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Conformal Kropina-Randers Change of Finsler Space with (α,β)-Metric of Douglas Type

Ramesha M[1] & Veeresh Malagi[2]
[1&2] Department of Mathematics,
School of Engineering and Technology,
Jain Global Campus, Jakkasandra-562112,
Ramanagara, Karnataka, India

Narasimhamurthy S. K[3]
[3]Department of P.G. Studies and Research in Mathematics,
Kuvempu University,
Shankaraghatta-577 451,
Shivamogga, Karnataka, India

Abstract:- A change of Finsler metric is $L(\alpha,\beta) \rightarrow \bar{L}(\bar{\alpha},\bar{\beta}) = e^{\rho(x)}\left[\frac{L^2(\alpha,\beta)}{\beta} + \beta\right]$ called conformal Kropina-Randers change, where σ is a function of position (x^i) only, α is Riemannian metric and β is a differential 1-form. In this article we mainly concentrated on Conformal Kropina-Randers change of a Finsler space. The present article is organized as follows: In the first part, we devoted to study about the necessary and sufficient condition for a Finsler space $\overline{F^n}$ which is obtained by Conformal Kropina-Randers change of a Finsler space F^n with (α,β) −metric of Douglas type to be also Douglas type. In the next part, we are discussing the conformal Kropina-Randers change of Finsler space with the Riemannian metric, Randers metric and finally Generalized Kropina metric.

***Keywords*:-** *Finsler space, (α,β) −metrics, Douglas space, Conformal Kropina-Randers change.*

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I. INTRODUCTION

The Finsler space $F^n = (M^n, L(x,y))$ is said to have an (α,β) −metric if L is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$. Basically the concept of Douglas space was introduced by S. Bacso and M. Matsumoto [2], as a generalization of Berwlad space from the view point of geodesic equations. A Finsler space issaid to be douglas space if $D^{ij} = G^iy^j - G^j y^i$ are homogeneous polynomial in (y^i) of degree three. M. Matsumoto [9], has found the condition that, the Finsler space with (α,β) −metric to be of Douglas space.

Conformal theory of Finsler space was introduced by M. S. Kneblem in 1929 and this theory has been investigated in detail by M. Hasiguchi [3], later on Y. D. Lee [6]. Ichijyo Y and Hasiguchi M [4], have studied the conformal change of (α,β) −metric. In [10], S. K. Narasimhamurthy, Ajith and C. S. Bagewadi, studied the necessary and sufficient condition for Douglas space with (α,β)-metric under conformal β-change. And also in [11], H. S. Sukla, O. P. Pandey and H. D. Joshi worked on conformal Kropina change of a Finsler space with (α,β) −metric of Douglas type.

In this article, we deduce the condition for a Finsler space $\bar{F}n$ which is transformed by conformal Kropina-Randers change of a Finsler space F^n with (α,β) −metric of Douglas type to be also Douglas type. And also we proved that, conformal Kropina-Randers change of a Finsler space $\bar{F}^n$ with the (α,β) −metrics to be of Douglas type.

II. PRELIMINARIES

Definition 2.1. *A Finsler space is the ordered pair $F^n = (M, L)$, where M is an n-dimensional differentiable manifold and L is a Finsler metric defined as a function $L : TM \rightarrow [0,\infty]$ with the following properties:*

i. *Regular: L is c^∞ on the entire tangent bundle TM_0,*
ii. *Positive homogeneous : $L(x,\lambda y) = \lambda L(x,y)$; $\forall \lambda \geq 0$,*
iii. *strong convexity : The $n \times n$ Hessian Matrix ;$g_{ij} = \frac{1}{2}[L^2]_{ij}$ is positive definite at every point on $TM_0$where TM_0 denotes the tangent vector y is non zero in the tangent bundle TM.*

The geodesic of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations:

$$\frac{d^2x^i}{dt^2}y^j - \frac{d^2x^j}{dt^2}y^i + 2(G^iy^j - G^jy^i) = 0, y^i = \frac{dx^i}{dt},$$ in a parameter *t*.

The function $G^i(x,y)$ is given by

$$2G^i(x,y) = g^{ij}\left(y^r\dot{\partial}_j\partial_r F - \partial_j F\right) = \gamma^i_{jk}y^jy^k$$

where$\partial_i = \frac{\partial}{\partial x^i}$,$F = \frac{L^2}{2}$ and $g^{ij}(x,y)$ be the inverse of the Finsler metric tensor $g_{ij}(x,y)$ with respect to (x^i). For an (α,β) −metric $L(\alpha,\beta)$, the space $R^n = (M,\alpha)$ is called the associated Riemannian space with $F^n = (M^n, L(\alpha,\beta))$ ([1], [8]). The covariant differentiation with respect to Levi-Civita connection $\{\gamma^i_{jk}\}$ } of R^n is denoted by (|).

