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Projective Change between Two Subclasses of (α, β)-Metrics

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Abstract: In this article, we characterize the projective relation between the special $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-metric $\boldsymbol{F} = \boldsymbol{\alpha} + \boldsymbol{\beta} + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and Randers metric with dim $\boldsymbol{n} \geq 3$. In the first part, we proved that both the metrics are Douglas Metrics. Further we studied the projective change between the two $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-metrics.

Keywords:- *Finsler space, Special (α, β)-Metric, Randers Metric, Projective Relation, Douglas Metric, Geodesic Sprays.*

AMS Subject Classification (2010): 53B40, 53C60, 53B10**;**

I. INTRODUCTION

A Finsler metric on a manifold M is a Function $F = F(x, y)$ defined as $F: TM \to [0, \infty)$ with the following properties:

A. $F = F(x, y)$ is C^∞ on $TM \setminus \{0\}$,
B. $F_x(y) = F(x, y)$ is positive 1-homogeneous on the fibres of the tangent bundle TM.
C. Strongly convex: The $n \times n$ hessian matrix

$g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}$ is strong definiteat every point of $TM \setminus \{0\}$.

Thus, the pair (M, F) is called a Finsler space.

In projective Finsler geometry, its important concept to study the projectively related Finsler spaces with (α, β)-metrics. It is one of the important problem in Projective Finsler geometry to study the metrics on open subset in the n-dimensional real vector space, whose geodesics are straight lines. FInsler metrics in which the geodesics are straight lines are said to be projective Finsler metrics. The projective relation between two Finsler space with (α, β)-metric have been studied by many authors [2], [9], [10], [13], [14].

In this article, the results can proved in two stages: In the first stage we proved that both the Finsler metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ are Douglas metrics. Further, in the next stage, we study the projective relation between (α, β)-metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$.

II. PRELIMINARIES

For a given Finsler metric $F = F(x, y)$, the geodesic of F satisfies the ODE's:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0$$

where $G^i = G^i(x, y)$ is called the geodesic coefficient, which is given by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}$$

The condition for a Finsler metric F is projectively related to $\bar{F}$ has been characterized by using spray coefficients [10],

$$G^i = \bar{G}^i + P(x, y)y^i, \qquad (2.1)$$

where $P(x, y)$ is a scalar function on $TM \setminus \{0\}$, with $P(x, \lambda y) = \lambda P(x, y), \lambda > 0$ and G and $\bar{G}$ are the spray coefficients of F and $\bar{F}$.

If $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is one form with $||\beta_x||_\alpha < b_0$ for $\forall x \in M$, then Finsler space with (α, β)-metric can be characterized as follows, $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$. The function $\phi(s)$is a c^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0; (|s| \leq b < b_0)$.(2.2)

In this case, the fundamental form of the metric tensor induced by F is positive definite.

Consider $r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i})$ where $b_{i|j}$ means the coefficients of the covariant derivative of β with respect to α. Clearly β is closed if and only if $s_{ij} = 0$. An (α, β)-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Furthermore, we denote $s_j = b^i s_{ij}, r^i_j = a^{il}r_{lj}, s^i_j = a^{il}s_{lj}, s_0 = s^i_{y}, s^i_0 = s^i_j y^j and\ r_{00} = r_{ij}y^i y^j$.

The geodesic coefficients G^i of F and geodesic coefficients $\bar{G}^i_\alpha$ of α is related as follows:

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$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\psi b^i + \theta\alpha^{-1}y^i\}, \quad (2.3)$$

Where

$$\theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}, \qquad Q = \frac{\phi'}{\phi - s\phi'},$$

$$\psi = \frac{1}{2}\frac{\phi''}{\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}.$$

Definition 2.1.
The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$, where

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right). \quad (2.4)$$

is called the Douglas tensor. A Finsler metric with vanishing Douglas tensor are called Douglas metrics. For an (α, β)-metric, Douglas tensor is characterized by [10],

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right), \quad (2.5)$$

Where $T^i = \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}b^i$. (2.6)

and

$$T^m_{y^m} = Q^i s_0 + \Psi\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] + 2\Psi[r_0 - Q^i(b^2 - s^2)s_0 - Qss_0]. \quad (2.7)$$

Here the terms with bar represents the terms of $\bar{F}$.

