On Semi-Uniformity, Quasi-Uniformity, Local Uniformity and Uniformity

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Abstract:- In this paper we shall study the notions of semi-uniformity, quasi-uniformity, local uniformity and uniformity and we can establish some relations among them. Here it can be shown that a uniform space has always a base of symmetric vicinities.

Keywords:- *Uniformity, Vicinity, Symmetric, Diagonal Filter.*

I. INTRODUCTION

If the symmetry, condition in the definition of a pseudometric is deleted, the notion of a quasi-pseudometric is obtained. Asymmetric distance functions already occurs in the work of hausdorff in the beginning of the twentieth century when in his book on set-theory he discusses what is know called the hausdroff metric of a metric space.

A family of pseudo-metrices on a set generates uniformity. Similarly, a family of quasi-pseudometrices on a set generates a quasi- uniformity.

In 1937 Weil- published his booklet on uniformites, which is now usually considered as the beginning of the modern-theory of uniformities. Three years later Tukey suggested an approach to uniformities via uniform coverings. The study of quasi-uniformities started in 1948 with Nachbin's investigations on uniform pre-ordered preorder is given by the intersection of the entourages of a (filter) quasi-uniformity u and sup-uniformity $u v u^{-1}$.

II. NOTIONS OF SEMI-UNIFORMITY, QUASI-UNIFORMITY, LOCAL UNIFORMITY AND UNIFORMITY

➢ *Definition :*

Let X **be a non-empty set and let** $\Delta=\{(x,x)\{x\in X\}$ **be the 'diagonal'. A subset** $A\subset X\times X$ **is said to be 'symmetric' if** $A=A^{-1}$**, where** $A^{-1}=\{(x,y)\in X\times X:(y,x)\in A\}$ **. For** $A,B\subset X\times X$ **we define** $A,B=[(x,y):(x,z)\in A]$ **and** $(z,y)\in B$ **for some** $z\in X\}$

The set A. A is also written as A^2

If $A, B, C\subset X\times X$ **then it is easy to see that**

(i) $A.\Delta=\Delta.A=A;$

(ii) $A.(B, C)=(A.B).C;$

(iii) $(A.B)^{-1}=B^{-1}.A^{-1}$

(iv) $A^{n+1}=A^n.A$ **and** $A^n.A^m=A^{n+m};$

(v) $(A^n)^{-1}=(A^{-1})^n$

(vi) $A\subset B \Rightarrow A^{-1}\subset B^{-1}$ *and* $A^n\subset B^n$

(vii) If A is symmetric so is $A^n$for each positive interger n.

Proposition :

Let B be a filter base on $X\times X$ such that for each $U\in B$ we have

(i) $\Delta\subseteq U$ (ii) $V^{-1}\subset U$ for some $V\in B$ and (iii) $V^2\subset U$ for some $V\in B$. Then the filter F (B) determined by B satisfies

(i)' $\Delta\subseteq U$ for all $U\in F(B)$

(ii)' $U^1\in F(B)$ for all $U\in F(B)$;

(iii)' For all $U\in F(B)$, there is a $V\in F(B)$ such that $V^2\subset U$.

Conversely, suppose F is a filter on $X\times X$ satisfying (i)' to (iii)'.

Then the conditions (i) to (iii) are satisfied by every filterbase B which determines F.

Proof:

(i)' is obvious for if $U\in F(B)$, then $U\geq U_1$ for some $U_1\in B$and by (i) we have $\Delta\subseteq U_1\subset U$. To prove (ii)' and (iii)', consider any $U\in F(B)$. Then $W\subset U$ for some $W\in B$. Using (ii) we have a $V\in B\subset F(B)$ satisfying

$V^1\subset W\subset U$ and so $V\subset U^1$; that is $U^1\in F(B)$. Also by (iii) there is a $V\in B$ such that $V^2\subset W\subset U$ i.e. $V^2\subset U$.

