ISSN No:-2456-2165

1-Parameter Group Structure of Integral Curves

Praveen Sharma
Department of Mathematics
University of Delhi

Abstract:- In this paper we have studied the integral curves of the field of vectors on the subsets of R^{n+1} and identify the 1-parameter transformation group structure associated to a particular case of a vector field with infinitely times differentiable component function and then generalizes it to any arbitrary field of vectors with infinitely times differentiable component function.

I. PRELIMINARIES

A. *Definition*

A vector at a point $p \in R^{n+1}$ is a pair $\vec{v} = (p,v)$ where $v \in R^{n+1}$.

Geometrically we can think v as the vector v translated in such a way that the tail of the vector $\vec{v}$ is at p instead of at origin. [2]

B. *Field of vectors*

A field of vectors on a subset $S \subseteq R^{n+1}$ is a function F from R^{n+1} to R_p^{n+1} defined by
$F(p) = (p,F(p))$
where $F : S \to R^{n+1}$ is some function.

C. *Smoothness of a function*

A function $f : S \to R$ where $S \subseteq R^{n+1}$ is open is said to be a smooth if all directional derivatives of f exists in all directions and are continous .

A function $f : S \to R^k$ is said to be smooth if each of its component real valued functions on S are smooth [1]

Remark A vector field F **on an open set** $S \subseteq R^{n+1}$ **is said to be a smooth field of vectors if the function** $F : S \to R^{n+1}$ **is a smooth function.**

D. *Parametrized Curve*

A parametrized curve in R^{n+12} is a smooth function $\alpha : I \to R^{n+3}$ where $I \subseteq R$ is an open interval.

---

1 . $\alpha(0) = p$
2 . if $\beta : I' \to S$ be any integral curve such that $\beta(0) = p$. Then $I' \subseteq I$ and $\beta(t) = \alpha(t)$ for all $t \in I'$.

Result(2) Let F be a smooth vector field on an open set $S \subseteq R^{n+1}$ and $p \in S$. Let $\beta : I \to S$ be the maximal integral curve of vector field F if $\alpha : I' \to S$ be any integral curve of the smooth vector field F such that $\beta(t_0) = p$ for some $t_0 \in I$. Then $\alpha(t) = \beta(t - t_0)\ \forall\, t \in I'$.

E. *Integral Curve*

A Parametrized curve $\alpha : I \to S$ is said to be an Integral curve of an vector field F on an open set $S \subseteq R^{n+45}$ if $F(\alpha(t)) = \alpha(\dot{t})$

i.e velocity vector at any point of the curve coincides with the value of the vector field at that point.

(Existence of the Integral curve of the smooth Vector field)
Let F be a smooth vector field on an open set $S \subseteq R^{n+1}$ and $p \in S$. Then there exist an maximal integral curve of F through p.
i.e. ∃ an open interval I in R containing 0 such that

II. 1- PARAMETER TRANSFORMATION GROUP STRUCTURE ASSOCIATED TO SMOOTH VECTOR FIELD

- **Theorem 2.1.** *Let* F *be a smooth vector field on* R^2 *given by*

$F(x^6,x_2) = (x_1,x_2,1,0)$

For $t \in R$ *and* $p \in R^7$ *let* $\varphi_t(p) = \alpha_p(t)$ *where* α_p *is the maximal integral curve through p . Then show* $\{\varphi_t : t \in R\}$ *is an 1-parameter transformation group associated to* F
Proof. Fix $p = (p_1,p_2) \in R^2$

Then $\alpha_p(t) = (x_1(t),x_2(t))$ is is a maximal integral curve of the vector field F if By definition

$$\frac{dx_1}{dt} = 1 \qquad and \qquad \frac{dx_2}{dt} = 0$$

$$\Rightarrow x_1(t) = t \qquad and \qquad x_2(t) = c(constant)$$

since $\alpha_p(0) = p \Rightarrow x_1(0) = 0 = p_1$ and $x_2(0) = c = p_2$ ∴ for each fixed $p \in R^2$ we have
$\alpha_p(t) = t(1,0) + p\ \forall\, t \in R$