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From the differential 1-form $\beta(x,y) = b_i(x)y^i$, we define:

$$2r_{ij} = b_{i/j} + b_{j/i}, r_j^i = a^{ih}r_{hj}, r_j = b_i r_j^i, \quad (2.1)$$
$$2s_{ij} = b_{i/j} - b_{j/i}, s_j^i = a^{ih}s_{hj}, s_j = b_i s_j^i,$$
$$b^i = a^{ih}b_h, \; b^2 = b^i b_i.$$

Definition 2.2. *A Finsler space F^n is said to be a Douglas space if*

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i, \quad (2.2)$$

is homogeneous polynomial in (y^i) of degree three.

According to [9], a Finsler space F^n is of Douglas type if and only if the Douglas tensor

$$D_{ijk}^h = D_{ijk}^h - \frac{1}{2}(G_{ijk}y^h + G_{ij}\delta_k^h + G_{jk}\delta_i^h + G_{ki}\delta_j^h)$$

vanishes identically, where $G_{ijk}^h = \dot{\partial}_k G_{ij}^h$ is the hv-curvature tensor of the Berwald connection

$$B\Gamma = (G_{jk}^i, G_j^i, 0), Gij = G_{ijr}^r$$
$$\text{and } Gijk = \dot{\partial}_k G_{ij}.$$

According to [7], For a Finsler space F^n with the (α,β) −metric, the function $G^i(x,y)$ can be written as,

$$2G^i = \gamma_{00}^i + 2B^i$$
$$B^i = \left(\frac{\alpha L_\beta}{L_\alpha}\right)S_0^i + C^*\left[\frac{\beta L_\beta}{\alpha L}y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i - \frac{\alpha}{\beta}b^i\right)\right] \quad (2.3)$$

where,

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}, \gamma^2 = b^2\alpha^2 - \beta^2 \quad (2.4)$$

Since $\gamma_{00}^i = \gamma_{jk}^i(x)y^i y^j$ are homogeneous polynomial in (y^i) of degree 2, From (2.3), we have

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s_0^i y^i - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*(b^i y^j - b^j y^i) \quad (2.5)$$

By means of (2.2) and (2.5), we use the following lemma proved by M. Matsumoto [9],

Lemma 2.1. *A Finsler space F^n with an (α,β) −metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are $hp(3)$.*

III. CONFORMAL KROPINA-RANDERS CHANGE OF FINSLER SPACE OF DOUGLAS TYPE

In this section, we are discussed the necessary and sufficient condition for a Finsler space $\overline{F^n}$ which is obtained by conformal Kropina-Randers change of Finsler space $F^n = (M^n, L)$ is of Douglas type.

Now, let us consider $F^n = (M^n, L)$ and $\overline{F^n} = (M^n, \bar{L})$ be the two Finsler spaces on the same underlying manifold M^n such that $\bar{L}(\bar{\alpha},\bar{\beta}) = e^\sigma\left[\frac{L^2}{\beta} + \beta\right]$, then F^n is called conformaly Kropina-Randers to $\bar{F}^n$, and the change $L \to \bar{L}$ of metric is called conformal Kropina Randers change of (α,β)- metric. A conformal Kropina Randers change of (α,β) −metric is expressed as

$$(\alpha,\beta) \to (\bar{\alpha},\bar{\beta}), \text{ where } \bar{\alpha} = e^\sigma\alpha, \bar{\beta} = e^\sigma\beta.$$

Therefore, we have

$$\bar{a}_{ij} = e^{2\sigma}a_{ij}, \quad \bar{a}^{ij} = e^{-2\sigma}a^{ij}$$
$$\bar{b}_i = e^\sigma b_i, \quad \bar{b}^i = e^{-\sigma}b^i$$
$$\bar{b}^2 = b^2, \; \bar{y}^i = y^i, \bar{y}_i = e^{2\sigma}y_i \quad . \quad (3.1)$$

Thus, we have

Proposition 1. *A Finsler space with (α,β) −metric the length b of b_i with respect to the Riemannian metric α is invariant under any conformal change of (α,β) −metric.*

