Derivation of Value of Root Over 2 across Ages

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Abstract:- The value of root over 2 ($\sqrt{2}$) was determined at various phases in the span of 2800 years. It began with "the *Vedangas*", a group of literatures as appendages of Vedas. Among six such corpuses of *Vedangas*, *Kalpa* (ritual) dealt with religious rituals. Śulbha Sūtras, within *Kalpa*, are instructions of ancient Vedic rituals through construction of sacrificial altars. These rituals required various geometrical shapes with strict rules of measurement Śulba content serves as the oldest known reference of engineering or constructional manual. In 323 numbers of aphorisms (Sūtras), the pioneer among the geometers was Baudhāyana who lived around 800-600 BCE, who developed various geometrical figures, their combinations, process of values of irrational numbers, value of π, and solutions of linear equation through geometry. One of his Sūtras included the determination of value of $\sqrt{2}$. Brahmagupta's Vargaprakṛti has also been discussed in this article. Isaac Newton (1642-1727 CE) was a genius whose contribution was a great leap forward for scientific revolution in the field of physics and mathematics. With his contemporary, Joseph Raphson (1648-1715 CE), root-finding algorithm for evolution of the value of $\sqrt{2}$ through successive approximations was a gift to coffer of Mathematics, which is narrated here. Srinivasa Ramanujan (Dec.1887-Apr.1920 CE), a prodigy of mathematics without any formal training and greatest mathematician of twentieth century had advanced various fields of mathematics, especially number theory. Numerical solution of $\sqrt{2}$ described here, was an offshoot of a solution out of 'Continued Fractions' dealt by him. Comparative evolution from other relevant ancient civilizations has also been referred.

Keywords:- *Baudhāyana, Śulbha Sūtras, Newton – Raphson, Ramanujan, mathematics, Root over 2 ($\sqrt{2}$).*

I. INTRODUCTION

This article depicts how ancient India's scholastic ability found a measure of incommensurable in$\sqrt{2}$. Vedic scholars were able to derive the value of $\sqrt{2}$ before 600 BCE with the help of geometry, whose root and development had already taken place in Indian subcontinent since 2000 BCE. Sūtras are thought to be oral wisdom passed on through generations. [1] Further, we discuss how application of modern mathematics since developed by famous physicist and mathematician, Isaac Newton with his contemporary Joseph Raphson derived values that vindicated the geometrical intellectual finesse of the ancient geometer like Baudhāyana. While comparing the values, derived values from the Vargapraḳiti in Brāhmaṣphutasiddhānta (Chapter: 18, Sūtra 64-65 and associated writings) is also taken for discussion.

Śulba Sūtras was recorded in written script in Sanskrit first by Baudhāyana and then developed further during the period of 500 BCE- to 200 BCE by scholars from different regions of India separately. In Baudhāyana Śulba Sūtras (B.S.S) [2], there are numbers of instructions for preparing sacrificial altars with various geometrical shapes. One Sūtra narrates how diagonal is related to its adjoining two sides of a square or rectangle. (B.S.S 1/48). These were recorded much ahead before Pythagorean Theorem came to light.

The Pythagorean Theorem, a milestone in the history of mankind was formulated by Pythagoras or any of his disciples. Contribution by Greece and neighboring countries to the Science from 600 BCE (Thales) to 400 AD (Diophantus)) for 1000 years was considered as an outstanding glorious period. However, the contemporary or earlier civilisations also had golden treasures. In this paper, article emphasized on to familiarize the 'captioned subject' on the knowledge and wisdom of great masters who recorded Sūtras indicating value of $\sqrt{2}$, even having lived in the early half of first millennium BCE. The derivations of widely acclaimed mathematicians are also added .

In the early medieval period, Brahmgupta (598-665 CE), an outstanding Mathematician commanded awe and respect worldwide for his contribution in delineating properties of Śūṇya. Apart from this, his contribution in astronomy, cyclic quadrilateral, quadratic equation, solutions of various kind of equation including those of indeterminate equation of second degree (Vargaprakṛti) in the form of $Nx^2 + z= y^2$ using algebra and arithmetic are milestones in the history of science. This form of equation is used here for numerical solutions of unknown root of x and y and its ratio (y/x) representing value of$\sqrt{2}$.

Conceptual form of infinitesimally small quantity was properly shaped and concretised by Isaac Newton and Gottfried Wilhelm Leibniz to give birth to Calculus. This new branch of mathematics in the seventeenth century CE helped many conceptual issues in their real solutions. Handling of non-linear equations of higher degrees too became easier. Here, equation of a Parabola in the form of $X^2– 2= 0$ was considered to derive the value of $\sqrt{2}$. In the above two theoretical premises, it would be shown how the values of $\sqrt{2}$ became almost same in its each stage during approaching nearer and nearer value through towards actual.[3]

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In the early period of twentieth century, G.H. Hardy (1877-1947) and J.E. Littlewood (1885-1977) made precious value addition in mathematical history. It was Hardy who adopted in their world a non-institutionalised natural , brilliant and intuitive mathematician, Srinivasa Ramanujan (1887-1920) from India. Ramanujan heightened the subject of numerical analysis, number theory and many innovative branches of mathematics with his profundity and uncanny abilities, even though his life span was so small. Here, $\sqrt{2}$ is being discussed with oblique reference to his solution in respect of solution of a problem through 'Continued Fraction'.

➢ *Understanding $\sqrt{2}$ with Baudhāyana Śulba Sūtras*

Baudhāyana Śulba Sūtras dealt with many aspects of rectangle, square circle, trapezium and rhombus etc. In Baudhāyana Śulba Sutra (B.S.S 1/ 45)[4], it was stated as following:

समचतुरस्रस्याक्ष्णायारज्जुर्द्विस्तावतींभूमिंकरोति॥१।४५,
(*samacaurasrākśṇayārajjudvistātiṃbhūmiṇkaroti) (1/45*)

which means that "The chord stretched across a square (i.e. in the diagonal) produces an area of double the size (i.e. The square of the diagonal of a square is twice as large as that square).

ABCD is a square. AC is diagonal; Taking AC as a side, a new square is drawn. In Sūtra, it says that area of ACEF is twice the area of square, ABCD.

