Derivations of BF- Algebras

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Abstract:- The concept of derivation in a BF- algebra has been introduced. In addition a left –right and a right-left derivation of BF, -algebras, left and right -derivation of ideal in BF- algebras are investigated. Different characterization of right-left-derivations, left-right - derivation, self map and fixed subalgebras have been discussed. We have also discussed derivation of BF-algebra if left and right -derivations are equal. In general different new theorems, Lemmas, Propositions and Corollaries have been proved.

Keywords:- BCI-algebra, B-Algebra, BG-algebra, Derivation of B-algebra, and BF-algebra.
Subject Classification Numbers: 06F35, 47L45, 08C05

I. INTRODUCTION


In this paper the derivation of BF-algebra with different properties and left and right -derivatives of ideals in BF-algebras have been introduced.

II. MATERIALS AND METHODS

In [7] a non-empty set A with a binary operation *, and a constant 0 is called a BCI-algebra, if it satisfies the following axioms:

1. \((a * b) * (a * c) * (c * b) = 0\)
2. \((a * (a * b)) * b = 0\)
3. \(a * a = 0\)
4. \(a * b = 0 \text{ and } b * a = 0 \Rightarrow a = b\).

Again in [5] a non-empty set A with binary operation *, and a constant 0 satisfying the following axioms:

1. \(a * a = 0\).
2. \(a * 0 = a\).
3. \((a * b) * c = a * (c * (0 * b)) \text{ for all } a, b, c \in A\) is said to be a B-algebra.

In [5] an algebra which satisfies conditions:
1. \((a * b) * (0 * b) = a\).
2. \(a * (b * c) = (a * (0 * c)) * b\).
3. \(a * b = 0 \text{ implies } a = b\).
4. \(0 * (0 * a) = a\).

is also called a B-algebra.

An algebra \((A, *, 0)\) is said to be aBH- algebra if it satisfies the following holds:
1. \(a * a = 0\).
2. \(a * 0 = a\).
3. \(a * b = 0 \text{ and } b * a = 0 \Rightarrow a = b\).

In addition a non-empty set A with a binary operation *, and a constant 0 is called a B-algebra if \(a, b \in A\) satisfies the following axioms:
1. \(a * a = 0\).
2. \(a * 0 = a\).
3. \(a = (a * b) * (0 * b)\).

Theorem 2.1. [5] If \((A, *, 0)\) is a B-algebra, then \((A, *, 0)\) is a B-algebra.

In [1] an algebra \((A, *, 0)\) of type(2,0) is called BF-algebra if it satisfies the following axioms for all \(a, b \in A\):
1. \(a * a = 0\).
2. \(a * 0 = a\).
3. \(0 * (a * b) = b * a\).

In [5] If a non-empty set A with a binary operation *, and a constant zero is called a B-algebra and \(a, b \in A\), then the following holds:
1. \(0 \ast (a \ast b) = b \ast a\).
2. \(a = (a \ast b) \ast (0 \ast b)\).
3. \(a \ast b = 0\) and \(b \ast a = 0 \Rightarrow a = b\).

Example 2.2. Let \((R, \ast, 0)\) be the algebra with the operation \(\ast\) defined by
\[
a \ast b = \begin{cases} 
a & \text{if } b = 0 \\
b & \text{if } a = 0 \\
0 & \text{otherwise}
\end{cases}
\]
Then \((R, \ast, 0)\) is a \(B\)-algebra. Where \(R\) is a real number.

In [1] a non-empty set \(A\) with a binary operation \(\ast\), and a constant \(0\) is said to be a \(BF\)-algebra and for all \(a, b \in A\), then the following holds:
1. \(0 \ast (0 \ast a) = a\)
2. \(0 \ast a = 0 \ast b \Rightarrow a = b\)
3. \(a \ast b = 0 \Rightarrow b \ast a = 0\)

In [1] a non-empty set \(A\) with a binary operation \(\ast\), and a constant \(0\) is called a \(BF\)-algebra if and only if for all \(a, b \in A\) the following holds:
1. \(a \ast a = 0\).
2. \(0 \ast (a \ast b) = b \ast a\).
3. \(a = (a \ast b) = b \ast a\).