For Finsler (α, β)-metrics F and $\bar{F}$, they have the same Douglas tensor, i.e., $D^i_{jkl} = \bar{D}^i_{jkl}$.

From (2.4) and (2.5) we have

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left[T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\right] = 0. \quad (2.8)$$

Now we define a scalar functions $H^i_{jk} = H^i_{jk}(x)$ represents

$$T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i = H^i_{00}, \quad (2.9)$$

where $H^i_{00} = H^i_{jk}(x)y^j y^k, T^i$ *and* $T^m_{y^m}$ are given by (2.6) and (2.7) respectively. Here we denote $\lambda = \frac{1}{n+1}$.

III. PROJECTIVE RELATION BETWEEN SPECIAL (α, β)-METRIC AND RANDERS METRIC

For an (α, β)-metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$, by (2.3) the geodesic coefficients of F are as follows:

$$\theta = \frac{9 - 54s^2 - 120s^3 + 45s^4 + 144s^5 - 24s^7}{6(3 + 3s + 6s^2 - s^4)\{1 - (6 + 4b^2)s^2 + 5s^4\}},$$

$$Q = \frac{3 + 12s - 4s^3}{3 - 6s^2 + 3s^4},$$

$$\Psi = \frac{2 - 2s^2}{\{1 + 4b^2 - (6 + 4b^2)s^2 + 5s^4\}}. \quad (3.1)$$

For Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, by (2.3), the geodesic coefficients $\bar{F}$ are as follows,

$$\bar{\theta} = 1, \quad \bar{Q} = \frac{1}{2(1+s)}, \bar{\Psi} = 0. \quad (3.2)$$

First, we prove the following lemma;

Lemma 3.1. Two Finsler metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ on an n-dimensional manifold $M(n \geq 3)$ have the same Douglas tensors if and only if they are all Douglas metrics.

Proof: Sufficient condition:

Two Finsler metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ be Douglas metrics and having Douglas tensors be D^i_{jkl} *and* $\bar{D}^i_{jkl}$. According to the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, i.e., both F and $\bar{F}$ have the same Douglas tensor.

Necessary condition:

If F and $\bar{F}$ have the same Douglas tensor, then (2.9) holds. Plugging (3.1) and (3.2) into (2.9), we have

$$H^i_{00} = \frac{\begin{pmatrix} A_{17}\alpha^{17} + A_{16}\alpha^{16} + A_{15}\alpha^{15} + A_{14}\alpha^{14} + A_{13}\alpha^{13} \\ + A_{12}\alpha^{12} + A_{11}\alpha^{11} + A_{10}\alpha^{10} \end{pmatrix}}{\begin{pmatrix} B_1\alpha^{16} + B_2\alpha^{14} + B_3\alpha^{12} + B_4\alpha^{10} + B_5\alpha^8 \\ + B_6\alpha^6 + B_7\alpha^4 + B_8\alpha^2 + B_9 \end{pmatrix}} + \frac{\begin{pmatrix} A_9\alpha^9 + A_8\alpha^8 + A_7\alpha^7 + A_6\alpha^6 + A_5\alpha^5 \\ + A_4\alpha^4 + A_2\alpha^2 + A_1 \end{pmatrix}}{\begin{pmatrix} B_1\alpha^{16} + B_2\alpha^{14} + B_3\alpha^{12} + B_4\alpha^{10} + B_5\alpha^8 + B_6\alpha^6 \\ + B_7\alpha^4 + B_8\alpha^2 + B_9 \end{pmatrix}} - \bar{\alpha}\bar{s}^i_0. \quad (3.3)$$

Where,

$A_{17} = 9(1 + 4b^2)\{(1 + 4b^2)s^i_0 - 4s_0 b^i\}$,

$A_{16} = (1 + 4b^2)[36\beta\{(1 + 4b^2)s^i_0 - 4s_0 b^i\} + 18r_{00}b^i - 36(r_0 + s_0)\lambda y^i]$,

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$A_{15} = -18\beta^2(1+4b^2)(7+8b^2)s_0^i + 36\beta^2(9+16b^2)s_0 b^i - (28+32b^2)\beta s_0 \lambda y^i,$