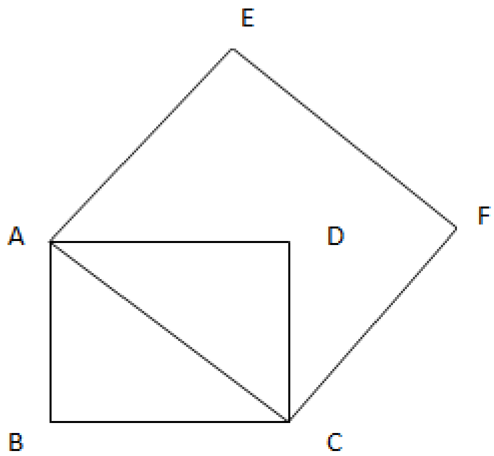

Fig 1:- Square on diagonal of a square.

In other words, two similar squares make square with side of diagonal of previous squares as following.

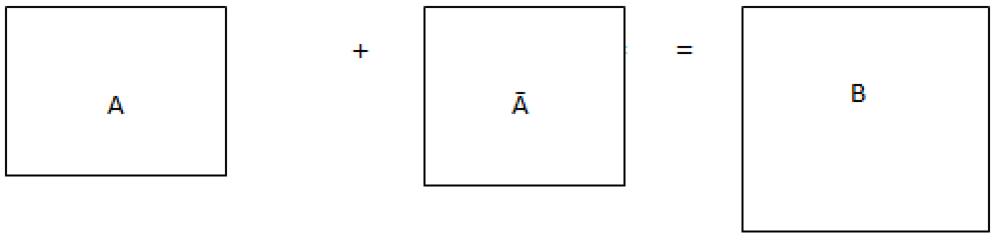

Fig 2:- Summation of two squares with equal area forming bigger square

A = Ā; A+Ā = B; Therefore area of A or A¯ is m^2, area of B is n^2.
Therefore, $2m^2= n^2$. Then $n^2/m^2= 2$, and $n/m = \sqrt{2}$,

Baudhāyana in Śulba Sūtra 61 and 62expressed the value of $\sqrt{2}$ as:

समस्याद्बिकरणीप्रमाणंतृतीयेनवर्धयेतच्चचतुर्थेनात्मचतुस्त्रिंशोनेन१।६१
samasyādvikaraṇī pramānaṃtṛtiyenavardhayettaccacaturthenātmacatuśtriṃśonena (1/61) [5]
सविशेषः१।६२*Saviśeṣah(1/62)* [6]

Meaning of above noted aphorism is, the measure of diagonal of a square is "*increase the measures (of which the dvikaraṇī is to be found) by its third part, and again by the fourth part (of this third part) less by thirty-fourth part of itself (that is of this fourth part). (The value thus obtained is called) the saviśeṣa.61.*

If we take 1 for the measure and increase it as directed, we get the following expression: $1+\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}-\frac{1}{3.4}\times\frac{1}{34}$.

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By this way, the value is found 577/408 which is equal to 1.4142156 whereas calculator shows 1.4142135. Professor George Thibaut[7] commented “comparing the two decimal fractions, we see the value of the Su*trakāras*, had found comes up remarkably close to the true value.” Not only, Baudhāyana deduced the value, he had clear vision that the value could be further fine tuned. Thereby, he added a single word, *Saviśesah,* which means such value is still more or less. A masterstroke indeed!

In the Introduction of the book “The Śulba Sūtras of Baudhāyana, Āpastamba, Kātyayāna and Mānava with Texts English Translation and Commentary” by very imminent scholars S N Sen and A.K Bag[8] (published by Indian National Science Academy, New Delhi, 1983) , authors did not hesitate to record “Neugebaurer had shown that these values are identical with those found in certain Babylonian cuneiform texts, given in sexagesimal system. He tried to imply that the Indian value after all represented the Babylonian one expressed only in decimal system or more accurately in fraction. As we have shown, there is certainly of no proof of such assertion and the Indian value is certainly derivable from the method contained in the Śulba Sūtras themselves.” Neugebaurer, however, unknowingly admitted that India knew the decimal system at the primitive stage of development of mathematics.

In “Hindu Arabic Numerals”, David Eugene Smith and Louis Charles Karpinski recorded the essence of partly ritualistic (the Brāhmaṇas) and partly philosophical (Upanishads) poetic literatures following Vedic period. According to them “Our special interest in the Sutras, a versified abridgements of the ritual and of ceremonial rules, which contain considerable geometric material used in connection with altar construction and also numerous examples of rational numbers the sum of whose square is also a square, i.e. “Pythagorean numbers,” although this was long before Pythagoras lived. Whitney* places the whole of the Vedas, the Brāhmanas, and the Sutras between 1500 B C to 800 B C, thus agreeing with Burk** who holds the knowledge of the Pythagorean Theorem revealed in the Sūtras goes back to the eighth century BC”. Further, they also refuted opinion held by Georg Cantor of a Greek origin on the basis of important research work by Von Schroeder[9].

Numbers for which two units of square is nearer to a square of a number were examined; few among them are 50,49 [50 (2x 5^2)≈49, (7^2)], 98,100 [98(2x7^2)≈100 (10^2] and 288, 289 [288 (2x12^2)≈289 (17^2)]. In fact, value of $\sqrt{2}$= 17/12 is an approximate value. Two 12 Units Square make 288 square units; One 17 units’ square makes 289 square units. Supposing, there are 289 units of squares are arranged in 17x17 arrays, just one unit is to be taken away in such a way that format will be square with its side $\sqrt{288}$ unit. It will produce the formula deduced by Baudhāyana (17x17 unit squares is not specified in scale).[10]

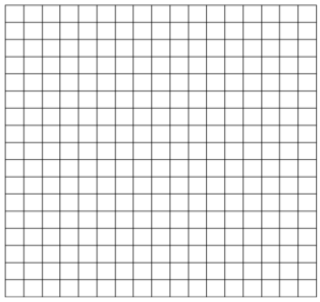

Fig 3:- A square of 17x17 consisting of 1x1 units

Supposing there are two equal squares with sides 12 units. Then these two squares jointly have the area of 288 square units. Alternatively, we may call it 288 units of squares. In the above figure, 17x17 (289) square units or 289 number of equal unit squares are arrayed. We have to deduct one unit of square and thereby reduce the length (and breadth) from the above square so that its side takes the value $\sqrt{288}$ (12.$\sqrt{2}$). One unit square out of 289 units be distributed (for reduction) at north and eastern side the square; thereby the side is to be reduced by 1/34 th part [1÷(17×2)] in east and north. It results the side i.e. 12$\sqrt{2}$= 17 – 1/34. We have arrived this value out of two 12 units square. For two one unit square, this value of $\sqrt{2}$ will be $\frac{1}{12}$(17- 1/34) = $\frac{17}{12}$ - $\frac{1}{34.12}$ = 1+$\frac{1}{3}$+$\frac{1}{3.4}$ - $\frac{1}{3.4.34}$ …. This method of determination of value may continue to eliminate the excess amount. It proves the Baudhāyana’s Sūtra in B. S. S. 1/61 [11] *W D Whitney: Saṇskrit Grammar, **Burk: Das Āpastamba Śulba- Sūtra.