Lemma 2.3. [1] Let \((A, \ast, 0)\) be a \(BG\) – algebra. Then the following holds for all \(a, b \in A\):
1. The right cancellation law holds in \(A\). That is \(a \ast b = c \ast b \Rightarrow a = c\).
2. \(0 \ast (0 \ast a) = a\).
3. If \(a \ast b = 0\), then \(a = b\).
4. If \(0 \ast a = a \ast b\), then \(a = b\).
5. \((a \ast (0 \ast a)) \ast a = a\).

Definition 2.4. [5] a non-empty set \(A\) with a binary operation \(\ast\), and a constant \(0\) is said to be 0-Commutative \(B\)-algebra if \(a \ast (0 \ast b) = b \ast (0 \ast a)\) for all \(a, b \in A\).

In [1] a \(BF\)-algebra \((A, \ast, 0)\) is 0-Commutative if \(a \ast (0 \ast b) = b \ast (0 \ast a)\).

Remark 2.5. If a \(BF\)-algebra is 0-Commutative, then for all \(a, b \in A\).
1. \(a \ast (a \ast b) = b\).
2. \(a \ast b = b \ast (b \ast a)\).

III. RESULTS

3.1. Derivation of \(BF\)-Algebras

Definition 3.1.1. If \((A, \ast, 0)\) be a \(BF\)-algebra, then we have the following:
1. By a left-right-derivation of \(A\) is a self-map \(d : A \rightarrow A\) satisfying the identity \(d(a \ast b) = (d(a) \ast b) \ast (a \ast d(b))\) for all \(a, b \in A\).
2. A right-left-derivation of \(A\) satisfying the identity \(d(a \ast b) = (a \ast d(b)) \ast (d(a) \ast b)\) for all \(a, b \in A\).
3. If \(d\) satisfy both a left-right and a right-left-derivation, then \(d\) is called a derivation of \(A\).

Remark 3.1.2. If \((A, \ast, 0)\) be a \(BF\)-algebra, then \(a \ast b = (b \ast a)\) for all \(a, b \in A\).

Example 3.1.3. Let \(A = \{0, a, b, c\}\) be a set defined by the table below:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>*</td>
<td>0</td>
<td>a</td>
<td>b</td>
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<tr>
<td>0</td>
<td>0</td>
<td>b</td>
<td>a</td>
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<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

Then \((A, \ast, 0)\) is a \(BF\)-algebra.

Define \(d : A \rightarrow A\) by \(d(e) = \begin{cases} c & \text{if } e = 0 \\
0 & \text{if } e = a \\
a & \text{if } e = b \\
0 & \text{if } e = c \end{cases}\)

Now,
\[d(a \ast b) = (d(a) \ast b) \ast (a \ast d(b)) = (0 \ast b) \ast (a \ast 0) = 0 \ast 0\]

Hence \(d\) is a left-right-derivation.

Again
\[d(a \ast b) = (a \ast d(b)) \ast (d(a) \ast b) = (a \ast 0) \ast (0 \ast b) = 0 \ast 0\]

Hence \(d\) is a \((R, L)\)-derivation of \(A\). Therefor \(d\) is a derivation of \(A\).

Definition 3.1.4. A self-map of a \(BF\)-algebra \(A\) is called regular if \(d(0) = 0\).

Proposition 3.1.5. Let \(d\) be a \((L, R)\)-derivation of \(BF\)-algebra \(A\). Then
1. \(d(0) = d(a) \ast a\), for all \(a \in A\).
2. \(d\) is one-to-one.
3. If \(d\) is regular, then it is the identity map.
4. If there is an element \(a \in A\) such that \(d(a) = a\), then \(d\) is the identity map.
5. If there is an element a in A such that d(b) * a = 0 or a * d(b) = 0, for all b in A, that is d is constant.

Proposition 3.1.6. Let d be (R, L) - derivation of BF-algebra A. Then
1. d(0) = a * d(a) for all a in A.
2. d(a) = d(a) * a for all a in A.
3. d is one -to-one.
4. If d is regular, then it is the identity map.
5. If there is an element a in A such that d(a) = a, then d is the identity map.
6. If there is an element a in A such that d(b) * a = 0 or a * d(b) = 0, for all b in A, then d(b) = a, for all b in A. That is d is constant.