From (3.1), the conformal change of Christoffel symbols is given by [3];

$$\bar{\gamma}_{jk}^i = \gamma_{jk}^i + \delta_j^i\sigma_k + \delta_k^i\sigma_j - \sigma^i a_{jk} \quad (3.2)$$

where $\sigma_j = \partial_j\sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (2.1),(3.1) and (3.2), we have the following under conformal change:

$$\bar{b}_{i/j} = e^\sigma(b_{i/j} + \rho a_{ij} - \sigma_i b_j)$$
$$\bar{r}_{ij} = e^\sigma\left[r_{ij} + \rho a_{ij} - \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)\right]$$
$$\bar{s}_{ij} = e^\sigma\left[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)\right],$$
$$\bar{s}_j^i = e^{-\sigma}\left[s_j^i + \frac{1}{2}(b^i\sigma_j - b_j\sigma^i)\right]$$
$$\bar{s}_j = s_j + \frac{1}{2}(b^2\sigma_j - \rho b_j) \quad \text{where } \rho = \sigma_r b^r \quad (3.3)$$

From (3.2) and (3.3), we can easily obtain the following:

$$\bar{\gamma}_{00}^i = \gamma_{00}^i + 2\sigma_0 y^i - \alpha^2\sigma^i$$
$$\bar{r}_{00} = e^\sigma[r_{00} + \rho\alpha^2 - \sigma_0\beta]$$
$$\bar{s}_0^i = e^{-\sigma}\left[s_0^i + \frac{1}{2}(b^i\sigma_0 - \beta\sigma^i)\right]$$
$$\bar{s}_0 = s_0 + \frac{1}{2}(b^i\sigma_0 - \rho\beta) \quad (3.4)$$

Next, to find the conformal Kropina-Randers change of B^{ij} given in (2.5), we first find the conformal Kropina-Randers change of C^* given in (2.4).

Since $\bar{L}(\bar{\alpha},\bar{\beta}) = e^\sigma\left[\frac{L^2}{\beta} + \beta\right]$, we have

$$\bar{L}_{\bar{\alpha}} = \frac{2LL_\alpha}{\beta}, \bar{L}_{\bar{\alpha}\bar{\alpha}} = \frac{2e^{-\sigma}}{\beta}\{(L_\alpha)^2 + LL_{\alpha\alpha}\},$$
$$\bar{L}_{\bar{\beta}} = \frac{2\beta LL_\beta - L^2}{\beta^2} + 1, \bar{\gamma}^2 = e^{2\sigma}\gamma^2 \quad (3.5)$$

By using (2.4), (3.4) and (3.5), we obtain,

$$\bar{C}^* = e^\sigma(C^* + D^*) \quad (3.6)$$

Where

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$$D^* = \frac{2[\alpha^2\beta^2L^2L_\alpha + \alpha^3\gamma^2L^2L_{\alpha\alpha} - \alpha^2\beta^4L_\alpha - \alpha^3\beta^2\gamma^2L_{\alpha\alpha}]s_0}{4[\beta^2LL_\alpha + \alpha\gamma^2(L_\alpha)^2 + \alpha\gamma^2LL_{\alpha\alpha}][\beta^2L_\alpha + \alpha\gamma^2L_{\alpha\alpha}]} + \frac{\begin{Bmatrix}(\alpha^2L^2 - 2\alpha^2\beta LL_\beta - \alpha^2\beta^2) \\ (b^2\sigma_0 - \rho\beta) \\ +2(\rho\alpha^2 - \sigma_0\beta)\alpha\beta LL_\alpha\end{Bmatrix}(\beta^2L_\alpha + \alpha\gamma^2L_{\alpha\alpha})}{4\{\beta^2LL_\alpha + \alpha\gamma^2(L_\alpha)^2 + \alpha\gamma^2LL_{\alpha\alpha}\}\{\beta^2L_\alpha + \alpha\gamma^2L_{\alpha\alpha}\}} - \frac{\alpha^2\beta\gamma^2(L_\alpha)^2\{r_{00}L_\alpha - 2\alpha L_\beta s_0\}}{4\{\beta^2LL_\alpha + \alpha\gamma^2(L_\alpha)^2 + \alpha\gamma^2LL_{\alpha\alpha}\}\{\beta^2L_\alpha + \alpha\gamma^2L_{\alpha\alpha}\}} \quad (3.7)$$

Hence under the conformal Kropina-Randers change, B^{ij} can be written in the form:

$$\bar{B}^{ij} = B^{ij} + C^{ij},$$

Where

$$C^{ij} = \frac{\alpha(\beta^2 - L^2)}{2\beta LL_\beta}(s_0^iy^j - s_0^jy^i) + \frac{2\alpha\beta LL_\beta + \alpha(\beta^2 - L^2)}{4\beta LL_\alpha}\begin{Bmatrix}\sigma_0(b^iy^j - b^jy^i) \\ -\beta(\sigma^iy^j - \sigma^jy^i)\end{Bmatrix} + \left[\frac{\alpha^2LL_{\alpha\alpha} + \alpha^2(L_\alpha)^2}{\beta LL_\alpha}D^* + \frac{\alpha^2L_\alpha}{\beta L}C^*\right](b^iy^j - b^jy^i) \quad (3.8)$$

Thus, the necessary and sufficient condition for Finsler space $\bar{F}^n$ to be of Douglas type is that C^{ij} are hp(3).