Being much influenced by the ancient geometry of Indian Sub-continent from books and papers of George Thibaut and Bibhuti Bhusan Dutta, David W Henderson of Cornell University put serious effort to add more. He delved how the ancient mathematical giant Baudhāyana derived this formula. He worked at Sankarcharyya Mutt in Kanchipuram, India in January 1990 on looking into related manuscripts.

Henderson presented a research paper on “Square root in the Śulba Sūtra”. Two geometrical figures from his paper are used here.[12]

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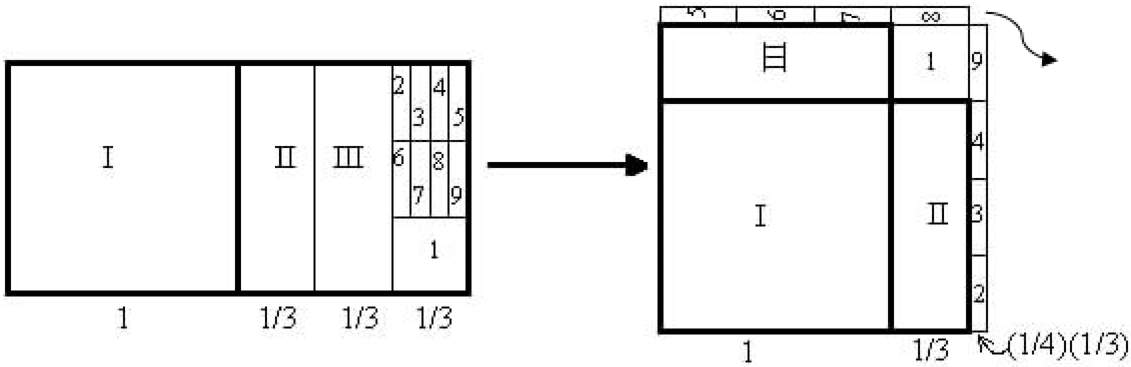

Fig 4:- Graphical representation of merging of two squares with depiction of excess (arrow mark)

In the figure above, there are two equal squares. Square number I keeping unchanged and dividing second one in three parts. Last part is divided in three parts. Part III and 1 of last part, (bottom one-third: one third of one-third unit square) are placed on square 1; [figure above; at right-hand I, III and 1]. In that case the side of new square is increased by one –third ($1+\frac{1}{3}$).

In the second stage, upper two boxes of one –third each of the second square is divided four equal parts [strips] (from serial 2,3,4,5 for first box, 6,7,8, and 9 for second box). These six strips are placed at north and eastern side (3 strips in each side) of the square with side ($1+\frac{1}{3}$). Linear measurement of each strip is $\frac{1}{3}.\frac{1}{4}$. The square gets its renewed side measuring now $1+\frac{1}{3}$ + $\frac{1}{4}\times\frac{1}{3}$. If we notice carefully, that we have added arrow marked portion square (with its side is $\frac{1}{12}$ of with area , $\frac{1}{12}.\frac{1}{12}$) . Baudhāyana in B.S.S 51/1 had kept provision for getting a square from subtraction of a small square from large square. But how could we subtract the excess area. The side of the square $1+\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}$(= 17/12) will be little less than that. The excess square with a measure can be distributed on two sides with a width of (say $w_{1.}$)

$W_1 = (\frac{1}{12}) \div (2.\frac{17}{12}) = \frac{1}{4.3.34} = \frac{1}{408}$. Now, the side of square (approximate value of $\sqrt{2} = 1+\frac{1}{3}+\frac{1}{4}\times\frac{1}{3}-\frac{1}{3.4}\times\frac{1}{34}, =\frac{577}{408}$.

Henderson had done further correction as shown in the following geometrical figure .[13]

