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A Numerical Statistical Solution to the Laplace and Poisson Partial Differential Equations

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Abstract:- Current work on the numerical solution of Laplace partial differential equation LPDE and of the Poisson partial differential equation PPDE is based on the replacement of the LPDE by an approximately equivalent system of n linear algebraic equations. The solution of this system has two distinct main approaches, namely direct methods and indirect or iterative techniques. In this article, we present a new statistical method based on the chain of recurrence relations of the so-called B matrix. Matrix B presents a chain recurrence relation where an algorithm for numerical calculations is simple.

The correctness and precision of the numerical results are remarkable and superior to conventional methods. The proposed stochastic matrix B and its series of summations E are well defined and proven capable of handling the diversity of situations in different domains of LPDE and PPDE in 2D and 3D configurations such as the study of electrostatic voltage in the Poisson problem with Dirichlet boundary conditions. In this article, we explain the underlying theory and discuss in detail some 2D and 3D applications of the new numerical method .

I. INTRODUCTION

We can find an efficient spatiotemporal statistical solution to the partial differential equation of Laplace and to the partial differential equation of Poisson [1,2] in addition to Heat diffusion equation all expressed as energy density distribution function U(x,t),

d U / d t) partial = Nabla ^{2} U + S(x,t).. . . . . (1)

with boundary conditions of Dirichlet or Neumann BC and initial conditions IC given by U (x, 0).

S (x, t) is the source / sink term of energy density .

In current digital methods, the most common procedure for numerically solving LPDE and PPDE,

(Nabla ^{2}) U = S

with the boundary conditions of Dirichlet / Neumann in Cartesian coordinates ,

The main current digital methods follow the following procedure:

i-discretize the finite region into a grid of nxm free nodes for the 2D domains or nxmxl for the 3D domain then,

ii-apply the finite difference approximation method to obtain an expression for partial derivatives and,

iii-Write the discrete linear system equations for all the nodes in a matrix format and solve the system of linear algebraic equations.

A U = b…. (2)

A is the mathematical transfer matrix resulting from a finite difference method.

b is the vector of the boundary conditions obtained by arranging and writing BC in the correct order.

In the proposed statistical method, we completely neglect PDE (1) as if it did not exist or never occurred. We can only try it for comparison with the numerical results of the proposed unconventional method.

The proposed numerical method is based on the statistical assumption of the replacement of Eq. 1 by the stochastic recurrence formula,

$U_{i,j,k}{}^{(N+1)} = B\,(U^{N} + b + S)$. . . (3)

B is the stochastic transition matrix of the model which generates the transfer matrix E.

b is the boundary condition value vector.

S is the source vector in PPDE while S = 0 in LPDE.

Eq. 3 is a rigorous physical hypothesis supposed to describe the nature of the diffusion process and means that the boundaries of the Laplace system act as a source / sink term from the first instant t = 0.

Note that the proposed method does not require solving the system of linear algebraic equations (2) but rather reduce it to a summation solution of the power matrix to numerically solve the discrete PPDE itself with a minimum number of operations.

In other words, bypassing the complexities exposed to solve the non-homogenius linear system of algebraic equations in matrix algebra

II. THEORY

We explain below the proposed procedure in 3 consecutive precise steps later followed by 3 illustrative applications in 2D and 3D configuration space.

First Step

Discretize the 2D or 3D domain and find the appropriate stochastic transition matrix B satisfying conditions i-iv below, and therefore hypothesis 3.

$U^{(N+1)} = B(U^N + b + S)$. . . (3)

Again,
B is the stochastic transition matrix of the model which generates the summation matrix E.

b is the boundary conditions value vector

S is the source vector in PPDE ,(S = 0 in LPDE).

$U^N(x, t)$ is the spatiotemporal solution of the situation described by Poisson PDE (1).
N represents the number of steps dt or iterations N.