Theorem 3.1. *Let $\bar{F}^n = (M^n, \bar{L})$ be a Finsler space which is transformed by a conformal Kropina Randers change of a Finsler space $F^n = (M^n, L)$ with the (α, β) −metric of Douglas type is also Douglas type if and only if C^{ij} is a homogeneous polynomial in (y^i) of degree.*

IV. CONFORMAL KROPINA-RANDERS CHANGE OF FINSLER SPACE OF DOUGLAS TYPE WITH (α, β) −METRICS

Here, we extend our study on conformal kropina-Randers change of Finsler space with (α, β) −metrics of Douglas type.

- Conformal Kropina-randers change of Finsler space with Riemannian metric:

Let F^n be Finsler space with Riemannian metric $L = \alpha$ and $\bar{F}^n = (M^n, \bar{L})$ a Finsler space which is obtained by conformal Kropina-Randers change of $F^n = (M^n, L)$. The Partial derivatives of Riemannian metric are

$$L_\alpha = 1,\ L_{\alpha\alpha} = 0,\ L_\beta = 0 \quad (4.1)$$

Now C^* and D^* are given by (2.4) and (3.7) respectively,

$$C^* = \frac{\alpha r_{00}}{2\beta},$$

$$D^* = \frac{2(\alpha^2\beta^2 - \beta^4)s_0 + \{(\alpha^2 - \beta^2)(b^2\sigma_0 - \rho\beta) + 2\beta(\rho\alpha^2 - \sigma_0\beta)\}\beta^2}{4b^2\alpha\beta^2} - \frac{2\beta(b^2\alpha^2 - \beta^2)r_{00}}{4b^2\alpha\beta^2} \quad (4.2)$$

Under the conformal Kropina-Randers change for Riemannian metric, B^{ij} can be written in the form:

$$\overline{B}^{ij} = B^{ij} + C^{ij},$$

where C^{ij} of (3.8) can be reduced to

$$C^{ij} = \frac{(\beta^2 - \alpha^2)\left[2(s_0^i y^j - s_0^jy^i) + \{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\}\right]}{4\beta} + \frac{r_{00}(b^iy^j - b^jy^i)}{2b^2} \quad +2\beta(\alpha^2 - \beta^2)s_0 + \beta\{(\alpha^2 - \beta^2)(b^2\sigma_0 - \rho\beta) + 2\beta(\rho\alpha^2 - \sigma_0\beta)\} - \frac{2(b^2\alpha^2 - \beta^2)r_{00}}{4b^2\beta^2}(b^iy^j - b^jy^i).$$
(4.3)

Now we have to show that C^{ij} is a hp(3). From the above equation (4.3), the terms

$$\frac{\beta}{2}(s_0^iy^j - s_0^jy^i) + \alpha^2 - \frac{\beta^2}{4}(\sigma^iy^j - \sigma^jy^i) + \left[\frac{\rho\beta^2}{4b^2} - \frac{\rho\alpha^2}{4b^2} + \frac{\rho\alpha^2 - \sigma_0\beta}{2b^2} + \frac{r_{00} - \beta s_0}{2b^2}\right](b^iy^j - b^jy^i) \quad (4.4)$$

are homogeneous polynomial of degree 3. So these terms may be neglected in our discussion and we consider only the terms

$V_{(3)}^{ij} = \frac{\alpha^2 s_0}{2b^2\beta}(b^iy^j - b^jy^i) - \frac{\alpha^2}{2\beta}(s_0^iy^j - s_0^jy^i)$, where $V_{(3)}^{ij}$ is $hp(3)$. (4.5)

The equation (4.5) can written as

$$2b^2\beta V_{(3)}^{ij} - \alpha^2 s_0(b^iy^j - b^jy^i) + b^2\alpha^2(s_0^iy^j - s_0^jy^i) \quad (4.6)$$

Take $n > 2, \alpha^2 \not\equiv 0(mod\beta)$ [9]. The terms of (4.6), which seemingly do not contain β are

$$b^2\alpha^2(s_0^iy^j - s_0^jy^i) - s_0(b^iy^j - b^jy^i) = \beta V_{(1)}^{ij}.$$