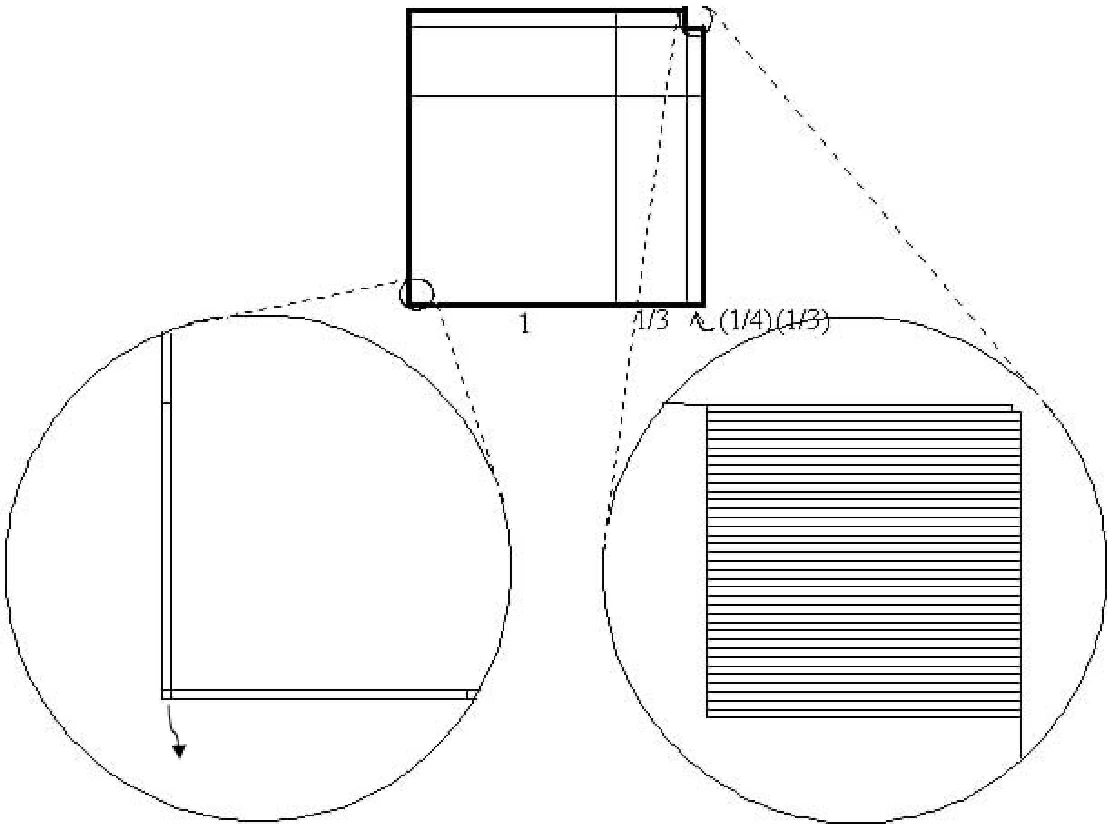

Fig 5:- Distributing excess area shown in Fig.4 with depiction of arrow-marked smaller excess area.

Again if we see minutely, the arrow marked portion, (shown after its enlargement in size,) a square of area $\frac{1}{408}.\frac{1}{408}$ is in excess which is to be subtracted on above mentioned similar method. We have to find out the thinner strip for smaller square $\frac{1}{408}.\frac{1}{408}$ on length of 577/408 spread over on two sides of the squares.

$W_2=(\frac{1}{408}.\frac{1}{408}) \div 2x (\frac{577}{408}) = \frac{1}{408x1154}$

The value of the side will, then, be $= \frac{577}{408} - \frac{1}{408.1154} = \frac{665857}{470832} = 1.41421356237469$

If we want to get more accuracy the repetitive process, the excess square to the tune of $\frac{1}{408.1154} x \frac{1}{408.1154}$ will have to be deducted from the square with side of $\frac{665857}{470832}$.

Then, $W_3 = (\frac{1}{408.1154} x \frac{1}{408.1154}) \div (2 \times \frac{665857}{470832}) = \frac{1}{627013566048}$, then the finer value of the side of square is $\frac{665857}{470832} - \frac{1}{627013566048}$, = 886731088897/627013566048= 1.4142135623730950488016896225025.

We can go to further step following the same process to arrive at the value as:

$\frac{1572584048032918633353217}{1111984844349868137938112} =$ 1.4142135623730950488016887242097

(The value of $\sqrt{2}$ is incommensurable)

- *Ancient Greece on $\sqrt{2}$ and its value*

Pythagoras (560 BCE to 420BCE) spent 22 years in the temples and other learning centres of Egypt to learn astronomy, mathematics, music, mysteries of Egypt . Persia invaded Egypt. He was brought to Persia in 525 BCE as a prisoner where he had to spend another 12 years. He was initiated to Chaldean mysteries, medicines and mathematics and studied Magi. He mastered over philosophies of two ancient civilisations, Egyptian and Babylonian. His teacher Thales also spent some time in Egypt. Babylon and Egypt had definitely influenced him in his journey towards pinnacle of academic excellence. He became a legend. Babylonian cuneiform based mathematics and astronomy could be adopted by pre- or post- Pythagorean philosophers of Greece and other neighbouring flourishing centres of mathematics faster than its travel to far away India. Theorem was named after him, 500 years after his death.

Pythagorean believed that numbers should be expressed as integers or ratio of integers. Value of hypotenuse of an isosceles right-angled tri-angle created a problem. Hippasus of Metapontum **(**530 BCE. 450 BCE) [14], was a Pythagorean philosopher, credited with the discovery of the existence of irrational numbers. The discovery of irrational numbers was shockingly irritating to the Pythagoreans and Hippasus was drowned at Mediterranean Sea, apparently as a punishment from the Gods for divulging this. However, biographer of Pythagoras, Lambichus made other two different accounts of death of Hippasus[14]. Bertrand Russell added "Even scientific and mathematical discoveries were deemed collective, and in a mystical sense due to Pythagoras even after his death, Hippasos of Metapontion, who violated this rule, was shipwrecked as a result of divine wrath at his impiety" [15]. They did not believe that value of √2 cannot be expressed other than the ratio of *p/q* where *p* and *q* are integers. He added 'Unfortunately for Pythagoras, his theorem led at once to the discovery of incommensurable, which appeared to disprove his whole philosophy '. For a certain period 'to them, surd was absurd.' The view on irrationality on numbers changed within 100-150 years. Theodorus (465 -398 BCE) studied irrationality of square root of non-square integers other than square root of 2.[16]

Bertrand Russell added that ingenious method of approximation of value of $\sqrt{2}$ was done by two columns of numbers; a and b, each starting with 1. The next a, at each stage, is formed by adding the last a and b already obtained; next b is formed twice the previous a to the previous b. The first six pairs so obtained are (1, 1), (2, 3) (5, 7) (12, 17) (29, 41), (70, 99)'. Pythagoreans' thought-process circled with axiom-based Euclidean geometry. While describing such sets of numbers, author did not mention period of creation of such "ingenious method". He further added that each pair satisfies $2a^2 - b^2 = \pm 1$. [17]

In the History of Mathematics, Volume II, David Eugene Smith added that Nicomachus of Geresa (probably the modern Jerash, a town situated fifty-six miles north east of Jerujalem) during first century CE developed number theory ; his contemporary Theon of Smyrna, a Greek city had developed the number series and larger number of each set was called " Theon Diameter". Smith further observed "It is interesting to observe a fact unknown to him, namely that the ratios d/n (larger to smaller number in a set) are the successive convergent of the continued fraction of $1+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\cdots}}}}$` and hence approach nearer and nearer the square root of 2." [18]. It may be said that Greece was more engaged in debate of irrationality of $\sqrt{2}$ than finding its value before first century CE. No doubt, Grecian scholars had huge contributions in Geometry.

- *Understanding $\sqrt{2}$ with Newton and Raphson Method .*

Mathematical giants used ideas of the calculus to generalize a method to find the zeros of an arbitrary equation $X^2 - 2 = 0$ As 2>0 is a positive real number, it will be shown that there is a real number X for which $X^2=2$. [19]

Let *r* be an actual square root of 2 depicted on X axis in the Graph[20]. We assume f (r) ≠0. Let X_1 be a number close to *r* (which may be obtained by looking at the graph of *f*(X). The tangent line to the graph of *f*(X) *at* ((X_1 *f*(X_1) has X_2 as its *x*-intercept.