The statistical transition matrix B = ($B_{i,j}$) itself is well defined by statistical assumptions.
For 2D Cartesian coordinates, the entries $B_{i,j}$ comply with or are subject to the following conditions:
i- B i, j = 1/4 for i adjacent to j .. and B i, j = 0 otherwise. probability aperiori equal.

ii- $B_{i,i}$ = RO, i.e. the main diagonal is made up of constant inputs RO

For the heat diffusion equation, RO can take any value in the closed interval [0,1] while for Laplace and Poisson PDE, RO = 0

That is to say that B is a null principal diagonal matrix which corresponds to the assumption of a null residue after each time step for all the free nodes

iii- $B_{i,j} = B_{j,i}$, for all i, j.

The matrix B is symmetrical to conform to the physical principle of detailed balance.

iv- The sumof $B_{i,j}$ = 1 for all the rows far from the borders and the sum $B_{i,j}$ <1 for all the rows connected to the borders meaning that the probability of the whole space = 1.

Obviously, the statistical matrix B is very different from the Laplacian mathematical matrix A and from the Markov transition matrix.

The physical nature of B is clear and briefly explained above through conditions i to iv which support hypothesis 3.

Second Step

Define b which is the boundary conditions vector by arranging BC in the proper order.

Compute the source /sink term vector in energy density J/m^3 rather than voltage in volts or temperature in degrees Kelvin (for the case of heat diffusion equation).

Third Step

Compute the transfer matrix E and the PPDE solution. The solution U(x,t) follows from the successive application of the recurrence formula:

$U^{(N+1)} = B(U^{N} + b + S)$… (3)

where N = 0,1,2, ... N.

That is to say the solution U(x,t) at iteration N or at time = N dt is given by:

$U^{N} = (B^{0} + B + B^{2} + \ldots + B^{N})(b + S)$ . . . . . . (4)

$B^{0} = I$, I= unit matrix (n x n)

Expressed in power series of matrix B ,

Define the statistical matrix E by the series of powers of the matrix B,

$E^{N} = B^{0} + B + B^{2} + \ldots + B^{N}$ . . . . (5)

Obviously, all the entries of the term matrix B^{N} converge towards zero as N tends towards infinity which is a necessary condition for the convergence of the matrix E.

At the limit where N tends to an infinitely large number, we arrive at the required steady state solution:

$U = E(b + S)$ ………… (6)

In fact, it is not complicated to calculate the matrix E. The infinite series (5) can be evaluated in two distinct equivalent ways, either i- by summing the series by matrix multiplication and adding for a large number N, or ii- by evaluating the of infinite power series using the formula:

$E_{infinite} = (I-B)^{-1}$. . . . . (7)

The simplicity and precision of the numerical statistical method are quite surprising, it suffices first to calculate the matrix B and the BC vector b according to the geometry of the field and the configuration of BC of the problem then to calculate the matrix E by the sum of the power series of B Eq. (5) or the use of equation (7).

In order not to worry too much about the details of the theory, let's go right into 2D and 3D illustrative applications.

III. APPLICATIONS

A.-2D CONFIGURATION SPACE

Consider the simple case of a rectangular domain with 9 equidistant free nodes, u1, u2, u3, ... u9 and 12 Dirichlet boundary conditions BC1 to BC12 as illustrated in Figure 1.

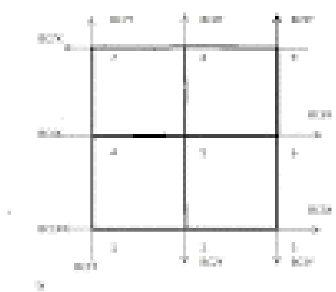

Fig.1 A 2D rectangular domain with 9 equidistant free nodes.

The 12 boundary conditions can be reduced to 9 BC for the 9 free nodes as follows,

BC1 = BC1X + BC1Y

BC2 = BC2X + BC2Y

. . . .

BC9 = BC9X + BC9Y

The transition matrix B 9x9 is built to satisfy the conditions i-iv for RO = 0 and is given by,

1- 0.000 0.250 0.000 0.250 0.000 0.000 0.000 0.000 0.000
2- 0.250 0.000 0.250 0.000 0.250 0.000 0.000 0.000 0.000
3- 0.000 0.250 0.000 0.000 0.250 0.000 0.000 0.000 0.000
4- 0.250 0.000 0.000 0.000 0.250 0.000 0.250 0.000 0.000
5- 0.000 0.250 0.000 0.250 0.000 0.250 0.000 0.250 0.000
6- 0.000 0.000 0.250 0.000 0.250 0.000 0.000 0.000 0.250
7- 0.000 0.000 0.000 0.250 0.000 0.000 0.000 0.250 0.000

8- 0.000 0.000 0.000 0.250 0.000 0.250 0.000 0.250 0.000

9- 0.000 0.000 0.000 0.000 0.000 0.250 0.000 0.250 0.000

And the matrix E calculated by equation (5) converges quickly for large N. Here are the digital inputs of the matrix E for N = 30.

