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Fast Way to Solve Ricatti Equation

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Abstract:- In mathematics a Ricatti equation in the narrowest sense is any first- order ordinary differential equation that is quadratic in the unknown function.

There are many methods to solve but all of them supposed arbitrary assumptions like $\frac{1}{u}$ *and* uIt but all of them elongate the solutions and complicate it .

My new way to solve ricatti differential equation is direct by using Asultani differential equation 2 directly and without needing to any substitution so decreasing the effort , time and papers ,

Keywords:- *Bernoulli differential equation, Ricatti equation, Alsultani differential equation 2, Alsultani differential equation [3].*

I. INTRODUCTION

First I will explain the old way to solve Ricatti equation

Ricatti equation denotes that

$$y' = A(x)y^2 + B(x)y + C(x)$$

where A(x) ,B(x) and C(x) are functions of (x) but the above equation cannot be solved without a particular solution $y_1=f(x)$.

Here we must take a point (y) on the curve

$y=y_1+\frac{1}{u}$ where (u) also a function of (x) then it's derivative will be

$$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{u^2}\frac{du}{dx}$$

$$A(x)y^2 + B(x)y + C(x) = A(x)y_1^2 + B(x)y_1 + C(x) - \frac{1}{u^2}\frac{du}{dx}$$

$$A(y_1+\frac{1}{u})^2+B(y_1+\frac{1}{u})+C(x)= A(x)y_1^2+By_1+C - \frac{1}{u^2}\frac{du}{dx}$$

$$(Ay_1^2 + \frac{2Ay_1}{u}+\frac{A}{u^2}) + B\left(y_1 + \frac{1}{u}\right) + C = Ay_1^2 + By_1 + C - \frac{1}{u^2}\frac{du}{dx}$$

$$2Ay_1u+A+Bu=-\frac{du}{dx}$$

$$\frac{du}{dx}+(2Ay+B)u= - A$$

$\frac{du}{dx}+P(x)u=Q(x)$ so the integrating factor=Ix= $e^{\int P(x)dx}$

$$\frac{du}{dx}\, e^{\int P(x)dx} +(p(x)\, e^{\int P(x)dx}u=Q(x)\, e^{\int P(x)dx}$$

$$D(u\, e^{\int P(x)dx})=Q(x)\, e^{\int P(x)dx}$$

$$.u=\frac{\int Q(x)\, e^{\int P(x)dx}dx}{e^{\int P(x)dx}}$$

As it is seen that because we choose the variable $(\frac{1}{u})$ so we must do this substitution and the procedure of the derivation through the solution of any problem of ricatti equation which leads to more time and may be more mistakes and more paper ,

I find by my new way that C(x) which is a function of (x) only is a part of the direvative of the particular solution (y_1) then if it exists or equal zero it will not inter in the integral processes and here we canconcider that : *Riccatti equation* is a first order linear D.E. consists of three terms [3].

Then to solve it by using Alsultani D.E. which denotes :

$$\frac{dy}{dx} + \frac{P(x)}{n}y = Q(x)y^k \text{ where } k = 1 - n$$

from Ay^2 we find that k=2 = 1-n then n= -1 i. e. (y^{-1}) is the primary solution in addition to

$$y_1 = f(x) \text{ as a particular solution}$$

so let (y_2) is a new point exits on the curve and

$$y_2 = y_1 + \frac{1}{y}$$

then it's derivative is :

$$y_2' = y_1' - \frac{1}{y^2}\frac{dy}{dx} = A(y_1 + \frac{1}{y})^2 + B\left(y_1 + \frac{1}{y}\right) + C(x)$$

$$.y_1' - \frac{1}{y^2}\frac{dy}{dx} = A(y_1^2 + \frac{2Ay_1}{y}+\frac{1}{y^2}) + By_1 + \frac{B}{y} + C(x)$$

but $y_1'= Ay_1^2+ By_1 + C(x)$

$$.i.e. - \frac{1}{y^2}\frac{dy}{dx} = \frac{A}{y^2} + \frac{2Ay_1+B}{y}$$

Now multiplying the above eq. by ($-y^2$) *we get*

$$\frac{dy}{dx} = -(2Ay_1 + B)y - A$$

(anyone can solve ricattiequation problem directly at once he knows A , B and the particular solution to save the time , papers and less mistakes)

$\frac{dy}{dx} + (2Ay_1 + B)y = -A$ I called this equation as :

$\boxed{\text{Alsultani D.E. 2}}$ so $P(x) = (2Ay_1 + B)$

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here we see that there is no need to the complicated substitutions.

