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Closed Form Stability Analysis of Solid Non-Prismatic Columns

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Abstract:- This paper presents the closed form stability analysis of solid non-prismatic columns. Using fundamental kinematics and Hooke's law, the total potential energy functional of a non-prismatic column was obtained. This was minimized with respect to deflection and the non-linear Euler-Bernoulli equation of equilibrium of a non-prismatic column was obtained. Two mathematical axioms were employed to completely integrate the non-linear governing equation. Individual deflection equations for four columns of various boundary conditions were obtained. Substituting the deflection equation into the non-linear governing equation and rearranging it gave the closed form formula for calculating buckling loads of non-prismatic columns. This formula was used to determine buckling loads for eight example problems. Results from four of the example problems were compared with results from earlier study that used an approximate method called weighted moment of inertia. The highest percentage difference recorded is 9.59%, which validates the present method since the result from earlier study is based on approximate method.

Keywords—closed form; non-prismatic; deflection; governing equations; buckling.

I. INTRODUCTION

Buckling analysis of non-prismatic column presents non-linear governing equation of equilibrium of forces. This makes it somewhat intractable to handle. In a bid to circumvent the integration of the non-linear governing equation, several scholars employ various technics in the analyses. The difficulty involved in the analysis of non-prismatic columns leads to use of approximate methods. However, it is pertinent to accurately analyze the non-prismatic columns because it (non-prismatic column construction) is used in getting economical column in practice [1]. [1] used modified matrix technique to determine the inelastic buckling loads of non-prismatic Columns. In the technique he included geometric residual and material non-linearity effects. In solving numerical problems, he employed iteration, which made his technique numerical method. [2] based his formulation on numerically solving governing equation of equilibrium of forces of non-prismatic column to obtain buckling loads. [3] used finite element model ANSYS 12.0 to perform the non-linear analysis of non-prismatic columns of square and circular sections. Their results indicate that tapering ratio and slenderness ratio affect the buckling loads of the non-prismatic columns. Another author that used non-analytical approach is [4]. More so, [5] proposed numerical approach in their paper for determining the buckling load of non-prismatic columns. This they did by discretizing the non-prismatic column on non-uniform elastic foundation into finite segments. They validated their method by comparing its results with results from analytical approach. In his work, he used Matlab coding in finite element analysis to formulate eigenvalue problem that determine the buckling loads of non-prismatic columns. Furthermore, [6], in their work determined the buckling loads of columns whose material obeys Ludwick's constitutive law by numerically solving ordinary differential equations of non-prismatic column.

However, some scholars tried using analytical approaches to solve the problem of non-prismatic columns. [7] used energy method and modified vibrational mode shape (MVM) in buckling analysis of non-prismatic columns. They used some numerical problems to demonstrate the accuracy and efficiency of their method. In their own, [8] considered the stability non-prismatic column under its self-weight and end load. They transformed the Euler–Bernoulli equation equilibrium of forces of columns into a functional from where they obtained critical buckling load. They used solution to numerical problems to validate their method. Furthermore, [9] presented a paper during a conference in Lisbon titled "buckling analysis of non-prismatic columns using slope-deflection method". Their work brought in a new analytical method for analysis of tapered columns that has fast convergence. To validate their method, numerical problems were solved using their method and finite element method. The two results compared well with each other. In their own work, [10] formulated exact elemental stiffness matrix of non-prismatic beam–column using Euler-Bernoulli beam theory. To demonstrate the accuracy and efficiency of the method numerical problems were solved. From the foregoing, earlier scholars have approached the analysis of non-prismatic columns numerically and analytically. However, they circumvented the integration of the governing equation of equilibrium of forces of non-prismatic column because of its non-linear nature. Hence, the essence of the present work is to

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integrate the non-linear governing equation of non-prismatic column to obtain its closed form solution.