Proof.
1. Let a in A. Then a * a = 0 and 
   d(0) = d(a * a) = (a * d(a)) * (a * d(a))
   = (d(a) * a) * [(d(a) * a) * (a * d(a))]
   = [(d(a) * a) * (a * d(a))] * (d(a) * a)
   = [(d(a) * a) * (a * d(a))] * (a * d(a)) a)
   = 0 * (d(a) * a) = a * d(a).
Hence d(0) = a * d(a).

2. Let (A, *, 0) be a BF-algebra. Then a * 0 = a by definition of BF-algebra.
   So that d(a) = d(a * 0) = (a * d(0)) * (d(a) * 0)
   = (a * d(0)) * d(a) * 0
   = d(a) [(d(a) * a) * d(0)]
   = d(a) [(d(a) * a) * d(a)] by (I)
   ⇒ d(a) * 0 = d(a) [d(a) * (a * d(a))].
We have d(a) [d(a) * (a * d(a))] = 0 implies d(a) = (a * d(a)) = d(a) * a.
   Hence d(a) = d(a) * a.

3. Let a, b in A such that d(a) = d(b), then by (I)
   d(0) = a * d(a).
   Also by (I) d(0) = b * d(b). Thus a * d(a) = b * d(b),
   But d(a) = d(b).
   We get a * d(a) = b * d(b).
   Hence a = b.
   Therefore d is one-to-one.

4. Let d be regular, and ain A. Then d(0) = 0, so we have 
   0 = a * d(a) by (I).
   Hence d(a) = a for all a in A. That is d is the identity map.

5. Assume d(a) = a , for some a in A. Then 
   a * d(a) = 0 and d(0) = 0.
   Thus d is the identity map.

6. Assume d(a) * a = 0 or a * d(b) = 0 for all b in A.
   Then d(0) = 0 and d(b) * a = 0.
   Hence d(b) = a.

On the similar manner, a * d(b) = 0 and d(0) = 0, implies d(b) = a.
   Thus d is the identity map.

Example 3.1.7. Let Q be a rational numbers and “+”, “•” the operations on Q. Then (Q, , 0) is a BF-algebra. Since
1. a – a = 0.
2. a – 0 = a.
3. 0 – (a – b) = -(a – b) = b – a.
Let d : Q → Q defined by d(a) = a -1, for all a in Q. Then
   d(a – b) = (d(a) – b) ∩ (a – d(b)).
   = ((a -1) – b) ∩ (a – (b -1)).
   = (a- b -1) ∩ (a – b +1).
   = (a - b +1) – ((a - b +1) – (a - b 1).
   = (a- b +1) – (a - b +1 + a - b +1).
   = (a- b -1) – (a - b +1 + a - b +1).
   = d(a – b) for all a in Q.
   So d is a left-right-derivation of A. But
d(1 -0) = (1 -d(0)) ∩ (d(1)-0) = (1 -(0-1)) ∩ ((1-1) -0).
   = 2 ∩ 0 = 0 – (0 -2) = 2.
   0 = 1-1 = d(1) = d(1 -0).
   Hence 2 ≤ 0. Therefore d is not a right-left-derivation of Q.

In addition
d(1-0) = d(1) -d(0) ∩ 1 -d(d(0)).
   = ((1-1)-0) ∩ (1 – (0-1)).
   = (0-0) ∩ (1- (0-1)) = 0 ∩ 2.
   = 2 – (2-0) = 2 -2 = 0.
   Hence 2 ≤ 0. Thus d is not derivation of Q.

Definition 3.1.8. Let (A, *, 0) be a BF₂ - algebra. Then
1. A left-right-derivation of BF₂ - algebra is a self-map
   d : A → A satisfy d(a * b) = (d(a) * b) ∩ (a * d(b))
   for all a, b in A.
2. A right-left-derivation of A satisfying the identity
   d(a * b) = (a * d(b)) ∩ (d(a) * b) for all a, b in A.
3. If $d$ satisfies both a left-right and a right-left derivation, then $d$ is called a derivation of $A$.