$A_{14} = -12\beta^3(1+4b^2)(43+52b^2)s_0^i + 144\beta^3(57+208b^2)s_0 b^i - 18\beta^2(10+20b^2)r_{00}b^i + 9b^2(28+32b^2)\beta r_{00}\lambda y^i + 36\beta^2(11+24b^2)r_0\lambda y^i + 18\beta^2(23-16b^2-23b^4)s_0\lambda y^i,$

$A_{13} = 9\beta^4\{35+208b^2+96b^4\}s_0^i - 36\beta^4(26+24b^2)s_0 b^i + 18\beta^3(28+56b^2+64b^4)s_0\lambda y^i,$

$A_{12} = 12\beta^5\{227+600b^2+352b^4\}s_0^i + 48\beta^5\{-87-88b^2\}s_0 b^i + 18\beta^4\{41+44b^2\}r_{00}b^i + 9\beta^3(8b^2-28)r_{00}\lambda y^i + 36\beta^4\{-45-60b^2\}r_0\lambda y^i + \beta^4\{828+3648b^4-2736b^2\}s_0\lambda y^i,$

$A_{11} = -18\beta^6\{82+136b^2+32b^4\}s_0^i + 36\beta^6\{14+32b^2\}s_0 b^i + 18\beta^5\{32-72b^2-32b^4\}s_0\lambda y^i,$

$A_{10} = 12\beta^7\{572+1076b^2+432b^4\}s_0^i + 48\beta^7\{134+72b^2\}s_0 b^i + 18\beta^6\{-97-80b^2\}r_{00}b^i + 18\beta^5\{36+2b^2+32b^4\}r_{00}\lambda y^i + 18\beta^6\{190+160b^2\}r_0\lambda y^i\ 12\beta^6[\{224+824b^2-292b^4\}+192b^2(1+4b^2)]\lambda s_o y^i,$

$A_9 = 9\beta^8\{191+168b^2+16b^4\}s_0^i + 36\beta^8\{-21-4b^2\}s_0 b^i + 18\beta^7\{-54-400b^2-288b^4\}s_0\lambda y^i,$

$A_8 = 12\beta^9\{737+816b^2+112b^4\}s_0^i + 48\beta^9\{-62-24b^2\}s_0 b^i + 18\beta^8\{115+60b^2\}r_{00}b^i + 18\beta^7\{6+8b^2-64b^4\}r_{00}\lambda y^i + 36\beta^8\{115+60b^2\}r_0\lambda y^i + \beta^8\{-12224-5456b^2+2176b^4\}s_0\lambda y^i,$

$A_7 = \beta^{10}\{90+360b^2\}s_0^i + \beta^{10}180 s_0 b^i + \beta^9\{522+72b^2\}s_0\lambda y^i,$

$A_6 = 12\beta^{11}\{-521-288b^2-16b^4\}s_0^i + \beta^{11}\{1728+192b^2\}s_0 b^i + 18\beta^{10}\{-81-24b^2\}r_{00}b^i + \beta^9\{-116b2+288b^4-1872\}r_{00}\lambda y^i + 36\beta^{10}\{61+8b^2\}\, r_0\, \lambda y^i + \beta^{10}\{10072+192b^2-704b^4\}\lambda s_0 y^i,$

$A_5 = 225\beta^{12}s_0^i - 360\beta^{11}\lambda s_0 y^i,$

$A_4 = \beta^{13}\{2220+480b^2\}s_0^i - 90\beta^{13}s_0 b^i + 18\beta^{12}\{31+4b^2\}r_{00}b^i + 9\beta^{11}\{292+48b^2\}r_{00}\lambda y^i + 36\beta^{12}\{14-4b^2\}r_0\lambda y^i\beta^{12}\{-1964+1536b^2\},$

$A_3 = 0,$

$A_2 = -90\beta^{14}r_{00}b^i + \beta^{13}\{1080-180b^2\}r_{00}\lambda y^i + 180\beta^{14}r_0\lambda y^i - 400\beta^{14}s_0\lambda y^i,$