II. MATHEMATICAL FORMULATION

A. Assumptions

The material used in the continuum is homogenous and isotropic. Shear strains and normal strains acting on x-y surface and x-z surface are small when compared with normal strain acting on y-z plane. Hence, neglecting them shall not affect the gross response of the column. Thickness of the continuum varies along the length (that along x axis) and the variation is continuous and differentiable.

A. Kinematics

The displacement considered are u, v and w along x, y and z axes respectively. From the assumptions of zero shear strains within x-z and y-z planes, displacements u and v relate to the displacement w as:

$$u = -z\frac{dw}{dx} = -S\frac{t}{L}\frac{dw}{dR} \qquad (1)$$

$$v = -z\frac{dw}{dy} = -S\frac{t}{b}\frac{dw}{dQ} \qquad (2)$$

where: R, Q and S are non-dimensional coordinates along x, y and z axes respectively defined as:

$$R = \frac{x}{L};\ Q = \frac{y}{b};\ S = \frac{z}{t} \qquad (3)$$

where: L, b and t are dimensions of column along x, y and z axes respectively.

The normal strain acting on y-z plane is defined as:

$$\varepsilon_{xx} = \frac{du}{dx} \qquad (4)$$

Substituting equations 2 and 3 into equation 4 gives:

$$\varepsilon_{RR} = -\frac{ts}{L^2} * \frac{d^2w}{dR^2} \qquad (5)$$

B. Hooke's Law

Hooke's law is the relation between stress and strain. Thus

$$\sigma_R \propto \varepsilon_R \qquad (6)$$

Introducing proportionality constant in equation 6 gives

$$\sigma_R = E\varepsilon_R \qquad (7)$$

Where E is the Young's modulus. Substituting equation 5 into equation 7 gives

$$\sigma_R = -\frac{Ets}{L^2} * \frac{d^2w}{dR^2} \qquad (8)$$

C. Strain Energy

Strain energy of a column is half of the product of strain and stress summed up in the domain of the column. That is:

$$U = \frac{1}{2}\iiint \varepsilon_R\,\sigma_R\,dx\,dy\,dz = \frac{btL}{2}\iiint \varepsilon_R\,\sigma_R\,dR\,dQ\,dS \quad (9)$$

Substituting equations 5 and 8 into equation 9 gives:

$$U = \frac{Eb}{2L^3}\iiint S^2t^3\left(\frac{d^2w}{dR^2}\right)^2 dR\,dQ\,dS \quad (10)$$

Rearranging equation 10 gives:

$$U = \int_0^1 dQ * \int_{-0.5}^{0.5} S^2\,dS * \int_0^1 t^3 * \left(\frac{d^2w}{dR^2}\right)^2 dR * \frac{Eb}{2L^3} \quad (11)$$

Carrying out the closed domain integration of equation 11 with respect to Q and S gives:

$$U = \frac{Eb}{24L^3}\int_0^1 t^3 * \left(\frac{d^2w}{dR^2}\right)^2 dR \quad (12)$$

A diagram of non-prismatic column is shown in figure 1

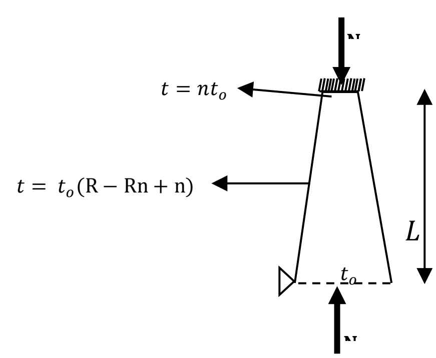

Fig 1:- Non-prismatic column

Here the base is the reference point and the thickness at that point is designated as to while at any other arbitrary point, t is a product of to and Fo

The thickness of the column at any arbitrary point along R axis is defined as

$$t = F_o * t_{so} \quad (13)$$

Where t_{so} is the thickness at a reference point and F_o is the function that defines how the thickness varies along R axis.