---

3 .6 **1-Parameter Transformation Group**

A one parameter family $\{\varphi_t : t \in R\}$ of linear transformations on R^2 is said to be 1-parameter transformation group if following properties are satisfied
4 . $\varphi t1{+}t2 = \varphi t1 \circ \varphi t2$
5 . $\varphi{-}t = \varphi{-}t\ 1$
6 .
$\varphi_0(p) = 0(1,0) + p = p\ \forall\, p \in R^2$
$\Rightarrow \varphi_0 = identity\ on\ R^2$
7 . $\varphi_{t1+t2}(p) = (t_1 + t_2)(1,0) + p = (t_2(1,0) + p) + t_1(1,0)$
$\varphi t1{+}t2(p) = \varphi t1 \circ \varphi t2(p)\ \forall\, p \in R2$
$\Rightarrow \varphi t1{+}t2 = \varphi t1 \circ \varphi t2$

To show for each fixed $t \in \mathbb{R}$ φ_t **is a one to one linear transformation from** $\mathbb{R}^2$ **to** $\mathbb{R}^2$

Fix $t \in \mathbb{R}$. Then
$\varphi_t(p) = \alpha_p(t) = t(1,0) + p \; \forall \; p \in \mathbb{R}^2$

Therefore for each fixed $t \in \mathbb{R}$ φ_t is the composition of first translation of vector (1,0) by t then translation by the vector $p \in \mathbb{R}^2$ which are both one one onto transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Hence φ_t is also a one to one linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ for each $t \in \mathbb{R}$ • **To show** $\{\varphi_t : t \in \mathbb{R}\}$ **is a 1-parameter transformation group associated to** F 3. $\varphi_t \circ \varphi_{-t} = \varphi_{t-t} = \varphi_0 = identity$
$\varphi{-t}\ 1 = \varphi{-t}$

Therefore $\{\varphi_t : t \in \mathbb{R}\}$ is a 1- parameter transformation group associated to F
□

- **Theorem 2.2.** *(Generalization) Let* F *be any smooth vector field on an open set*

$S \subseteq \mathbb{R}^{n+1}$ then $\{\varphi_t : t \in \mathbb{R}\}$ is the 1-Parameter group associated to F.
Proof. We only needs to show it satisfies the two properties that

- $\varphi t1{+}t2 = \varphi t1 \circ \varphi t2$
- $\varphi{-t} = \varphi{-t}\ 1$

Now Let $p \in \mathbb{R}^2$ is arbitrary and $\varphi_{t2}(p) = \alpha_p(t_2)$

where α_p is the maximal integral curve of the smooth vector field F through p i.e $\alpha(0) = p$ Let
$\alpha_p(0) = \gamma \in S$

Then by the Existence of the maximal integral curve of a smooth vector field ∃ the maximal integral curve $\beta : I \to S$ of vector field F where $0 \in I$ and $\beta(0) = \gamma$

Now since α_p is also an integral curve of smooth vector field F such that
$\alpha_p(t_2) = \beta(0) = \gamma$
Then by the result 2 we have
$\beta(t) = \alpha_p(t + t_2) \; \forall \; t \in I$
Now
$\varphi t1 \circ \varphi t2(p) = \beta(t1) = \alpha p(t1 + t2) = \varphi t1{+}t2(p)$
since $p \in \mathbb{R}^2$ is arbitrary hence
$\varphi t_1{+}t_2 = \varphi t_1 \circ \varphi t_2$
Also
$\varphi_{-t2} \circ \varphi_{t2}(p) = \beta_{-t2}(p) = \alpha_p(0) = p$
$\Rightarrow \varphi_{-t} \circ \varphi_t = identity$
$\Rightarrow \varphi{-t} = \varphi{-t}\ 1$
Hence the result

III. CONCLUSION

We have established the one parameter group structure on any vector field whose component functions are infinitely many times differentiable. this helps us to study some of the measure properties of the maximal integral curves of a vector field with component functions are infinitely many times differentiable and ease the study of level sets and graph of such arbitrary functions

REFERENCES

[1]. Robert G Bartle and Donald R Sherbert. *Introduction to real analysis*, volume 2. Wiley New York, 2000.
[2]. John A Thorpe. *Elementary topics in differential geometry*. Springer Science & Business Media, 2012.