0>0$ and a sequence (k_i) of positive integer such that

$$f_{k_i}(t_0) \le M < \infty. \qquad \ldots\ldots (4)$$

Define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} t_0, & \text{if } i, j = ki \quad i = 1, 2, 3 \ldots\ldots; \\ 0 & \text{otherwise} \end{cases}$$

Thus $x \in {}^2\ell_\infty^I(F, \Delta_m^n)$ by (4).

But $x \notin {}^2c_0^I(\Delta_m^n)$, so that (3) must hold.

If $\ell_\infty^I(F, \Delta^n) \subseteq c_0^I(\Delta^n)$.

Conversely, let (3) hold. If $x \in {}^2\ell_\infty^I(F, \Delta_m^n)$,

then $f_{i,j}(|\Delta_m^n x_{ij}|) \le M < \infty$, for i,j = 1, 2, 3 ….. Suppose that $x \in c_0^I(\Delta^n)$.

Then for some number $c_0 > 0$ and positive integer k_0 we have $|\Delta^n x_{ij}| < c_0$ for $i,j \ge k_0$.

Therefore $f_{i,j}(c_0) \ge f_{i,j}(|\Delta^n x_{ij}|) \le M$ for $i,j \ge k_0$, which contradicts (3).

Hence $x \in {}^2c_0^I(\Delta_m^n)$.

Theorem 2.4: The inclusion ${}^2\ell_\infty^I(\Delta_m^n) \subseteq {}^2c_0^I(F, \Delta_m^n)$ holds if and only if

$$\lim_{i+j\to\infty} f_{i,j}(t) = 0, \quad \text{for } t > 0 \qquad \ldots\ldots.. (5)$$

Proof: Suppose that ${}^2\ell_\infty^I(\Delta_m^n) \subseteq {}^2c_0^I(F, \Delta_m^n)$ but (5) does not hold
Then

$$\lim_{i+j\to\infty} f_{ij}(t_0) = i \ne 0, \text{ for some } t_0 > 0 \qquad \ldots\ldots\ldots.(6)$$

Define the sequence $x = (x_{ij})$ by

$$x_{ij} = t_0 \sum_{u=0}^{i-n} \sum_{v=0}^{i-n} (-1)^{u+v} \begin{bmatrix} n-i-v-i & n-i-u-1 \\ i-v & j-v \end{bmatrix}$$

for i,j = 1, 2, 3 ….. Then $x \notin {}^2c_0^I(F, \Delta_m^n)$ by (6). Hence (5) must hold, conversely, let $x \in {}^2\ell_\infty^I(\Delta_m^n)$ and suppose that (5) holds.

Then $|\Delta_m^n x_{ij}| \le M < \infty$ for K=1, 2, 3, …..

There for $f_{ij}(|\Delta_m^n x_{ij}|) \le f_{ij}(M)$ for i,j = 1, 2, 3 ….. and

$$\lim_{i+j\to\infty} f_{ij}(|\Delta_m^n x_{ij}|) \le \lim_{i+j\to\infty} f_{ij}(M) = 0 \text{ by (5).}$$

Hence $x \in {}^2c_0^I(F, \Delta_m^n)$.

REFERENCES

[1]. Bektas, C.A. and R. Colak : Generalized difference sequence spaces defined by a sequence of moduli, Soochow. J. Math., 29(2) (2003), 164, 215-220.

[2]. Colak, R. and M. ve Et. : On some generalized difference sequence spaces and related matrix transformations, Hokkaido Math. J., 26(3) (1997), 483-492.

[3]. Demerici, K.: I-limit superior and limit inferior, Math. Commun., 6(2001), 165-172.

[4]. Dems K. : On I-Cauchy sequences, Real Analysis Exchange, 30(2005), 123-128.

[5]. Esi, A. and M. Isik : Some generalized difference sequence spaces, Thai J. Math., 3(2) (2005), 241-247.

[6]. Fast, H. : Sur la convergence statistique, Colloq. Math., 2(1951), 241-244.

[7]. Friday, J.A. : On statical convergence, Analysis, 5(1985), 301-313.

[8]. Fridy, J.A. : Statistical limit points, *Proc.* Amer. Math. *Soc.*, 11(1993), 1187-1192.

[9]. Gaur, A.K. and Mursaleen, M. : Difference sequence spaces defined by a sequence of moduli, Demonstratio Math., 31(1908), 275-278.

[10]. Kizmaz, H. : On certain sequence spaces, Canadian Math. Bull., 24(1981), 169-176.

ISSN No: - 2456 – 2165

[11]. E. Kolak, On strong boundedness and summability with respect to a sequence of moduli, Acta Comment. Univ. Tatrtn., 960(1993), 41-50.

[12]. Kolak, E. : Inclusion theorems for some sequence spaces defined by a sequence of modulii, Acta Comment. Univ. *Tartn.*, 970(1194), 65-72.

[13]. Kostyrko, P.; Salat, T. and Wilczynski, W. : I-convergence, Real analysis exchange, 26(2) (2000), 669-686.

[14]. Maddox, I.J. : Sequence spaces defined by a modulus, Math. Camb. Phil. Soc., 100(1986), 161-166.

[15]. Nakano , H.; Concave modulars, J. Math. Soc. Japan, 5(1953), 29-49.

[16]. Ruckle, W.H.: On perfect Symmetric BK-spaces, Math. Ann.,175(1968), 121-126.

[17]. Ruckle, W.H. : Symmetric coordinate space and symmetric bases, Canada, J. Math., 19(1967), 828-838.

[18]. Ruckle, W.H. : FK-spaces in which the sequence of coordinate vector is bounded, Canada, J. Math., 25(5) (1973), 973-975.

[19]. Salat, T. : On statical convergent sequences of real numbers, Math. Solvaca., 30(1980), 139-150.

[20]. Salat, T.; Tripathy, B.C. and Ziman, M. : On some properties of I-convergence, Tatra Mt. Math. Publ., 28(2004), 279-286.