Substituting equation 13 into equation 12 gives:

$$U = \frac{Eb}{24L^3}\int_0^1 {F_0}^3 * {t_{so}}^3 * \left(\frac{d^2w}{dR^2}\right)^2 \quad dR \quad (14)$$

Simplifying equation 14 gives:

$$U = \frac{EI_{so}}{2L^3}\int_0^1 F_2 * \left(\frac{d^2w}{dR^2}\right)^2 \quad dR \quad (15)$$

Where the second moment of area at the reference point denoted as I_{so} and the function that defines how the second moment of area varies along R axis denoted as F_2 are defined as:

$$I_{so} = \frac{b\,{t_{so}}^3}{12} \quad (16)$$

$$F_2 = {F_0}^3 \quad (17)$$

Total potential energy functional for buckling analysis of non-prismatic column

The external work for buckling analysis of column is given by [11] as:

$$V_N = \frac{N}{2L}\int_0^1 \left(\frac{dw}{dR}\right) \quad dR \quad (18)$$

Subtracting equation 18 from equation 15 gives the total potential energy functional as:

$$\Pi = \frac{EI_{so}}{2L^3}\int_0^1 F_2 * \left(\frac{d^2w}{dR^2}\right)^2 \quad dR - \frac{N}{2L}\int_0^1 \left(\frac{dw}{dR}\right) \quad dR \quad (19)$$

Rearranging equation 19 gives:

$$\Pi = \frac{EI_{so}}{2L^3}\int_0^1 \left[F_2 * \left(\frac{d^2w}{dR^2}\right)^2 \quad dR - \frac{NL^2}{EI_{so}}\left(\frac{dw}{dR}\right)\right] \quad dR \quad (20)$$

Euler-Bernoulli equilibrium of forces equation of line continuum of varying thickness.

Minimization of total potential energy functional with respect to deflection gives Euler-Bernoulli equilibrium of forces (resultant of forces, FR acting on the column in state of equilibrium). The domain (d1 ≤ R ≤ d2) of the Euler-Bernoulli equilibrium of forces is the bent part of the column, whose length is called the effective length, Le. Hence, minimizing equation 20 gives:

$$F_R = \frac{d\pi}{dw} = \frac{E\,I_{so}}{L^3}\int_{d_1}^{d_2}\left(F_2\frac{d^4w}{dR^4} + \frac{NL^2}{E\,I_{so}}.\frac{d^2w}{dR^2}\right)dR = 0 \quad (21)$$

III. SOLUTION NON-LINEAR EULER-BERNOULLI EQUATION OF EQUILIBRIUM OF FORCES FOR BUCKLING OF NON-PRISMATIC COLUMN

Let buckling load n be defined as a product of two functions as:

$$N = N_1 * N_2 \quad (22)$$

Substituting equation 22 into equation 21 gives:

$$\int_{d_1}^{d_2}\left(F_2\frac{d^4w}{dR^4} + N_2\frac{N_1L^2}{E\,I_{so}}.\frac{d^2w}{dR^2}\right)dR = 0 \quad (23)$$

Integrating equation 23 with respect to R and rearranging the outcome gives:

$$\iint_{d_1}^{d_2} F_2 * \frac{d^4w}{dR^4}\,dRdR = -\iint_{d_1}^{d_2} N_2\frac{N_1L^2}{E\,I_{so}} * \frac{d^2w}{dR^2}dRdR = 0 \quad (24)$$

Using the following mathematical axiom (or maxim), equation 24 can be simplified. Let the axiom be:

$$\iint G_1G_2\,dR\,dR = n_1\int G_1dR * \int G_2dR \quad (25)$$

Where n_1 is a constant.