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Conversely, suppose that F is filter on $X \times X$ which satisfies conditions (i)' to (iii)' and let B be a filter base on $X \times X$ which determine F. since $B \subset F(B)$, condition (i) is satisfied. If $U \in B$ then $U^{-1} \in F(B)$ and $V \subset U^{-1}$ for some $V \in B$. Thus $V^{-1} \subset U$ and condition (ii)' is satisfied. Finally, let $U \in B \subseteq F(B)$ and let $W^2 \subset U$. Since $W \in F(B)$, one has $V \subset W$ for some $V \in B$. In particular, we have $V^2 \subset W^2 \subset U$. Hence condition (iii)' is satisfied. This completes that proof.

➢ *Definition :*

Let X be a non-empty set.

A filter $\mathcal{U}$ of subsets of $X \times X$ is said to be a

(a) <u>semi-uniformity</u> for X if $\Delta \subseteq U$ and $U^{-1} \in \mathcal{U}$ for all $U \in \mathcal{U}$,

(b) <u>quasi-uniformity</u> for X if $\Delta \subseteq U$ and there exists a $V \in \mathcal{U}$ such that $V^2 \subset U$ for all $U \in \mathcal{U}$,

(c) Uniformity for X if $\mathcal{U}$ is both a semiuniformity and a quasi-uniformity.

(d) Local uniformity for X if it is semi-uniformity and for each $U \in \mathcal{U}$ and $x \in X$, there exists a $V \in \mathcal{U}$ such that $V^2[x] \subset U[X]$, $\mathcal{U}$ is said to be Hausdorff (or separated) if $\bigcap\{U : U \in u\} = \Delta$. The members $U \in \mathcal{U}$ are called the Vicinities. Moreover, $x, y \in X$ are called U-close (or close of order U) if $(x, y) \in U$ (where $U \in \mathcal{U}$).

➢ *Definition :*

Uniform space: A uniform space is a pair (X, $\mathcal{U}$) where X is a non-empty set and $\mathcal{U}$ is a uniformity for X, i.e. $\mathcal{U}$ is a filter on $X \times X$ satisfying the following conditions:

$[V_1]$ Every $U \in \mathcal{U}$ contains the diagonal Δ;

$[V_2]$ If $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$

$[V_3]$ for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ with $V^2 \subset U$

III. COMPARISON OF UNIFORMITIES :

➢ *Definition :*

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are both uniformities for a non-empty set X, we say that $\mathcal{U}_1$ is weaker or coarser than $\mathcal{U}_2$ (and $\mathcal{U}_2$ if finer or stronger than $\mathcal{U}_1$) if $\mathcal{U}_1 \subseteq \mathcal{U}_2$. that is, if every vicinity for $\mathcal{U}_1$ is also a vicinity for $\mathcal{U}_2$.

➢ *Definition :*

A base for a uniformity $\mathcal{U}$ on X is a filterbase β such that $U \in \mathcal{U}$ iff U contains some $B \in \beta$.

Proof (6.2) :

A uniform space (X, $\mathcal{U}$) has always a base of symmetric vicinities.

Proof :

Let β = the family of all symmetric vicinities i.e. those with $U = U^{-1}$

For $U \in \mathcal{U}$. Then β is a base for $\mathcal{U}$ since the symmetric vicinity $U \bigcap U^{-1}$ is contained in the vicinity U for each $U \in \mathcal{U}$.

➢ *Definition :*

Let β_1 and β_2 be two bases for a uniformity U on a non-empty set X we say that β_1 and β_2 are equivalent if for each $B_1 \in \beta_1$ there exists a $B_2 \in \beta_2$ such that $B_2 \subseteq B_1$ and for each $B_2' \in \beta_2$ there exists $B_1' \in \beta_1$ such that $B_1' \subseteq B_2'$.

Example (1) :

Given a non-empty set X, the indiscrete uniformity $\mathcal{U}_1 = X \times X$ is the weakest uniformity for X and the discrete uniformity $\mathcal{U}_D = \{U \subset X \times X ; \Delta \subseteq U\}$ is the strongest uniformity for X. We note that $\{\Delta\}$ is a base for discrete uniformity for X where as $\{X \times X\}$ is a base for indiscrete uniformity for X.

IV. CONCLUSION

Hence, A uniform space (X, $\mathcal{U}$) has always a base of symmetric vicinities.

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