1	1.1964247567313038	0.39284951346260755	0.12499618530273438	0.39284951346260755	0.24999237060546875	0.10713522774832995	0.12499618530273438	0.10713522774832995	5.3567613874164977E-002
2	0.39284951346260755	1.3214209420340381	0.39284951346260755	0.24999237060546875	0.49998474121093750	0.24999237060546875	0.10713522774832995	0.17856379917689935	0.10713522774832995
3	0.12499618530273438	0.39284951346260755	1.1964247567313038	0.10713522774832995	0.24999237060546875	0.39284951346260755	5.3567613874164977E-002	0.10713522774832995	0.12499618530273438
4	0.39284951346260755	0.24999237060546875	0.10713522774832995	1.3214209420340381	0.49998474121093750	0.17856379917689935	0.39284951346260755	0.24999237060546875	0.10713522774832995
5	0.24999237060546875	0.49998474121093750	0.24999237060546875	0.49998474121093750	1.4999847412109375	0.49998474121093750	0.24999237060546875	0.49998474121093750	0.24999237060546875
6	0.10713522774832995	0.24999237060546875	0.39284951346260755	0.17856379917689935	0.49998474121093750	1.3214209420340381	0.10713522774832995	0.24999237060546875	0.39284951346260755
7	0.12499618530273438	0.10713522774832995	5.3567613874164977E-002	0.39284951346260755	0.24999237060546875	0.10713522774832995	1.1964247567313038	0.39284951346260755	0.12499618530273438
8	0.10713522774832995	0.17856379917689935	0.10713522774832995	0.24999237060546875	0.49998474121093750	0.24999237060546875	0.39284951346260755	1.3214209420340381	0.39284951346260755
9	5.3567613874164977E-002	0.10713522774832995	0.12499618530273438	0.10713522774832995	0.24999237060546875	0.39284951346260755	0.12499618530273438	0.39284951346260755	1.1964247567313038

Note that the general relationship between matrix B and matrix E,

$E^{-1} = I-B$. . . . . (7) holds.

Now it suffices to multiply the matrix E by any BC arbitrary vector b to obtain the solution required for the electrostatic voltage distribution.

Mathews [3] classically solved the system resulting from 9 linear algebraic equations using Gaussian elimination method in a more efficient scheme by extending the tridiagonal algorithm to the more soffesticulated pentadiagonal algorithm for his arbitrary chosen BC vector,

b = [100,20,20,80,0,0,260,180,180] T.. . . . . (8)

and arrived at the solution vector:

U=[55,7143,43,2143,27,1429,79,6429,70,0000,45,3571,112,357,111,786,84,2857] T.. . . (9)

Now, vector BC for the proposed statistical solution corresponding to Ref. 3 Eq. (8) is simply rewritten,

[100/4, 20/4, 20/4, 20/4, 80/4, 0, 0.260 / 4, 180/4, 180/4] T.. . . . . (10)

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The calculated transfer matrix E can be multiplied by the vector BC (b) of Eq. 10, we get,

U=[55,7132187 43,2126846 27,1417885 79,6412506 69,9978638 45,3555412 112,856079 111,784111 84,2846451]T. . . . ..(11)

If we compare the proposed statistical solution (11) with that of Mathews, we find a striking precision.

B. 3D CONFIGURATION SPACE, 8 FREE NODES

Consider the simplest of the 3 D configuration, a rectangle with 8 free nodes u1, u2, ..u8 and 24 Dirichlet BC boundary conditions as shown in figure 2.