Hence we must have $hp(1)V_{(1)}^{ij}$ such that the above expression is equal to $\alpha^2\beta^2V_{(1)}^{ij}$. Thus

$$b^2(s_0^iy^j - s_0^jy^i) - s_0(b^iy^j - b^jy^i) = \beta V_{(1)}^{ij} \quad (4.7)$$

By putting $V_{(1)}^{ij} = V_k^{ij}(x)y^k$, the terms of (4.7) can reduced to

$$b^2[s_h^i\delta_k^j + s_k^i\delta_h^j - s_h^i\delta_k^j - s_k^i\delta_h^j] - [(s_h\delta_k^j + s_k\delta_h^j)b^i - (s_h\delta_k^i + s_k\delta_h^i)b^j] = b_hV_k^{ij} + b_kV_h^{ij} \quad (4.8)$$

Contracting (4.8) by $j = k$, we have

$$nb^2s_h^i - nb^is_h = b_hV_r^{ir} + b^rV_h^{ir}. \quad (4.9)$$

Again contracting (4.8) by b_jb^h, we have

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$b^2(b^2 s^i_k - s^i b_k - s_k b^i) = b^2 b_r V^{ir}_s + b_k b_r V^{ir}_s\ b^s.$ (4.10)

Again contracting (4.10) by b^k, we get
$b_r V^{ir}_s\ b^s = -b^2 s^i$, provided that $b^2 \neq 0$. (4.11)

Plugging (4.11) in (4.10), we have
$b_r V^{ir}_k = b^2 s^i_k - s_k b^i$. (4.12)

Again Plugging $b_r V^{ir}_h$ from (4.12) in (4.9), we have
$b^2 s^i_h = \frac{1}{n-1} V^{ir}_r b_h + b^i s_h$. (4.13)

Suppose if we plug $V^i = \frac{1}{n-1} V^{ir}_r$, then the terms of (4.13) can be written as $b^2 s^i_h = v^i b_h + b^i s_h$ which implies $b^2 s_{ij} = v_i\, b_j + b_i s_j$, where $v_i = a_{ij} v^j$. Since is skew-symmetric tensor, we have $v_i = -s_i$ easily. Thus
$s_{ij} = 1/b^2\ (b_i s_j - b_j s_i)$. (4.14)

Thus, we have

Theorem 4.2. A Finsler space $\overline{F}^n (n > 2)$ which is obtained by conformal Kropina-Randers change of a Riemannian space F^n with $b^2 \neq 0$ is of Douglas space if and only if (4.14) is satisfied.

- *Conformal Kropina-Randers Change of Finsler Space with Randers Metric:*

Let F^n be a Finsler space with Randers metric $L = \alpha + \beta$ and $\overline{F}^n = (M^n, \overline{L})$ be a Finsler space which is obtained by conformal Kropina-Randers change of $F^n = (M^n, L)$. The Partial derivatives of Randers metric are

$L_\alpha = 1, \quad L_{\alpha\alpha} = 0, \quad L_\beta = 1.$ (4.15)

According to [9], Finsler space with Randers metric is Douglas space if and only if $s_{ij} = 0$.

Now C^* and D^* are given by (2.4) and (3.7) respectively,

$C^* = \frac{\alpha r_{00}}{2\beta}$,
$D^* = \frac{\alpha\beta[(\alpha^3 - 2\alpha\beta^2)(b^2\sigma_0 - \rho\beta) + (2\alpha\beta + 2\beta^2)(\rho\alpha^2 - \sigma_0\beta)] - 2\alpha^2(b^2\alpha^2 - \beta^2) r_{00}}{4\beta(b^2\alpha^3 + \beta^3)}$, (4.16)

Under the conformal Kropina-Randers change for Randers metric, B^{ij} can be written in the form:
$\overline{B}^{ij} = B^{ij} + C^{ij}$,

where C^{ij} of (3.8) can be reduced to

$$C^{ij} = \frac{\alpha\beta(b^2\alpha^3+\beta^3)(2\beta^2-\alpha^2)\left\{\begin{matrix}\sigma_0(b^iy^j - b^jy^i) - \\ \beta(\sigma^iy^j - \sigma^jy^i)\end{matrix}\right\} + 2\alpha^3(\beta^3+\alpha\beta^2) r_{00}(b^iy^j - b^jy^i)}{4\beta^2(\alpha+\beta)(b^2\alpha^3+\beta^3)}$$

$+ \frac{\alpha^3\{(\alpha^3\beta - 2\alpha\beta^3)(b^2\sigma_0 - \rho\beta) + (2\alpha\beta^2 + 2\beta^3)(\rho\alpha^2 - \sigma_0\beta)\}}{4\beta^2(\alpha+\beta)(b^2\alpha^3+\beta^3)}(b^iy^j - b^jy^i)$. (4.17)