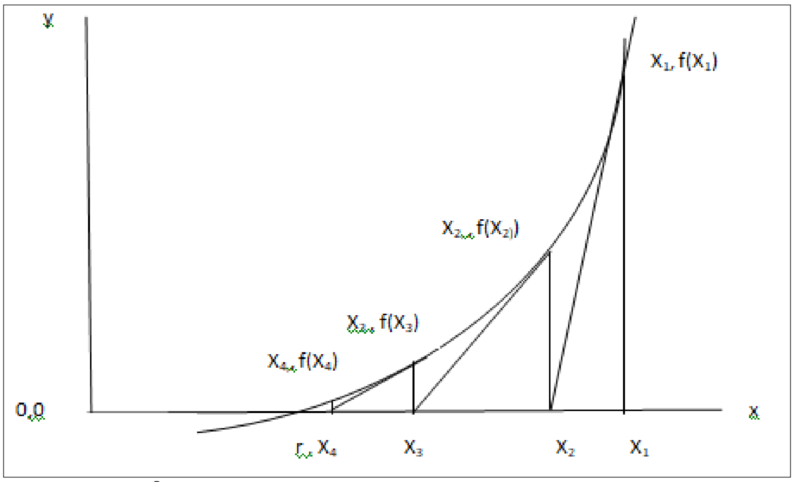

Fig 6:- Curve of Parabola, (X^2-2) =0 depicting real value r and tangents touching X-axis approaching 'r'.

Let f (X):=X^2−2. We want to solve the equation f (X) = 0, Say, we have some approximation X_1 to a solution. Newton showed how to get a better approximation with X_2. At point X_1, the coordinate X_1, f (X_1).

The equation of the tangent line at point X_1 is
[Y- f (X_1)]/ [X-X_1] = m= f '(X1) => Y = f (X_1) + f '(X1) (X-X_1). Through iteration, this process will generate a sequence of numbers X which approximates *r*. (square root of 2) [21]

We take this opportunity for approximation is close enough to the solution through iteration process. Let us explore values through reiterative process as following. We know that 2> √2>1. Value of X, equal to 1 was considered arbitrarily . At next stage value becomes : $\frac{3}{2}$. The real value lies between 1 < √2< $\frac{3}{2}$.

X	*f* (x)[=X^2-2=0]	*f* '(X) =2x	X− [*f* (X)/ *f* '(X)]	Value of X− [(*f* X)/*f* '(X)] in decimal
1	−1	2	1− (−1/2)=3/2	1.5
3/2	1/4	3	3/2− (1/4)/3 =17/12	1.416666
17/12	1/144	17/6	17/12 – (1/144)/(17/6)	577/408 = 1.41421568
577/408	1/166464	577/204	577/408− (1/166464)/(577/204)	665857/470832=1.4142135624

Table 1:- Analysis of values emerging out of tangents of parabola touching the X axix approaching real value 'r" [22]

This technique of successive approximations of real zeros is called **Newton's method**, or the **Newton-Raphson Method**.

Baudhāyana had invented the value of root over 2. After 2200 years, Isaac Newton with his contemporary mathematician Joseph Raphson discovered such value in the seventeenth century with the help of calculus, the subject itself was developed in the 17th century by Isaac Newton himself and Gottfried Wilhelm Leibniz.

As we noticed earlier, one small square unit out of 289 small square units of (1x1 each unit) is excess. To exclude such unit from 17x17 square areas, further fine tuning was done. In similar manner, among 332939 (=577x577) smaller square units, one square unit is in excess. For, further iterative process, the square units will go smaller to smaller with always an excess square unit. This is expressed in very fascinating manner like $(17)^2$ - $2(12)^2$= 289-288=1, $(577)^2$ – 2x $(408)^2$=1, $(665857)^2$–2x $(470832)^2$=1, we may generalize the equation evolved as p^2 −2q2 = 1. This form is familiarized as Pell- Fermat equation in the nature of Diophantine equation .This form is now known as Brahmagupta-Pell equation.

Brahmagupta's (598-668 AD) contribution of solution of indeterminate equation of second degree was utilized to derive value of root over 2 with higher perfection without recourse to geometry.

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- *Brahmgupta's Bhāvanā and deriving value of* $\sqrt{2}$.

In Brāhmaṣputasiddhānta, chapter: 18, under head '*Vargaprakṛti* 'Saṇskrit text with English translation for Sutra 64 and 65, is following:

मूलं द्विघेष्टवर्गाद् गुणकगुणादिष्ट युत बिहीनच्च।
आद्धवधो गुणकगुण: सहान्त्यघातेन कृतमन्त्यम्॥ ६४
वज्रवधैक्यं प्रथमं प्रक्षेप: क्षेपवघतुल्य:॥
प्रक्षेपशोघकहते मूले प्रक्षेपकै रूपे॥ ६५

The nature of squares:

18.64. [Put down] twice the square-root of a given square by a multiplier and increased or diminished by an arbitrary [number]. The product of the first [pair], multiplied by the multiplier, with the product of the last [pair], is the last computed.
18.65. The sum of the thunderbolt products is the first. The additive is equal to the product of the additives. The two square-roots, divided by the additive or the subtractive, are the additive *rupas*. [23]

Brahmagupta had developed formula for solving indeterminate quadratic equation 1000 years' ahead of what was done by John Pell (1611-1685) and Pieree De Fermat (1607-1665]. In brief, discussion is confined to the context, solution of the equation in the form of $Nx^2 + z = y^2$;

Brahmgupta *Vargaprakṛti* solved problem of an equation whose square term x^2 is multiplied with a given non-square term N (here N=2, z =1) and adding an integer produces another square term . Brahmagupta's *Bhāvanā* is the method of finding an identity combining two solutions of unknown sets of values (x_1,y_1,z_1) and (x_2,y_2,z_2) of the *Vargaprakṛti* . If (x_1, y_1) and (x_2, y_2) are two roots of $Nx^2 + z = y^2$, then,

$Nx_1^2 + z_1 = y_1^2$; $Nx_2^2 + z_z = y_2^2$, $(x_1y_2 + x_2y_1, y_1y_2 + Nx_1x_2)$ and $(x_1y_2 - x_2y_1, y_1y_2 - Nx_1x_2)$ are roots of solutions of $Nx^2 + z_1z_2 = y^2$. In other words, we have two identities (called Brahmagupta's identities) $(y_1^2 - Nx_1^2)(y_2^2 - Nx_2^2) = (y_1y_2 \pm Nx)^2 - N(x_1y_2 \pm x_2y_1)^2$. Now, this identity will be used for third roots .[24]