When the product of A and B is the same as product of C and D and quantity A is the same as quantity C, the it logically follows that quantity B is the same as quantity D. this is mathematically expressed as:

$$If\ A \times B = C \times D\ and\ A = C \text{ then } B = D \quad (26)$$

With the axiom of equation 25, the left hand side of equation 24 is simplified as:

$$\int\int_{d_1}^{d_2} F_2 * \frac{d^4w}{dR^4} dRdR = n_1 \int F_2\, dR * \int_{d_1}^{d_2} \frac{d^4w}{dR^4} dR \quad (27)$$

Similarly, with the axiom of equation 25, the right hand side of equation 24 is simplified as:

$$\int\int_{d_1}^{d_2} N_2 \frac{N_1 L^2}{E\, I_{so}} * \frac{d^2w}{dR^2} dRdR = n_1 \int N_2\, dR * \frac{N_1 L^2}{E\, I_{so}} \int_{d_1}^{d_2} \frac{d^2w}{dR^2} dR \quad (28)$$

Substituting equations 27 and 28 into equation 24 gives:

$$n_1 \int F_2\, dR * \int_{d_1}^{d_2} \frac{d^4w}{dR^4} dR = -n_1 \int N_2\, dR * \frac{N_1 L^2}{E\, I_{so}} \int_{d_1}^{d_2} \frac{d^2w}{dR^2} dR = 0 \quad (29)$$

With the axiom of equation 26, equation 29 gives two independent equations as:

$$n_1 \int F_2\, dR = n_1 \int N_2\, dR \quad (30)$$

$$\int_{d_1}^{d_2} \frac{d^4w}{dR^4} dR = -\frac{N_1 L^2}{E\, I_{so}} \int_{d_1}^{d_2} \frac{d^2w}{dR^2} dR = 0 \quad (31)$$

Solving equation 30 gives:

$$N_2 = F_2 \quad (32)$$

Rearranging equation 31 gives:

$$\int_0^1 [F_2\, dR - N_2\, dR] = 0 \quad (33)$$

$$\int_{d_1}^{d_2} \left[\frac{d^4w}{dR^4} + \frac{N_1 L^2}{E\, I_{so}} * \frac{d^2w}{dR^2}\right] dR = 0 \quad (34)$$

For the integral of equation 34 to be zero, its integrand must be zero. That is:

$$\frac{d^4w}{dR^4} + \frac{N_1 L^2}{E\, I_{so}} * \frac{d^2w}{dR^2} = 0 \quad (35)$$

The ready solution of equation 35 is:

$$w = a_0 + a_1 R + a_2 \cos\left(\frac{N_1 L^2}{E\, I_{so}}\right) R + a_3 \sin\left(\frac{N_1 L^2}{E\, I_{so}}\right) R \quad (36)$$

A. Satisfying the boundary conditions of the non-prismatic columns

Four non-prismatic columns of various boundary conditions are considered. The boundary conditions are denoted as SS, CC, CS and CF standing for simply supported at both ends, clamped at both end, clamped at one end and simply support at the other end, and clamped at one end and free of support at the other end respectively. After satisfying the boundary conditions, the individual deflection functions for the columns, which are presented on Table 1 become:

$$w = A\, \mathrm{h} \quad (37)$$

Where: A is the deflection coefficient and h is the shape function.

Columns/domain	Boundary conditions	Shape function of the deflection
SS $0 \le R \le 1$	$w(0) = \frac{d^2w(0)}{dR^2} = 0;$ $w(1) = \frac{d^2w(1)}{dR^2} = 0$	$Sin(\pi R)$
CC $0.25 \le R \le 0.75$	$w(0) = \frac{dw(0)}{dR} = 0;$ $w(1) = \frac{dw(1)}{dR} = 0$	$\text{Cos}\, 2\pi \text{R} - 1$
CS $0.3 \le R \le 1$	$w(0) = \frac{dw(0)}{dR} = 0;$ $w(1) = \frac{d^2w(1)}{dR^2} = 0$	$g_1 - g_1.R - g_1 \cos g_1 R + \sin g_1 R$ $Where: g_1 = 4.49340946$
CF $-1 \le R \le 1$	$w(0) = \frac{dw(0)}{dR} = 0;\ M(1) = V(1) = 0$	$\text{Cos}\frac{\pi R}{2} - 1$