Lemma 3.1.9. Let $(A, \ast, 0)$ be a $BF_2$-algebra and let $a$ in $A$. Then $a \ast a = 0$.

Proposition 3.1.10. Let $(A, \ast, 0)$ be a $BF_2$-algebra and let $d$ be a derivation of $BF_2$-algebra $A$. Then
1. $d(a) = d(a) \ast 0$.
2. $d(a \ast b) = d(0)$ if and only if $a = b$, for all $a, b$ in $A$.

Lemma 1.1.11. Let $(A, \ast, 0)$ be a $BF_2$-algebra and let $d$ be a derivation of $BF_2$-algebra. Then $d(a \ast b) = d(0)$ if and only if $d(a) = a$, $d(b) = b$, and $a \ast b = 0$.

Proof. Let $(A, \ast, 0)$ be a $BF_2$-algebra and $d : A \rightarrow A$ be a derivation of $BF_2$-algebra. Then assume $d(a \ast b) = d(0)$, $a, b, 0$ in $A$.

Now, $d(a \ast b) = (d(a) \ast b) (a \ast d(b))$.

$= (a \ast d(b)) (a \ast d(b)) (a \ast d(b))(a \ast d(b))$

$= [a \ast d(b)] (0 \ast d(b)) (a) (b \ast d(a))$

$= 0 \ast (b \ast d(a)) = d(a) \ast b$ by $(L, R)$ derivation.

The rest of the proof follow trivially.

Proposition 3.1.12. Let $A$ be a $BF_2$-algebra and let $d$ be derivation of $A$. Then the following holds.
1. If $d$ is regular, then $d(a \ast b) = 0$ for some $a, b$ in $A$.
2. If $d$ is one-to-one.
3. If $d$ is regular, then $d$ is the identity map.
4. If there is an element $a$ in $A$ such that $d(a) = a$, then $d$ is the identity map.

Definition 3.1.13. Let $A$ be a $BF_2$-algebra and let $d$ be a derivation of $A$. Then the fixed derivation of $A$ is defined by $Fix_d(a) = \{a \in A : d(a) = a\}$.

Proposition 3.1.14. Let $A$ be a $BF_2$-algebra and let $d$ be a derivation of $A$. Then $Fix_d(a)$ is a subalgebra of $A$.

Definition 3.1.15. Let $A$ be a $BF$-algebra and $d : A \rightarrow A$ be a self-map. Then $d$ is called a left derivation of $BF$-ideal of $A$ if it satisfies the following conditions:
1. $0 \in P$.
2. $d(a \ast b) \in P$ and $d(a) \in P$ implies $d(b) \in P$, for any $a, b$ in $A$. Similarly $d$ is called a right derivation of $BF$-algebra if it satisfies the following conditions:
1. $0 \in P$.
2. $d(a \ast b) \in P$ and $d(b) \in P$ implies $d(a) \in P$, for any $a, b$ in $A$.

Example 3.1.16. Let $A = \{0, a, b, c\}$ and $\ast$ be defined by the table below:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
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<td>a</td>
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<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence $(A, \ast, 0)$ is a $BF$-algebra.

Let $P = \{0, a, c\}$ be ideals of $A$. Define $d : A \rightarrow A$ by $d(a) = \begin{cases} 0, & \text{if } a = 0, a, b, c \\ a, & \text{if } a = c \end{cases}$, we have
1. $0 \in P$.
2. $d(a \ast c) = d(0) = 0 \in P$ and $d(a) = 0 \in P$, implies $d(c) = a \in P$.

Hence $d$ is a left derivation of ideal of $A$.

IV. DISCUSSION

In this paper we introduced derivation of BF-algebra which is important for the growth of the theory towards applications in algebraic coding theory which become new area of research.

V. CONCLUSIONS

In this paper we introduced the concepts of derivations in BF-algebra, the left-right and right-left derivation of $BF_2$-algebra has been introduced. In addition, left-right derivation of ideals of $BF_2$-algebra has been investigated. Finally, different characterization Theorems, Lemmas and corollaries have been proved.

ACKNOWLEDGMENTS

The authors would like to thank the referee’s for their valuable comments.

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IJSRT20OCT038 www.ijsr.com 332
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