$Nx^2 + z = y^2$ to produce a third solution (x_3,y_3,z_3), given by $x_3 = x_1y_2 + x_2y_1$, $y_3 = Nx_1x_2 + y_1y_2$ and $z_3 = z_1z_2$. With this identity , we can interpolate in $2q^2+1 = p^2$ and find out value of p and q using q,p and 1 in place of x, y and z respectively ; where N=2

Now for putting x= q, y= p , N=2, z=1 , we may put q_1, p_1 and q_2, p_2, two sets of value produce value of q_3, p_3. Interpolating , q for x, p for y for the composition operation, Brahmagupta's identity law takes the form.
$(q_1,p_1;1) \odot (q_2,p_2; 1)$ then $q_3 = (q_1p_2 + q_2p_1$, $p_3 = Nq_1q_2 + p_1p_2$; $z_3 = z_1.z_2 = (1x1)$.

Here, equation is $2q^2+1 = p^2$, minimum positive integer value, if q =2, then p=3, the equation is satisfied. For the beginning let us consider the same values for $(q_1,p_1;)$ and (q_1,p_2) to find out $(q_3, p_3.)$

$(2, 3,1) \odot (2,3,1)$, for $q_3 = 2x3+3x2=12$ and $p_3 = 2.2.2+ 3.3 = 17$ ……(A)
$(p_{3/Q\,3}) = (17/12)$

For, sets of values (2,3,1) (12,17,1), $q_4 = 2.17 +3.12 =70$, $p_4 = 2.2.12+ 3x17= 99$ ………(B)
$(p_4/q_4) = (99/ 70)$………. (C)

For sets $(2,3,1) \odot (70,99,1)$ $q_5 = 2x99+3x70= 408$, $p_5 = 2.2.70+3x99= 577$. $P_5/q_5 = 577/408$…….. (D)
For $(12,17, 1) \odot (70,99)$.$q_6 = 12x99+70x17= 2378$,$p_6 = 2. 12.70+17x99= 3363$, $p_6/q_6 = 3363/2378$……… (E)
For $(70,99\ 1) \odot (2378, 3363, 1)$ $q_7 = 70x3363+ 99x2378=470832$ $p_7 = 2. 70.2378+ 99x3363 = 665857$. $P_7/q_7 = 665857/470832$……… (F). With any two sets from A,B,C,D,E,F number of sets can be found.

Brahmagupta's Bhāvanā provided the values of $\sqrt{2}$ in the ratio of p/q are 3/2, 17/12, 577/408, 3363/2378, 665857/470832 through numerical method.

- *Understanding* $\sqrt{2}$ *with Srinivasa Ramanujan*

Srinivasa Ramanujan was the greatest mathematician of his time and a legendary number theorist, ever born. **Robert Kanigel** portrayed an anecdote on a Sunday morning of December, 1914 meet with Prasanta C Mahalanobis, (Scientist and later founder of Indian Statistical Institute , Calcutta) his Indian contemporary, preparing for natural sciences , Tripos . Strand, the popular English Magazine devoted a page entitled "Perplexities" on intriguing puzzle. Mahalonobis narrated the mathematical puzzle from the current issue of Strand for a solution. The problem was at a long street, houses numbered on one side as one, two, three, and so on, and that all the numbers on one side of a particular number of house added up exactly the same as all the numbers on the other side of the house. Number of houses was stated between 50 to 500."

Those who have elementary knowledge of Mathematics, they would go for the sum in the following way.

Let there be n number of houses. Particular house number is 'm'; then on its left side, (m -1) number of houses and on right side, (n – m) number of houses were there. [(m – 1) +1 (for m^{th} number of house) + rest number of houses on right side, (n – m) = n number of houses].

Sum of all the left side numbered houses of 'm' is 1+2+3+4……… (M-1) = [(m-1)].m/2…………(A)

Sum of all the right side numbered houses 'm' from (m+1) + (m+2) + (m+3)……….upto n. = (m+1) + (m+1) + (m+2) +….. + [n−(m +1) +1]
=>m (n – m) + [1+2+3+4…. + (n-m)] => m (n- m) + [(n –m) (n- m-+ 1)]/2

=>(n –m).[m+(n –m+1)]/2 => [(n –m) .(2m+n-m+1)]/2 => (n- m). (m+n +1)/2……. (B)
Now, A= B, hence, m (m-1)/2 = (n- m). (m+n +1)/2 => m (m -1) = (n- m). (m+n +1)
$m^2 – m = n^2 – m^2 + n – m$ => $2 m^2 = n^2 + n = n (n+1)$ => $m^2 = [n (n+1)]/2$………. (C)

We get an equation of indeterminate form. Mahalanobis figured it out through trial and error. The house number was 204. If we add from 1 to 203 all natural numbers, it will come to (1+2+3+4+5 ………….. 203) 203x204/2 = 20706. Again 205+206+207+208….. 288. Here 84 (288-204) houses whose first (number) term is 205, Sum is 84/2 (410+ 83) = 42x 493= 20706.

This equation has been manipulated as following.
$m^2 = [n (n+1)]/2$=> $(n^2+n)/2$, => $2m^2 = n^2+n$, => $8m^2 = 4n^2+4n$ (multiplying by 4 on both sides).
$8m^2+1 = 4n^2+4n+1$, (by adding 1 by both sides).
$8m^2+1 = (2n+1)^2$………. (D). Let p= 2n+1, q = 2m, then $2q^2+1= p^2$, then $p^2 - 2q^2 =1$, [26]

This equation is the same that we have achieved during generalising the formula out of deriving the value of $\sqrt{2}$ before.

In this context, square Triangular numbers are relevant for discussion. Square Triangular numbers are numbers which are both square numbers and also triangular numbers. Pythagorean triangular number: n is 8 ,then total number of dots in the triangle is (8x9)/2= 36, which is square of m with 6 dots, similarly, when n=49, total number of dots in the triangle (49x50)/2=1225 which is square of 35 dots.[27] Other values of m and n are following.