Table 1: Shape functions of the deflection (w = Ah) for columns of various boundary conditions

M = moment; V = shear force; A = deflection coefficient; h = shape function

B. Calculation of the buckling loads of non-prismatic columns

Substituting equation 37 into equation 34 gives:

$$A \int_{d_1}^{d_2} \left[\frac{d^4h}{dR^4} + \frac{N_1 L^2}{E\, I_{so}} * \frac{d^2h}{dR^2}\right] dR = 0 \quad (38)$$

Rearranging equation 38 and making the buckling load the subject gives:

$$N_1 = \frac{E\, I_{so}}{L^2} * \frac{\int_{d_1}^{d_2} \frac{d^4h}{dR^4} dR}{-\int_{d_1}^{d_2} \frac{d^2h}{dR^2} dR} \quad (39)$$

Substituting equations 32 and 39 into equation 22 gives buckling load of non-prismatic column:

$$\text{N} = \frac{E\, I_{so}}{L^2} * \frac{\int_{d_1}^{d_2} F_2. \frac{d^4h}{dR^4} dR}{-\int_{d_1}^{d_2} \frac{d^2h}{dR^2} dR} \quad (40)$$

Substituting the shape function for ss non-prismatic column from Table 1 into equitation 40 gives:

$$N = \frac{E\, I_{so}}{L^2} * \frac{\int_0^1 F_2. [\pi^4\, Sin(\pi R)]\, dR}{-\int_0^1 [-\pi^2\, Sin(\pi R)]\, dR} = \left(\pi^2 * \frac{E\, I_{so}}{L^2}\right) * \frac{L}{L_e} * \int_0^1 F_2\, dR \quad (41)$$

Substituting the shape function for cc non-prismatic column from Table 1 into equitation 40 gives:

$$N = \frac{E\, I_{so}}{L^2} * \frac{\int_{0.25}^{0.75} F_2. [16\pi^4\, \text{Cos}\, 2\pi\text{R}]\, dR}{-\int_{0.25}^{0.75} [-4\pi^2\, \text{Cos}\, 2\pi\text{R}]\, dR} = \left(4\pi^2 * \frac{E\, I_{so}}{L^2}\right) * \frac{L}{L_e} * \int_{0.25}^{0.75} F_2\, dR \quad (42)$$

Substituting the shape function for cs non-prismatic column from Table 1 into equitation 40 gives:

$$N = \frac{E\,I_{so}}{L^2} * \frac{\int_{0.3}^{1} F_2.[{g_1}^4(-g_1\cos g_1 R + \sin g_1 R)]\,dR}{-\int_{0.3}^{1}[-{g_1}^2(-g_1\cos g_1 R + \sin g_1 R)]\,dR}$$
$$= \left({g_1}^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * \int_{0.3}^{1} F_2\,dR \quad (43)$$

Substituting the shape function for cf non-prismatic column from Table 1 into equitation (40) gives:

$$N = \frac{E\,I_{so}}{L^2} * \frac{\int_{-1}^{1} F_2.\left[\frac{\pi^4}{16}\mathrm{Cos}\frac{\pi R}{2}\right] dR}{-\int_{-1}^{1}\left[-\frac{\pi^2}{4}\mathrm{Cos}\frac{\pi R}{2}\right] dR}$$
$$= \left(\frac{\pi^2}{4} * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * \int_{-1}^{1} F_2\,dR \quad (44)$$

From equations 41, 42, 43 and 44, it is concluded that the buckling load on non-prismatic column is a function of buckling load of prismatic column and is defined (generally) as:

$$N = N_{so} * \frac{L}{L_e} * \int_{d_1}^{d_2} F_2\,dR \quad (45)$$

Where: N_{so} is the buckling load of prismatic column.