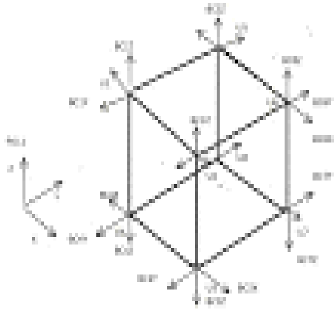

Fig. 2 A 3 D, rectangular configuration with 8 free nodes.

The vector of the boundary conditions is reduced to b = (BC1, BC2,…, BC8) where,

BC1 = BC1X + BC1Y + BC1Z

BC2 = BC2X + BC2Y + BC2Z

. . . .

BC8 = BC8X + BC8Y + BC8Z

The statistical transition matrix B is constructed according to the four conditions i-iv except that B i, j = 1/6 instead of 1/4 for i adjacent to j .. and B i, j = 0 otherwise.

The digital inputs of the matrix E 8x8, obviously independent of BC, were calculated for 22 series terms ,(Eq. 5) with N = 22, and the results obtained are as follows:

1- 1.1047618332254650 0.20952377941600242 0.20952377941600242 7.6190402902389667E-002 0.20952377941600242 7.6190402902389667E-002 7.6190402902389667E-002 3.8095202586520306E-002

2- 0.20952377941600242 1.1047618332254650 7.6190402902389667E-002 0.20952377941600242 7.6190402902389667E-002 0.20952377941600242 3.8095202586520306E-002 7.6190402902389667E-002

3- 0.20952377941600242 7.6190402902389667E-002 1.1047618332254650 0.20952377941600242 7.6190402902389667E-002 3.8095202586520306E-002 0.20952377941600242 7.6190402902389667E-002

4- 7.6190402902389667E-002 0.20952377941600242 0.20952377941600242 1.1047618332254650 3.8095202586520306E-002 7.6190402902389667E-002 7.6190402902389667E-002 0.20952377941600242

5- 0.20952377941600242 7.6190402902389667E-002 7.6190402902389667E-002 3.8095202586520306E-002 1.1047618332254650 0.20952377941600242 0.20952377941600242 7.6190402902389667E-002

6- 7.6190402902389667E-002 0.20952377941600242 3.8095202586520306E-002 7.6190402902389667E-002 0.20952377941600242 1.1047618332254650 7.6190402902389667E-002 0.20952377941600242

7- 7.6190402902389667E-002 3.8095202586520306E-002 0.20952377941600242 7.6190402902389667E-002 0.20952377941600242 7.6190402902389667E-002 1.1047618332254650 0.20952377941600242

8- 3.8095202586520306E-002 7.6190402902389667E-002 7.6190402902389667E-002 0.20952377941600242 7.6190402902389667E-002 0.20952377941600242 0.20952377941600242 1.1047618332254650

The matrix E has interesting statistical physical properties, the most important of which is the mirror symmetry of its main diagonal and the mirror symmetry of the left diagonal reflecting the geometric symmetry of the Laplace domain considered.

Obviously, we can arrive or return to the original stochastic transition matrix B by applying the formula,

E ^ -1 = I-B. . . . . . . (7)

As a digital test, choose arbitrary BC at 100 volts for half of the bottom face and zero for the other 5 faces Eq. 6 gives the following solution,

for b = (100 / 6,100 / 6,0,0,0,0,0,0) T, we obtain,
U = (65.7142792 65.7142792 14.2857084 14.2857094 14.2857084 14.2857094 5.71428061 5.71428013) T

In addition, the source vector S can take values other than zero to transmit the description of the PDE problem from Laplace to the Poisson situation. For example, let the source vector S change,

S = (0,0,10,30,0,0,0,0)

in units of voltage rather than free charge density divided by the Epsilon permittivity. and apply

$V^{N} = E^{N} (b + S)$. . . . . . . (8)

We obtain the numerical results of this Poisson problem after 22 iterations converging to the following vector solution,

V^{N} = [70.0952301 72.7618942 31.6190395 49.5238037 16.1904678 16.9523735 10.0952301 12.7618980] T

Here, as expected, there is a noticeable increase in V at nodes 3 and 4 where the free space charges are placed.