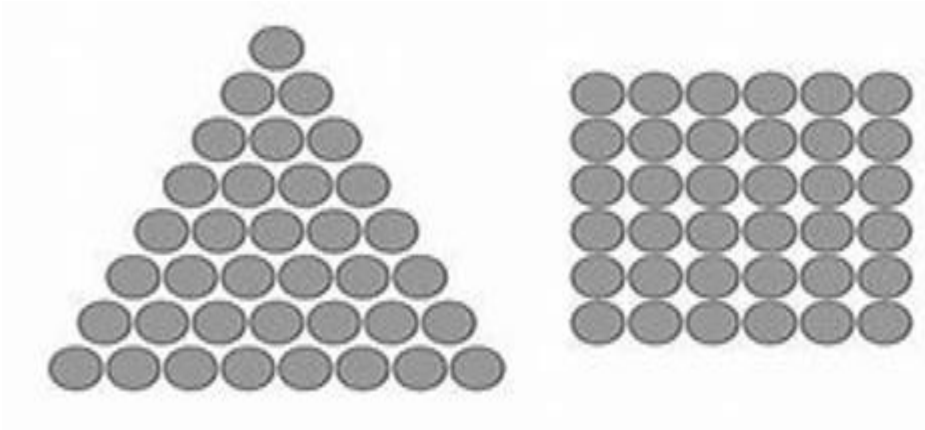

Fig 7:- The first triangular number in the series after 1, where base of the triangle is with 8 dots (total dots are 36) equalize with a square with 6 dots. Source: https://en.wikipedia.org.

Dots in the triangle is [n (n+1)]/2 where n representing number of dots in the base and m is a side of a square = m^2

This is the same equation as it is mentioned in (C). How could we find the value of m or n in an indeterminate equation?
Let , N= n(n+1)/2 => 2N = n(n+1) => $n^2 + n – 2N=0$ > n= [- 1 ±√(1 – 4.1– 2N)]/2 = [√(8N +1)-1]/2 [avoiding negative value of −√(8N +1)-1]/ 2] . In other words

n = √(8N +1)-1]/2 => n= √($8m^2$ +1)-1]/2 (As N = m^2). We are to ensure now that square root of +1 is an integer. Now, m= 1 ,6 35,204,1169,6930, we get corresponding n= 8,49,288,1681,9800 respectively. If there is i) one house, m=n, both case , it is 1, ii) 8 houses, 6th is the number, then (1+2+3+4+5=15 , again after 6, 7 and 8 which are totaled to 15, iii) if there is 49 houses 35th number house is the reply; 1+2+3+….. 34= 34x35/2= 595, 36+37+38…… +48+49 = [14 x (72+ 13.1)]/2 = 7x85= 595. Here , answer was 204 (mth numberof house) and total number of houses are 288. Sum total is discussed before. But , this is a trial and error method.

Ramanujan found the solution in a very unusual way. His spontaneous intuitive ability helped find a solution through a continued fraction. Instantaneously, his thought-process pinpointed the nature of continued fraction and the exact number involved for the series. He asked his friend to take down the notes. It was not an answer of a particular problem , it was solution of ‘whole class of problem’ . [28] The series is

$$\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{6-\cfrac{1}{1\ldots\ldots}}}}}$$

The series is also unique in feature as minus (−) sign in the continued fraction was used instead commonly used plus (+) sign .

This series will continue for ever .Any portion of this continued fraction provided the value of m as said before ; If we take each tier to its finality , the following situation will arise .

Tier with	Values of m	Values of n
First	1/6(1,6)	1,8
Second	1/(6 - $\frac{1}{6}$), 6,35	8, 49
Third	1/(6 - $\frac{1}{6-\frac{1}{6}}$) 35, 204	49, 208
Fourth	1/(6 - $\frac{1}{6-\frac{1}{6-\frac{1}{6}}}$) 204,1189	208, 1681
Fifth	1/(6 - $\frac{1}{6-\frac{1}{6-\frac{1}{6-\frac{1}{6}}}}$) 1189, 6930	1681, 9800
Sixth	1/(6- $\frac{1189}{6930}$)* , 6930, 40391,	9800,57121
Seventh	1/6- $\frac{6930}{40391}$ * , 40391,235416	57121, 332928

*Value upto previous tier is placed.

Table 2:- Value of m with resultant value of n from the continued fraction from each tier from the beginning .

With these values of m and n, we get value of p and q, the higher the values of p and q nearer the perfection of the value of $\sqrt{2}$ is.

m	n	p= (2n+1)	q = 2m	p/q ($\sqrt{2}$)	p^2–$2q^2$
1	1	3	2	3/2=1.5	3^2 – 2,2^2 =1
6	8	17	12	17/12	17^2- 2.12^2= 289-288 =1
35	49	99	70	99/70	99^2 – 2.70^2= 9801-9800=1
204	288	577	408	577/408	577^2 – 2.408^2= 332929- 332928=1
1189	1681	3363	2378	3363/2378	11309769- 11309768 =1
6930	9800	19601	13860	19601/13860	384199201- 384199200 =1
40931	57121	114243	80782	1142431/80782	13051463049 – 13051463048=1
235416	332928	665857	470832	665857/470832	443365544449 – 443365544448=1

Table 3:- Value of p/q: m and n out of m^2= n(n+1)/2. p and q are resulted thereof.

Geometry –based derivation of Baudhāyana had provided values those are quite apart from each other; more towards the perfection, more is the gap between two consecutive values. With other three methods, we get more number of stages towards perfecting the said value. Ramanujan's innovative creations of thousands of identities and formulas in numerical and infinite series were astoundingly advanced. One of such is used here in the expression of continued fraction .

II. CONCLUSION

Measure of value of first irrational number was derived by Baudhāyana Śulba Sūtra through Geometry. One liner cryptic form of aphorism in archaic Saṇskrit , 2800 years ago was astonishingly validated by algebraic form of *Bhāvanā* of Brahmgupta of early seventh century ; after 1400 years of its creation. In the seventeenth century, Isaac Newton with Joseph Raphson applied calculus, the new branch of Mathematics invented during same period, determined the values of the same. Modern mathematics satisfied with the result derived from ancient form. In the early two decades of last century, Srinivasa Ramanujan provided incites in abundant diversified field. One of such corroborated the issue involved in the present paper.

Formulation for deriving square-root of positive integers or fractions in its generalized form had taken place in India since Bakhsali Manuscrpits, [Folios 56 (recto) and 57 (verso 57) of the manuscript] (300-400 CE) [29]. From early medieval period to late medieval period, astronomers-mathematicians, Āryabhaṭa, Brahmgupta, Mahāvīra, Srīpati, Srīdhara, Nārayaṇa (500 CE-1400 CE) did their contribution .[30] Other civilizations had also put forward the same. Here, specific analysis for the value of $\sqrt{2}$ based on historical perspective is presented.