The buckling load is expressed as:

$$N = N_d * \frac{E\,I_{so}}{L^2} \quad (46)$$

Where: N_d is the non dimensional buckling load of the column.

IV. EXAMPLE PROBLEMS

Some problems given by [12] to determine buckling loads are:

A. *Example 1 (Ex P1).*

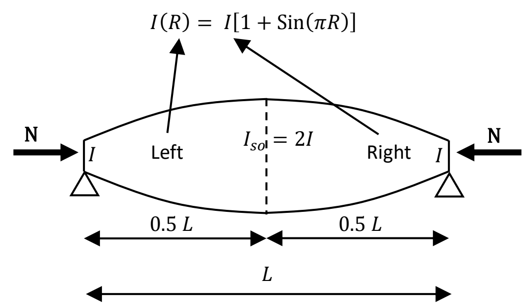

Fig 2:- Simply supported beam under buckling for Ex P1

$$\int_0^1 F_2\,dR = \int_0^{0.5}[1+\mathrm{Sin}(\pi R)]\,dR + \int_{0.5}^{1}[1+\mathrm{Sin}(\pi R)]\,dR$$
$$= \left(1+\frac{2}{\pi}\right) \quad (47)$$

$$N = \left(\pi^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * \left(1+\frac{2}{\pi}\right) = (\pi^2 + 2\pi)\frac{E\,I_{so}}{L^2} \quad (48)$$

B. *Example 2 (Ex P2).*

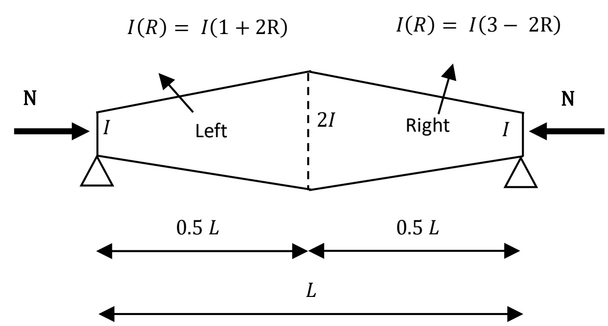

Fig 3:- Simply supported beam under buckling for Ex P2

$$\int_0^1 F_2\,dR = \int_0^{0.5}(1+2\mathrm{R})\,dR + \int_{0.5}^{1}(3-2\mathrm{R})\,dR = 1.5 \quad (49)$$

$$N = \left(\pi^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * 1.5 = 1.5\pi^2 * \frac{E\,I_{so}}{L^2} \quad (50)$$

C. Example 3 (Ex P3).

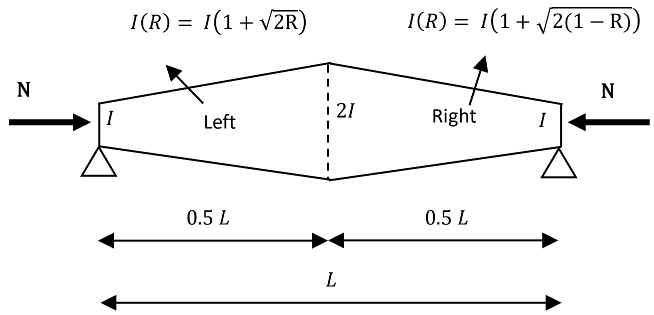

Fig 4:- Simply supported beam under buckling for Ex P3

$$\int_0^1 F_2\,dR = \int_0^{0.5}\left(1+\sqrt{2R}\right)dR + \int_{0.5}^{1}\left(1+\sqrt{2(1-R)}\right)dR = \frac{10}{6} \quad (51)$$

$$N = \left(\pi^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * \frac{10}{6} = \frac{10}{6}\pi^2 * \frac{E\,I_{so}}{L^2} \quad (52)$$