Now we have to show that C^{ij} is a $hp(3)$. The terms of (4.17) can be rewritten as follows:
$4\beta^2(\alpha+\beta)(b^2\alpha^3+\beta^3)C^{ij} + \alpha\beta(\alpha^2 - 2\beta^2)(b^2\alpha^3 + \beta^3)\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} - 2\alpha^3(\alpha\beta^2 + \beta^3)r_{00}(b^iy^j - b^jy^i) - \alpha^3\{(\alpha^3\beta - 2\alpha\beta^3)(b^2\sigma_0 - \rho\beta) + (2\alpha\beta^2 + 2\beta^3)(\rho\alpha^2 - \sigma_0\beta)\}(b^iy^j - b^jy^i) = 0.$ (4.18)

Since α is irrational in (y^i), the terms of (4.18) gives rise to two equations as follows:

$(2b^2\alpha^4\beta^2 + 4\beta^6)C^{ij} + (b^2\alpha^6\beta - 2b^2\alpha^4\beta^3)\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} - 2\alpha^4\beta^2 r_{00}(b^iy^j - b^jy^i) - \alpha^3\{(\alpha^3\beta - 2\alpha\beta^3)(b^2\sigma_0 - \rho\beta) + 2\alpha\beta^2(\rho\alpha^2 - \sigma_0\beta)\}(b^iy^j - b^jy^i) = 0,$ (4.19)

$(4\beta^2 + 4b^2\alpha^2)C^{ij} + (\alpha^2\beta - 2\beta^3)\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} - 2\alpha^2 r_{00}(b^iy^j - b^jy^i) - 2\alpha^2\{(\alpha^3\beta - 2\alpha\beta^3)(b^2\sigma_0 - \rho\beta) + 2\alpha\beta^2(\rho\alpha^2 - \sigma_0\beta)\}(b^iy^j - b^jy^i) = 0.$ (4.20)

Eliminate C^{ij} from the above two equations, we have
$(4b^4\alpha^8 - 8b^4\alpha^6\beta^2 - 4\alpha^2\beta^6 + 8\beta^3)\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} + [8\alpha^2\beta^3(\beta^2 - \alpha^2)r_{00} + \{8\alpha^2\beta^3(\beta^2 - \alpha^2)(\rho\alpha^2 - \sigma_0\beta) + (4b^2\alpha^8 + (4 - 8b^2)\alpha^6\beta^2 - 8\alpha^4)(b^2\sigma_0 - \rho\beta)\}](b^iy^j - b^jy^i) = 0$. (4.21)

Suppose for $n > 2, \alpha^2 \neq 0(mod\beta)$, the terms of (4.21) which seemingly does not contain α^2 as a factor are $8\beta^8\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\}$. Hence we must have a $hp(0), V^{ij}(x)$ such that,

$\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} = \alpha^2 V^{ij}(x).$ (4.22)

The terms of (4.22) can be written as
$[(\sigma_h\delta^j_k + \sigma_k\delta^j_h)b^i - (\sigma_k\delta^i_h + \sigma_h\delta^i_k)b^j] - [(b_h\delta^j_k + b_k\delta^j_h)\delta^i - (b_h\delta^i_k + b_k\delta^i_h)\delta^j] = a_{hk}V^{ij}$. (4.23)

Contracting (4.23) by $j = h$, we have
$n(\sigma_k b^i - b_k\sigma^i) = V^i_k$. (4.24)

which implies
$Vij(x) = n(b_i\sigma_j - b_j\sigma_i).$ (4.25)

Thus, we state that
Theorem 4.3. A Finsler space $\overline{F}^n (n > 2)$ which is obtained by conformal Kropina-Randers change of a Finsler space F^n with the Randers metric of Douglas type remains to be of Douglas type if and only if (4.25) is satisfied.

- Conformal Kropina-Randers Change of Finsler Space with Generalized Kropina Metric:

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Let F^n be a Finsler space with Generalised Kropina metric $L = \frac{\alpha^{m+1}}{\beta^m}, m \neq 0, -1$ and $\overline{F}^n = (M^n, \overline{L})$ be a Finsler space which is obtained by conformal Kropina-Randers change of $F^n = (M^n, L)$. The Partial derivatives of Generalised Kropina metric are

$$L_\alpha = \frac{(m+1)\alpha^m}{\beta^m}, \quad L_{\alpha\alpha} = m(m+1)\frac{\alpha^{m-1}}{\beta^m}, \quad L_\beta = -m\frac{\alpha^{m+1}}{\beta^{m+1}} . \qquad (4.26)$$