$A_1 = 180\beta^{15}r_{00}\lambda y^i,$

and

$B_1 = 9(1\ +\ 4b^2)2, B_2 = 18(1+4b^2)(-8-12\beta^2)\beta^2,$

$B_3 = 9(324+136b^2+240b^4)\beta^4, B_4 = 18(-165-380b^2-160b^4)\beta^6,$

$B_5 = 9(830+\ 920b^2+208b^4)\beta^8, B_6 = 8(-328-324b^2-48b^4)\beta^{10},$

$B_7 = 9(361+248b^2+16b^4)\beta^{14}, B_8 = 18(-80-20b^2)\beta^{14}, B_9 = 225\beta^{16}.$

Further, (3.3) is equivalent to

$$(A_{17}\alpha^{17} + A_{16}\alpha^{16} + A_{15}\alpha^{15} + A_{14}\alpha^{14} + A_{13}\alpha^{13} + A_{12}\alpha^{12} + A_{11}\alpha^{11} + A_{10}\alpha^{10} + A_9\alpha^9 + A_8\alpha^8 + A_7\alpha^7 + A_6\alpha^6 + A_5\alpha^5 + A_4\alpha^4 + A_2\alpha^2 + A_1) = (H_{00}^i + \bar{\alpha}\,\bar{s}_0^i)(B_1\alpha^{16} + B_2\alpha^{14} + B_3\alpha^{12} + B_4\alpha^{10} + B_5\alpha^8 + B_6\alpha^6 + B_7\alpha^4 + B_8\alpha^2 + B_9). \quad (3.4)$$

Replacing (y^i) by $(-y^i)$ in (3.4) yields

$$(-A_{17}\alpha^{17} + A_{16}\alpha^{16} - A_{15}\alpha^{15} + A_{14}\alpha^{14} - A_{13}\alpha^{13} + A_{12}\alpha^{12} - A_{11}\alpha^{11} + A_{10}\alpha^{10} - A_9\alpha^9 + A_8\alpha^8 - A_7\alpha^7 + A_6\alpha^6 - A_5\alpha^5 + A_4\alpha^4 + A_2\alpha^2 + A_1) = (H_{00}^i - \bar{\alpha}\,\bar{s}_0^i)(B_1\alpha^{16} + B_2\alpha^{14} + B_3\alpha^{12} + B_4\alpha^{10} + B_5\alpha^8 + B_6\alpha^6 + B_7\alpha^4 + B_8\alpha^2 + B_9). \quad (3.5)$$

subtracting (3.5) from (3.4), we get

$$(A_{17}\alpha^{17} + A_{15}\alpha^{15}A_{13}\alpha^{13} + A_{11}\alpha^{11} + A_9\alpha^9 + A_7\alpha^7 + A_5\alpha^5) = (H_{00}^i + \bar{\alpha}\,\bar{s}_0^i)(B_1\alpha^{16} + B_2\alpha^{14} + B_3\alpha^{12} + B_4\alpha^{10} + B_5\alpha^8 + B_6\alpha^6 + B_7\alpha^4 + B_8\alpha^2 + B_9). \quad (3.6)$$

From (3.6), $B_9\bar{\alpha}\,\bar{s}_0^i$ has the factor α^2, i.e., the term $225\beta^{16}\bar{\alpha}\,\bar{s}_0^i$ has the factor α^2.

Now we studied two cases for α.

Case(i): If $\bar{\alpha} \not\equiv \mu(x)\alpha$, then α^2 is one of term in $B_9\bar{\alpha}\,\bar{s}_0^i = 225\beta^{16}\bar{\alpha}\,\bar{s}_0^i$. But β^2 has no factor α^2. Then the only possibility is that $\beta\bar{s}_0^i$ has the factor α^2. Then for each i there exists a scalar function $\tau^i = \tau(x)$ such that $\beta\bar{s}_0^i = \tau^i\alpha^2$ which implies that $b_j\bar{s}_k^i + b_k\bar{s}_j^i = 2\tau^i\alpha_{jk}$.

When $n \geq 3$, if we assume $\tau^i \neq 0$, then

$$2 \geq rank(b_j\bar{s}_k^i) + rank(b_k\bar{s}_j^i)$$

$$> rank(b_j\bar{s}_k^i + b_k\bar{s}_j^i)$$

$$= rank(2\tau^i\alpha_{jk}) \geq 3. \quad (3.7)$$

From (3.7) satisfies only when $\tau^i = 0$. Then $\beta\bar{s}_0^i = 0$. but $\beta \neq 0$, implies $\bar{s}_0^i = 0$. That implies $\bar{\beta}$ is closed.