C. 3D CONFIGURATION SPACE, 27 FREE NODES

Let us consider the more complicated case of the 3 D, rectangular configuration with 27 free nodes u1, u2, ..u27 as well as their Dirichlet boundary conditions BC as shown in figure 3. The 27 free nodes with their corresponding 52 BC are classified and numbered in the appropriate order as shown in Fig.3.

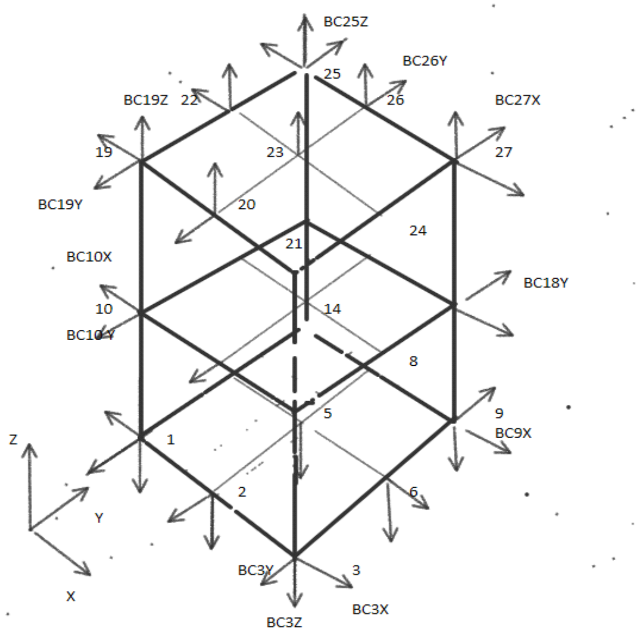

Fig.3. A 3 D rectanguloid with 27 free nodes and 52 BC.

For this 3D example with n = 27 equidistant free nodes as shown in figure 3, we have followed exactly the three consecutive steps described in applications B above to obtain the matrices B and E.

As a numerical test, suppose the rectangle is placed inside a larger of BC = 0 volts for 5 sides while the bottom is held at 100 volts.

We calculated the numerical values of the 729 inputs of the matrix E (27x27) and applied the Eq. 6 to obtain the solution vector for U.

The numerical results obtained are,

[43.224176413 51.468114187 43.224176413 51.468114187 61.93832292 51.468114187 43.224176413 51.468114187 43.224176413 13.79382080 17.907666783 13.79382080 17.907666783 23.38564376 17.907666783 13.79382080 17.90766678 13.79382080 4.09601968 5.53205655 4.09601968 5.53205655 7.491538148 5.53205655 4.09601968 5.53205655 4.09601968]T

It should be mentioned that the vector of intensity of the electric field **E = -Grad V** in 2D and 3D is essential in the description of the gas discharge physics [4,5]. However, the presented statistical method simplifies the calculations of E thanks to the precise calculation of V.

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The new statistical method can be applied to many situations in 2D and 3D gas discharge physics problems [4,5,6].It simplifies field calculations and can more powerfully replace many traditional methods currently used in gas discharge problems. [4,5,6].

Throughout this article, we have analyzed LPDE and PPDE in 2D and 3D Cartesian coordinates, but obviously the proposed statistical method can be extended to spherical and cylindrical coordinates.

IV. CONCLUSION

We propose and examine an unconventional technique to find a numerical statistical solution of the Laplace and Poisson partial differential equations (LPDE and PPDE).

We present a stochastic transition matrix B for the numerical STATISTICAL solution of the LPDE and PPDE equation with Dirichlet or Neumann BC boundary conditions and arbitrary initial conditions IC.

The resulting matrix E can be used to calculate the voltage distribution under transient and stationary equilibrium conditions for the boundary value problem of Poisson and Laplace PDEs..

The accuracy of the proposed matrix solutions has been tested in three different 2D and 3D applications by comparing them to existing solutions..

It is shown that the unconventional method presented has the advantage of directly solving the problem by bypassing the PDE itself and the associated systems of linear algebraic equations or any other conventional method. In addition, the stability, precision and rapid convergence of the solution are ensured.

We believe that this new technique is a small step in a long way, when it is developed and generalized through the use of supercomputers, many of the existing classic digital PDE solutions can become a thing of the past.

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