$\frac{577}{408}$, a ratio evolved out of Śulba Sūtra for a diagonal of a square was recorded first by Baudhāyana, the great ancient master of Geometry. Mathematicians, all over the World, (especially of Indian root) will think over the issue in naming $\frac{577}{408}$ as Baudhāyana ratio or Baudhāyana constant.

Mathematicians have been working for finding out the actual value on adding billions or trillions numbers of digits following 1.4142135624……..

REFERENCES

[1]. John F. Price: Applied Geometry of the Śulba Sūtras.* School of Mathematics University of New South Wales Sydney', NSW 2052, Australia.

[2]. Dwarka Nath Yavan and George Thibaut :Baudhyāna Śulbasutram with Sanskrit Commentary by Dwarka Nath Yajvan and English Translation and Critical Notes by G Thibaut. Edited by Pt. Ram Swarup Sharma, Director, The Research Institute of Ancient Scientific Studies and Dr Satyaprakash : The Research Institute of ancient scientific studies , New Delhi -8 , published in 1968 Ratna Kumari Publication series:.4

[3]. John F. Price : Applied Geometry of the Sulba Sutras* School of Mathematics Uiniversitv of New South Wales Sydnel', NSW 2052, Australia wherein reference of Newton-Raphson Method was mentioned.

[4]. Dwarka Nath Yavan and George Thibaut :Baudhyā̄na Śulbasutram with Sanskrit Commentary by Dwarka Nath Yajvan and English Translation and Critical Notes by G Thibaut. Edited by Pt. Ram Swarup Sharma, Director, The Research Institute of Ancient Scientific Studies and Dr Satyaprakash : The Research Institute of ancient scientific studies , New Delhi -8 , published in 1968 Ratna Kumari Publication series.,4 page :51

[5]. Ibid : page : 61

[6]. Ibid: page :61

[7]. Ibid : page 61

[8]. Sen S N and A.K Bag : The Śulbasūtras of Baudhyāna, Ạ̄pastamba, Katyāyana and Mānava with Text, English translation and commentary published by Indian National Science Academy, in New Delhi, 1983., page:11 . For English translations of Sūtras, this book is also consulted.

[9]. David Eugene Smith and Louis Charles Karpiniski: Hindu Arabic Numerals: Pages; 12-13 of chapter -II Boston and London Ginn and Company Publications: 1911. Reprinted by Sadesh ,Kolkata:700006,India.

[10]. Datta Bibhuti Bhuasn ; Ancient Hindu Geometry, The Science of Śulba (Calcutta University, 1932), page:190

[11]. ibid: pages:191-194

[12]. Henderson David W: Square root in the Śulba Sutra, Department of Mathematics, Cornell University (published in 1992).

[13]. ibid: Two Geometrical figures showing unit of square subtracted in each stages. paper is used.

[14]. Hippasus of Metapontum: https://en.wikipedia.org/wiki/Hippasus.

[15]. Bertand Russell : Western Philosophy , Paperback Edition,1981, Unwin Paperbacks, page no 52 and of chapter III , Pythagoras

[16]. Spiral of Theodorus : https://en.wikipedia.org/wiki

[17]. Bertand Russell : Western Philosophy , Paperback Edition,1981, Unwin Paperbacks, , page 218-219,, Chapter XIV ;Early Greek Mathematics and Astronomy.

[18]. David Eugene Smith: History of Mathematics : Volume II , Dover Publication INC , New York, The New edition ,1958, page ;5 and 6..

[19]. Newton–Raphson Method, : Elisa T. Lee and John Wenyu Wang, Wiley on-line Library.

[20]. For Graphical representation of curve. https://www.bing.com/images/search?q=newton+raphson+method+graph&id=819E40456ADAEE8B8FE36367A4461754ED3E3B70&FORM=IQFRBA.

[21]. html 20https://en.wikipedia.org/wiki/Newton%27s_method

[22]. Computing the square root of 2 . https://www.youtube.com/watch?v=2158QbsunA8

[23]. Brahmgupta's from Brāhmaṣphutosiddhānta Vargaprakiti (Volume 4, Chapter: 18).

[24]. The *bhāvanā* in Mathematics by Amartya Kumar Dutta .: Bhāvanā: a publication of Mathematical Magazine.

[25]. Robert Kanigel : "The Man who knew infinity", a life of genius Ramanujan , Washington Square Press, 1991 (page 214-215)

[26]. Talk by Professor Ashok Kumar Mullik on Magical Mathematics of Ramanujan on YouTube. https://www.youtube.com/watch?v=M3_ThpQiq68&t=1855s

[27]. https://en.wikipedia.org/wiki/Square_triangular_number on Square Triangular Numbers.

[28]. Talk by Professor Ashok Kumar Mullik on Magical Mathematics of Ramanujan on YouTube. https://www.youtube.com/watch?v=M3_ThpQiq68&t=1855s

[29]. The Bakhsali Manuscript: Edited by Swami Satya Prakash Saraswati and Dr Usha Jyotishmati, ; Dr Ratna Kumari :Savdhyaya Sansthan, Aliahabad Page 36-38 , dealing with.

[30]. Datta Bibhuti Bhusan and A. N Singh , History of Hindu Mathematics ; Volume :I, Asia Publishing House 1935 and 1938, Pages: 169-175

➢ *Further References:*

[31]. George Thibaut : Mathematics in the making in Ancient India, Edited with introduction by Debiprasad Chattopadhyaya. K P Bagchi & CO., Calcutta and Delhi : 1984

[32]. Life of Pythagoras by Iamblichus of Chalcis, Pythagoras Contributor)Thomas Taylor (Translator). Inner Tradition : 1986

[33]. Pythagoras: His Life, Teaching, and Influence 2nd Edition by Christoph Riedweg, , translated by Steven Rendall .

[34]. History of Hindu Mathematics: Bibhuti Bhusan Datta and Avadesh Narayan Singh; Asia Publishing House 1935 and 1938 (Combined Volume; I and II)

[35]. Mathematics in India: Kim Plofker: Princeton University Press, 2009

[36]. Mactutor History of Mathematics: University of St. Andrews. U K. Edited by J J O'Connor and E F Robertson . Relevant Areas .