D. Example 4 (Ex P4).

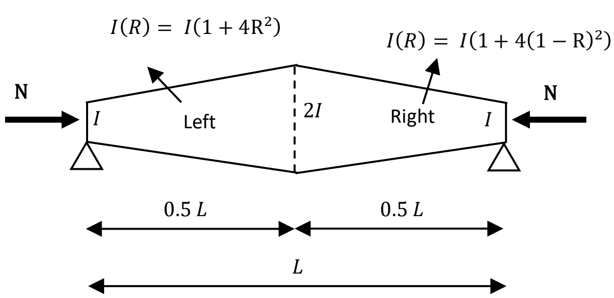

Fig 5:- Simply supported beam under buckling for Ex P4

$$\int_0^1 F_2\,dR = \int_0^{0.5}(1+4R^2)\,dR + \int_{0.5}^{1}(1+4(1-R)^2)\,dR = \frac{4}{3} \quad (53)$$

$$N = \left(\pi^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{L_e} * \frac{4}{3} = \frac{4}{3}\pi^2 * \frac{E\,I_{so}}{L^2} \quad (54)$$

Other example problems are

E. Example 5 (Ex P5).

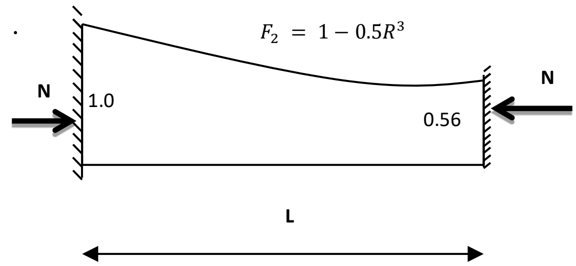

Fig 6:- Clamped supported beam under buckling for Ex P5

$$\int_{0.25}^{0.75} F_2\,dR = \int_{0.25}^{0.75}(1-0.5R^3)\,dR = 0.4609375 \quad (55)$$

$$N = \left(4\pi^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{0.5L} * \int_{0.25}^{0.75} F_2\,dR = \left(8\pi^2 * \frac{E\,I_{so}}{L^2}\right) * 0.4609375 = 3.6875\pi^2 * \frac{E\,I_{so}}{L^2} \quad (56)$$

F. Example 6 (Ex P6).

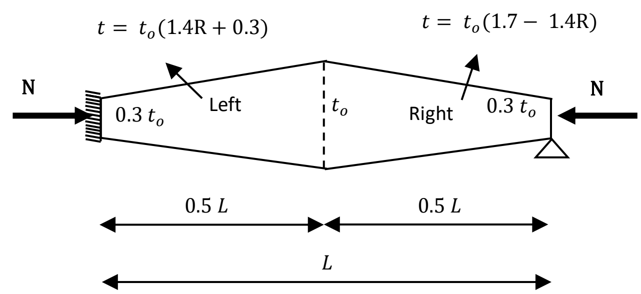

Fig 7:- Clamped supported beam under buckling for Ex P6

$$\int_{0.3}^{0.1} F_2\,dR = \int_{0.3}^{0.7}(1.4R+0.3)^3\,dR + \int_{0.7}^{1}(1.7-1.4R)^3\,dR = 0.3077074 \quad (57)$$

$$N = \left({g_1}^2 * \frac{E\,I_{so}}{L^2}\right) * \frac{L}{0.7L} * 0.3077074 = 0.439582{g_1}^2 * \frac{E\,I_{so}}{L^2} \quad (58)$$