So $\boxed{\frac{dy}{dx} + P(x)y = -A}$ $linear\ ODE.\ in\ (y)$

which is the fast way to solve Riccati D. E..

First find a particular solution y_1 then it is easy to transform it to a first order linear ODE

Notes :

- If we use the solution ($y_1 + u$) *then Bernoulli D.E. will apear* and we must use another substitution with it's complications.
- It is clear that we need the particular solution (y_1) three times the first in the question , second in the differential equation and in (y_2) and this proofs that any problem hasn't (y_1) is not Ricattie quation but similar to it and we can solve it by another procedure like the separation for example .

Main results:

I will solve some problems to proof the exactness of the new way .

Example 1

Solve $y' = (y-x)^2$, the particular solution is $x+1$

Solution

$$y' = (y-x)^2 = y^2 - 2xy + x^2$$

A=1 , B= -2x and C=x^2

At the beginning we see that the particular solution y_1= x+1 and this will cause that

$$y' = 1 \text{ and not zero .}$$

$$\frac{dy}{dx} + (2Ay_1 + B)y = -A \boxed{\text{Alsultani D. E. 2}}$$

$\frac{dy}{dx} + [2(1+x) - 2x\ ]$ y= -1

$\frac{dy}{dx} + 2y = -1$ then P(x) = 2

$$I_x = e^{\int 2dx} = e^{2x}$$

$$y\, e^{2x} = -\int e^{2x}dx + c \quad \text{and} \quad y\, e^{2x} = -\frac{1}{2}e^{2x} + c$$

So $y = \frac{-\frac{1}{2}e^{2x}+c}{e^{3x}} = \frac{-e^{2x}+2c}{2e^{2x}} = \frac{-1+N}{2}$ where

$$N = 2c$$

$$\frac{1}{y} = \frac{2}{-1+Ne^{-2x}}$$

Then y_2= x+1+$\frac{2}{-1+Ne^{-2x}}$ solution

Example 2

Solve $y' + y^2 = \frac{2}{x^2}$

solution

$$y' = -y^2 + \frac{2}{x^2}$$

A=-1 B=0 C= $\frac{2}{x^2}$

Let the particular solution is $\frac{c}{x}$ $so\ y' = \frac{-c}{x^2}$

Then $\frac{-c}{x^2} + (\frac{c}{x})^2 - \frac{2}{x^2} = 0$ so $c^2 - c - 2 = 0$

(c-2)(c+1)=0 and c=-1 or c=2 we will take c=2

Then by using $\boxed{Alsultani\ D.E.2}$ we get

$$\frac{dy}{dx} + (2Ay_1 + B)y = -A$$

$$\frac{dy}{dx} + \left[2\left(\frac{-2}{x}\right) + 0\right]y = -(-1)$$

$$\frac{dy}{dx} - \frac{4}{x}y = 1$$

Then $I_x = e^{\int \frac{-4dx}{x}} = e^{-4lnx} = \frac{1}{x^4}$

So $\frac{y}{x^4} = \int \frac{dx}{x^4} + c$ then $\frac{y}{x^4} = \frac{-1}{3x^3} + c$

.y= $\frac{-x}{3} + cx^4 = \frac{x(3cx^3 - 1)}{3}$

$$\frac{1}{y} = \frac{3}{x(3cx^3 - 1)}$$

$y_2 = y_1 + \frac{1}{y} = \frac{2}{x} + \frac{3}{x(3cx^3-1)} = \frac{6cx+1}{x(3cx^3-1)}$ let $3c = N$

.y= $\frac{2Nx+1}{x(Nx^3-1)}$

$$\frac{1}{y} = \frac{x(Nx^2 - 1)}{2Nx + 1}$$

y_2=$\frac{-2}{x} + \frac{x(Nx^2-1)}{2Nx+1}$ solution

Example 3

Solve $y' + 6y^2 = \frac{1}{x^2}$

Solution

here since B=0 so it not Riccati equation and there is no particular solution

$$y' = -6y^2 + \frac{1}{x^2} \quad \text{so} \quad k = 2 = 1 - n \quad \text{then} \quad n = -1$$