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Case (ii): If $\bar{\alpha} \equiv \mu(x)\alpha$, then (3.6) reduces to

$\mu(x)\bar{s}_0^i\beta_9 = [A_{17}\alpha^{14} + A_{15}\alpha^{12}A_{13}\alpha^{10} + A_{11}\alpha^8 + A_9\alpha^6 + A_7\alpha^4 + A_5\alpha^2 - \mu(x)\{B_1\alpha^{14} + B_2\alpha^{12} + B_3\alpha^{10} + B_4\alpha^8 + B_5\alpha^6 + B_6\alpha^4 + B_7\alpha^2 + B_8\}]\alpha^2.$
(3.8)

In (3.8), we observe that, α^2 is one of the factor in $\mu(x)B_9\bar{s}_0^i$.i.e., α^2 is a factor of $\mu(x)B_9\bar{s}_0^i = 225\mu(x)\bar{s}_0^i\beta^{16}$. Here $\mu(x) \neq 0,\ \forall x \in M$, and α^2 has no factor of β^2. Then the only way is that α^2has a factor of $\beta\bar{s}_0^i$. With reference to case-(i), we have $\bar{s}_0^i = 0$, when $n \geq 3$, which says that $\bar{\beta}$ is closed.

And we know that the Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric [4].

In the view of above discussions, we conclude that the both F and $\overline{F}$ are Douglas metric.

Now we prove the following main Theorem:

Theorem 3.1. The projective change among these two Finsler (α, β)-metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$, can be characterized if and only if they satisfy the following conditions:

$G_\alpha^i = \bar{G}_\alpha^i + \theta y^i - \tau\alpha^2 b^i,$

$b_{i|j} = \frac{\tau}{2}\left[(1 + 4b^2)a_{ij} - 5b_i b_j\right],$

$d\bar{\beta} = 0,$
(3.9)

where $b = \left||\beta|\right|_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to $\alpha, \tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a1-form on a manifold M with dimension $n \geq 3$.

Proof: Necessary condition.
As we know that, In projective change, Douglas tensor is invariant. From the above lemma 3.1, we proved that the both the Finsler metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ are Douglas metrics.

We know that Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed [10], that is
$d\bar{\beta} = 0.$
(3.10)

and $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ is a Douglas metric if and only if
$b_{i|j} = \frac{\tau}{2}\left[(1 + 4b^2)a_{ij} - 5b_i b_j\right],$
(3.11)

For some scalar function $\tau = \tau(x)$ [3]. In this case, β is closed. Since β is closed, $s_{ij} = 0$, implies $b_{i|j} = b_{j|i}$. Thus
$s_0^i = 0,\ s_0 = 0.$

By using (3.11), we have $r_{00} = \frac{\tau}{2}[(1 + 4b^2)\alpha^2 - 5\beta^2].$

Plugging (3.11) into (2.3) with (3.1) yields,

$G^i = G_\alpha^i + \tau\alpha^2 b^i + \tau\left[\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta^1+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right]y^i$

(3.12)

Since F is projective to $\bar{F} = \bar{\alpha} + \bar{\beta}$, this is a Randers change between F and $\bar{\alpha}$. Noticing that $\bar{\beta}$ is closed, then F is projectively related to $\bar{\alpha}$. Thus there is a scalar function $P = P(y)$ on $TM \setminus \{0\}$ such that
$G^i = G_{\bar{\alpha}}^i + Py^i.$
(3.13)

From (3.12) and (3.13), we have
$\left[P - \tau\left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta^1+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\right]y^i = G_\alpha^i - G_{\bar{\alpha}}^i + \tau\alpha^2 b^i.$
(3.14)

Here RHS of (3.14) is a quadratic form. Then there exist a one form $\theta = \theta_i y^i$ on M, suchthat

$P - \tau\left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta^1+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right) = \theta$.

Thus (3.14) becomes
$G_\alpha^i = G_{\bar{\alpha}}^i + \theta y^i - \tau\alpha^2 b^i$.
(3.15)

By (3.10) and (3.11) together with (3.15), completes the proof of the necessity condition.