G. Example 7 (Ex P7).

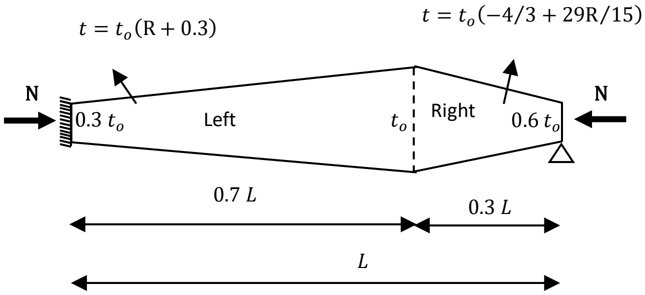

Fig 8:- Clamped supported beam under buckling for Ex P7

$$\int_{0.3}^{1} F_2\, dR = \int_{0.3}^{0.7} (\mathrm{R} + 0.3)^3\, dR + \int_{0.7}^{1} (-4/3 + 29\mathrm{R}/15)^3\, dR = 0.2343586 \quad (59)$$

$$N = \left(g_1{}^2 * \frac{E\, I_{so}}{L^2}\right) * \frac{L}{0.7L} * 0.2343586 = 0.334798 g_1{}^2 * \frac{E\, I_{so}}{L^2} \quad (60)$$

H. Example 8 (Ex P8).

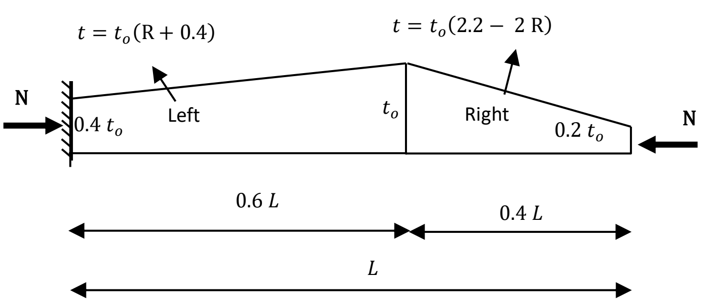

Fig 9:- Cantilever beam under buckling for Ex P8

$$\int_{-}^{1} F_2\, dR = \left(\int_{-0.6}^{0} \mathrm{F_{2L}\, dR} + \int_{0}^{0.6} \mathrm{F_{2L}\, dR}\right) + \left(\int_{-1}^{-0.6} F_{2R}\, dR + \int_{0.6}^{1} F_{2R}\, dR\right) = 2\int_{0}^{0.6} (\mathrm{R} + 0.4)^3\, dR + 2\int_{0.6}^{1} (2.2 - 2\,\mathrm{R})^3\, dR = 0.7368 \quad (61)$$

$$N = \left(\frac{\pi^2}{4} * \frac{E\, I_{so}}{L^2}\right) * \frac{L}{2\mathrm{L}} * 0.7368 = 0.0921\pi^2 * \frac{E\, I_{so}}{L^2} \quad (62)$$

V. RESULTS AND DISCUSSIONS

The results from the present study obtained using the closed form formula are compared with results from [12] obtained using numerical methods. The summary of the results is presented on Table 2, which contains the non-dimensional buckling loads. The method used by [12] in solving problems Ex p1, Ex P2, Ex P3 and Ex P4 is an approximate method called weighted moment of inertia. It is expected that his results should be close to exact results. However, the closeness of his results and the results from the closed form formula is an indication that the results from the present study ought to be desired exact results. This fit is assumed since the non-linear governing equation was actually integrated without circumventing it (integration), the outcome of the integration gave the closed form formula used in getting the results presented on Table 2. The authors are recommending for future studies to consider using the approach used herein for bending analyses of non-prismatic beams and vibration analyses of non-prismatic beams.

Example Problem	**Ex P1.**	**Ex P2.**	**Ex P3.**	**Ex P4.**	**Ex P5.**	**Ex P6.**	**Ex P7.**	**Ex P8.**
Present Result	16.153	14.804	16.449	13.159	36.394	8.8755	6.760	0.909
Manicka Results	17.866	16.106	17.757	14.43	-	-	-	-
Percentage difference	9.59	8.08	7.36	8.80	-	-	-	-

Table 2:- Non-dimensional buckling loads from present study and from [12].

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