Now C^* and D^* are given by (2.4) and (3.7) respectively,

$$C^* = \frac{\alpha\{(m+1)\beta r_{00} + 2m\alpha^2 s_0\}}{2(m+1)\{mb^2\alpha^2 - (1-m)\beta^2\}},$$

$$D^* = \frac{2(m+1)\begin{bmatrix}(1-m)\alpha^{3m+4}\beta^3 + mb^2\alpha^{3m+6}\beta \\ +(m-1)\alpha^{m+2}\beta^{2m+5} - mb^2\alpha^{m+4}\beta^{2m+3}\end{bmatrix} s_0}{\frac{4(m+1)^2\alpha^m\alpha^{2m+1}\left\{\begin{matrix}\beta^3 + (m+1)\beta(b^2\alpha^2-\beta^2) \\ +m\beta(b^2\alpha^2-\beta^2)\end{matrix}\right\}}{\alpha^m\{mb^2\alpha^2-(1-m)\beta^2\}}}$$

$$+ \frac{\frac{\begin{bmatrix}\{\alpha^{2m+4}\beta + 2m\alpha^{2m+4}\beta - \alpha^2\beta^{2m+3}\}(b^2\sigma_0 - \rho\beta) \\ +2(m+1)\alpha^{2m+2}\beta^2\end{bmatrix}}{(m+1)\alpha^m\{mb^2\alpha^2-(1-m)\beta^2\}}}{\frac{4(m+1)^2\alpha^m\alpha^{2m+1}\left\{\begin{matrix}\beta^3 + (m+1)\beta(b^2\alpha^2-\beta^2) \\ +m\beta(b^2\alpha^2-\beta^2)\end{matrix}\right\}}{\alpha^m\{mb^2\alpha^2-(1-m)\beta^2\}}}$$

$$- \frac{2(m+1)^2(b^2\alpha^2-\beta^2)\alpha^{(2m+2)}\beta\alpha^m\{(m+1)\beta r_{00} + 2m\alpha^2 s_0\}}{\frac{4(m+1)^2\alpha^m\alpha^{2m+1}\{\beta^3 + (m+1)\beta(b^2\alpha^2-\beta^2) + m\beta(b^2\alpha^2-\beta^2)\}}{\alpha^m\{mb^2\alpha^2-(1-m)\beta^2\}}}. .$$

Under the conformal Kropina-Randers change for Generalised Kropina metric, B^{ij} can be written in the form:

$$\overline{B}^{ij} = B^{ij} + C^{ij},$$

where C^{ij} of (3.8) can be reduced to

$$4(m+1)^2\alpha^{2m}\alpha^{2m+1}\beta\{\beta^3 + (m+1)(b^2\alpha^2-\beta^2)\beta + m\beta(b^2\alpha^2-\beta^2)\beta\}\{mb^2\alpha^2 - (1-m)\beta^2\}C^{ij} = (m+1)\alpha^{2m+1}\{\beta^{2m+2} - (2m+1)\alpha^{2m+2}\}\{mb^2\alpha^2 - (1-m)\beta^2\}\{\beta^3 + (m+1)(b^2\alpha^2-\beta^2)\beta + m\beta(b^2\alpha^2 - \beta^2)\beta\}\{\sigma_0(b^iy^j - b^jy^i) - \beta(\sigma^iy^j - \sigma^jy^i)\} + 2(m+1)\alpha^{2m+1}\{\beta^{2m+2} - \alpha^{2m+2}\}\{mb^2\alpha^2 - (1-m)\beta^2\}\{\beta^3 + (m+1)(b^2\alpha^2-\beta^2)\beta + m\beta(b^2\alpha^2 - \beta^2)\beta\}(s_0^iy^j - s_0^jy^i) + (2m+1)\alpha^{m+1}[2(m+1)\{(1-m)\alpha^{3m+4}\beta^3 + (m-1)\alpha^{m+2}\beta^{2m+5} + mb^2\alpha^{3m+6}\beta - \alpha^{m+2}\beta^{2m+5} - mb^2\alpha^{m+4}\beta^{2m+4}\}s_0 + (m+1)\alpha^m[\{\alpha^{2m+4}\beta + 2m\alpha^{2m+4}\beta - \alpha^2\beta^{2m+3}\}\{(m+1)\alpha^m\beta r_{00} + 2m\alpha^{m+2}s_0\}](b^iy^j - b^jy^i)] + +2(m+1)^2\alpha^{4m+3}\{(m+1)\beta r_{00} + 2m\alpha^2 s_0\}\{\beta^3 + (m+1)(b^2\alpha^2-\beta^2)\beta + m(b^2\alpha^2-\beta^2)\}(b^iy^j - b^jy^i) = 0 \quad . \qquad (4.27)$$