Sufficient Condition:
Here we know that $\bar{\beta}$ is closed, it suffices to prove that F is projectively related to $\bar{\alpha}$. Inserting (3.11) into (2.3) with (3.1) ressults (3.12).

By (3.12) and (3.15), we get
$G^i = G_{\bar{\alpha}}^i + \left[\theta + \tau\left(\frac{\{(1+4b^2)\alpha^2-5\beta^2\}\{9\alpha^7-54\alpha^5\beta^2-120\alpha^4\beta^3+45\alpha^3\beta^4+144\alpha^2\beta^5-24\beta^7\}}{12\{3\alpha^4+3\alpha^3\beta^1+6\alpha^2\beta^2-\beta^4\}\{\alpha^4-(6+4b^2)\alpha^2\beta^2+5\beta^4\}}\right)\right]y^i$

(3.16)

i.e., by (3.16) we came to a conclusion that F is Projectively related to $\bar{\alpha}$.

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From theorem (3.1), we define the following corollary:

If we assume that the Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b}^i y^i$ is 1-form with $\bar{b}^i = constants$. Then (3.9) can be written as

$$G_\alpha^i = G_{\bar{\alpha}}^i + \theta y^i - \tau\alpha^2 b^i,$$

$$b_{i|j} = \frac{\tau}{2}\left[(1+4b^2)a_{ij} - 5b_i b_j\right].$$
(3.17)

Thus, we state that

Corollary 3.2. The Finsler metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ is projectively related $\bar{F} = \bar{\alpha} + \bar{\beta}$, Then the Finsler metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ projective flat if and only if (3.17) holds.

IV. CONCLUSION

After overall discussion of this article, the results can be summarized as follows:

- Two Finsler metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$ on an n-dimensional manifold $M(n \geq 3)$ have the same Douglas tensors if and only if they are all Douglas metrics.
- The projective change among these two Finsler (α, β)-metrics $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ and $\bar{F} = \bar{\alpha} + \bar{\beta}$, can be characterized if and only if they satisfy the following conditions:

$$G_\alpha^i = \bar{G}_\alpha^i + \theta y^i - \tau\alpha^2 b^i,$$

$$b_{i|j} = \frac{\tau}{2}\left[(1+4b^2)a_{ij} - 5b_i b_j\right],$$

$$d\bar{\beta} = 0.$$

- The Finsler metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ is projectively related $\bar{F} = \bar{\alpha} + \bar{\beta}$, Then the Finsler metric $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{\alpha^3}$ projective flat if and only if (3.17) holds.

REFERENCES

[1]. P. L. Antonelli, R. S. Ingarden and M. Matsumto, *The Theory of Sprays and Finsler spaces with application in Physics and biology*, kluwer acad. publ., Dordrecht, Boston, London, (1985).

[2]. S. Bacso and M. Matsumoto, *Projective changes between Finsler space with (α, β)-metrics*, ibibd., 55(1994), 252-257.

[3]. M. Hashiguchi and Y. Ichijyo, *Randers space with rectilinear geodesics*, Rep. Fac. Sci. Kagoshima Uni.(Math.Phys.Chen.)., 13(1980), 33-40.

[4]. B. Li, Y. Shen and Z. Shen, *On a Class of Douglas metrics*, Studia Scientiarum Mathematicarum Hungarica, 46 (3) (2009), 355-365.

[5]. M. Matsumto, *Finsler Space with (α, β)-metric of Douglas type*, Tensor N.S. 60 (1998).

[6]. M. Matsumto, *On C-reducible Finsler spaces*, Tensor N.S. 24 (1972), 29-37.

[7]. S. K. Narasimhamurthy, *Projective changes between Matsumoto metric and Randers metric*, Proc. jangjeon. Math. Soc., 17 (2014), No. 03, 393-402.

[8]. Ningwei Cui and Yi-Bing Shen, *Projective changes between two classes of (α, β)-metrics*, Differential geometry and its applications., 27 (2009), 566-573.

[9]. A. Tayabi, E. Peyghan and H. Sadeghi, *On two subclasses of (α, β)-metrics being projectively related*, Journal of geometry and physics., 62 (2012),292-300.

[10]. M. Zohrehvand and M. M. Rezaii, *On projectively related of two special classes of (α, β)-metrics*, Differential Geometry and its applications., 29 (2012), 660-669.