Now we have to show that C^{ij} is a hp(3). Suppose for $n > 2, \alpha^2 \neq 0 (mod \beta)$, the terms of (4.27) which seemingly does not contain β as a factor are $\{m(m+1)(2m+1)b^2\alpha^{4m+7}\, s_0(b^iy^j - b^jy^i) - m(m+1)(2m+1)b^4\alpha^{4m+7}(s_0^iy^j - s_0^jy^i)\}$. Hence we must have a $hp(1), V_{(1)}^{ij}$ such that,

$$s_0(b^iy^j - b^jy^i) - b^2(s_0^iy^j - s_0^jy^i) = \beta V_{(1)}^{ij}. \qquad (4.28)$$

Let $V_{(1)}^{ij} = V_k^{ij}(x)y^k$, then the terms of (4.28) can be written as

$$[(s_h\delta_k^j + s_k\delta_h^j)b^i - (s_h\delta_k^i + s_k\delta_h^i)b^j] - b^2[s_h^i\delta_k^j + s_k^i\delta_h^j - s_h^i\delta_k^j - s_k^i\delta_h^j] = b_hV_k^{ij} + b_kV_h^{ij}. \qquad (4.29)$$

Contracting (4.8) by $j = k$, we have

$$nb^is_h - nb^2s_h^i = b_hV_r^{ir} + b^rV_h^{ir}. \qquad (4.30)$$

Again contracting (4.29) by b_jb^h, we have

$$-b^2(b^2s_k^i - s^ib_k - s_kb^i) = b^2b_rV_k^{ir} + b_kb_rV_s^{ir}\, b^s. \qquad (4.31)$$

Again contracting (4.31) by b^k, we get

$b_rV_s^{ir}\, b^s = -b^2s^i$, provided that $b^2 \neq 0$. (4.32)

Plugging (4.32) in (4.31), we have

$$b_rV_k^{ir} = s_hb^i - b^2s_h^i. \qquad (4.33)$$

Again Plugging $b_rV_h^{ir}$ in (4.30), we have

$$b^2s_h^i = b^is_h - \frac{1}{n-1}V_r^{ir}b_h. \qquad (4.34)$$

Suppose if we plug $V^i = \frac{1}{n-1}V_r^{ir}$, then the terms of (4.34) can be written as $b^2s_h^i = b^is_h - v^ib_h$ which implies $b^2s_{ij} = b_is_j - v_i\, b_j$, where $v_i = a_{ij}v^j$. Since is skew-symmetric tensor, we have $v_i = s_i$ easily. Thus

$$s_{ij} = \frac{1}{b^2}\,(b_is_j - b_js_i). \qquad (4.35)$$

Thus, we state that

Theorem 4.2. A Finsler space $\overline{F}^n (n > 2)$ which is obtained by conformal Kropina-Randers change of a Riemannian space F^n with Generalised Kropina metric of Douglas space remains to be of Douglas type if and only if (4.35) is satisfied.

V. CONCLUSION

Let $F^n = (M^n, L)$ be a Finsler space equipped with the fundamental function $L(x, y)$ on the smooth manifold M^n. Let $\beta = b_iy^i$ be a one form on manifold M^n, then $\overline{L} \to \frac{L^2}{\beta} + \beta$ is called Kropina-Randers change of Finsler metric. If we write $\overline{L} \to \frac{L^2}{\beta} + \beta$ and $\overline{F}^n = (M^n, \overline{L})$, then the Finsler space $\overline{F}^n$ is obtained from F^n by Kropina-Randers change.

For this Kropina-Randers change, we are applying the conformal theory. i.e., $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L} = \frac{L^2}{\beta} + \beta)$, be two Finsler spaces on the same underlying manifold M^n. If $\sigma(x)$ be a function in each coridinate

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neighborhood of M^n such that $\bar{L}(\bar{\alpha},\bar{\beta}) = e^{\sigma(x)}\left[\frac{L^2}{\beta} + \beta\right]$, then F^n is called Conformaly Kropina-Randers to $\overline{F}^n$, and the change $L \to \bar{L}$ of metric is called conformal Kropina-Randers change of (α,β)- metric.

From the above concept, we are trying to generalise the condition for a Finsler space $\overline{F}^n$ which is obtained by conformal Kropina-Randers change of Finsler space F^n of Douglas type to be also Douglas type. And also we are discussed conformal Kropina-Randers change of Finsler space with particular (α,β)- metric is of Douglas type.

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