.y_2=y^{-1} and $y_2' = \frac{-1}{y^2}\frac{dy}{dx} = -6y^{-2} + \frac{1}{x^2}$ eq. (1)

now multiply eq, (1) by $(y)^2$

$$-\frac{dy}{dx} = -6 + \left(\frac{y}{x}\right)^2 \text{ and let } \frac{y}{x} = v$$

$$\text{so } -\frac{dy}{dx} = \frac{-vdx - xdv}{dx} = -v - \frac{xdv}{dx}$$

$$= -6 + v^2 \quad \text{then} \quad -\frac{xdv}{dx}$$

$$= v^2 + v - 6$$

$$\frac{xdv}{dx} = (v+3)(v-2) \quad \text{and}$$

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$-\frac{dx}{x} = \frac{dv}{(v+3)(v-2)}$

$$\text{let} \quad \frac{1}{(v+3)(v-2)} = \frac{a}{v+3} + \frac{b}{v-2}$$

$$\text{then} \quad a = -b = -\frac{1}{5}$$

$$\int -\frac{dx}{x} = \int -\frac{dv}{5(v+3)} + \int \frac{dv}{5(v-2)}$$

$$\frac{1}{5}\ln|v+3| - \frac{1}{5}\ln|v-2| = \ln|x| + \ln c \qquad (c > 0)$$

$$\ln\left|\frac{v+3}{v-2}\right| = \ln(c^5|x|^5)$$

$$\frac{v+3}{v-2} = \pm c^5 x^5$$

$N = \pm c^5$

$\frac{v+3}{v-2} = Nx^5$ then let $\frac{1}{y} = vx$ so $v = \frac{1}{xy}$

$\frac{\frac{1}{xy}+3}{\frac{1}{xy}-2} = \frac{1+3xy}{1-2xy} = Nx^5$ when N=0 then $y=\frac{1}{3x}$ so $y' = \frac{-1}{3x^2}$

$$\frac{-1}{3x^2} + 6\left(-\frac{1}{3x}\right)^2 = \frac{1}{X^2}$$

$$\frac{1}{3x^2} + 6 \cdot \frac{1}{9x^2} = \frac{1}{x^2} \quad \text{so} \quad \frac{1}{3x^2} + \frac{2}{3x^2} = \frac{1}{x^2}$$

$$\frac{1}{x^2} = \frac{1}{x^2}$$

Please compare the above solution with the old way and see the differences

Example 4

Solve $y' = x^3(y-x)^2 + \frac{y}{x}$, particular solution is (x)

Solution

1. by the new way

$y' = x^3(y^2 - 2xy + x^2) + \frac{y}{x} = x^3y^2 - 2x^4y + x^5 + \frac{y}{x}$

A= x^3 ; B = $\frac{1}{x} - 2x^4$; C = x^5 ; $y_1 = x$

$$\frac{dy}{dx} + (2Ay + B)y = -A$$

$\frac{dy}{dx} + \left[2x^3(x) + \frac{1}{x} - 2x^4\right] y = -x^3$

$\frac{dy}{dx} + \frac{1}{x} y = -x^3$

$$I_x = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$xy = \int -(x)x^3 dx = -\frac{x^5}{5} + c$$

$.y = \frac{5c - x^5}{5x}$ and $\frac{1}{y} = \frac{5x}{5c - x^5}$ then

$$y_2 = y_1 + \frac{1}{y} = x + \frac{5x}{N - x^5} = \frac{Nx - x^6 + 5x}{N - x^5}$$

$$= \frac{5x(c+1) - x^6}{N - x^5}$$

Where N=5c

Example 5

Solve

$x^2y' = x^2y^2 + xy - 3 \quad eq,$ (1) and $y_1 = \frac{1}{x}$

divide eq.(1) by (x^2) *we get*

$$y' = y^2 + \frac{y}{x} - \frac{3}{x^2}$$

A=1 , B=$\frac{1}{x}$ *and* $C = \frac{-3}{x^2}$

$$\frac{dy}{dx} + (2Ay_1 + B)y = -A$$

$\frac{dy}{dx} + \left(\frac{2}{x} + \frac{1}{x}\right) y = -1$, p(x)=$\frac{3}{\square}$

I(x)=$e^{\int \frac{3dx}{x}} = e^{3\ln x} = x^3$

,$yx^3 = \int -x^3 dx + c = -\frac{x^4}{4} + c$

.y=$\frac{-x^4}{4x^3} + cx^{-3} = \frac{-x}{4} + \frac{c}{x^3} = \frac{4c - x^4}{4x^3}$ and $\frac{1}{y} = \frac{4x^3}{4c - x^4}$

.$y_2 = \frac{1}{x} + \frac{4x^3}{4c - x^4} = \frac{4c - x^4 + 4x^4}{x(4c - x^4)} = \frac{3x^4 + N}{Nx - x^5}$ where N = 4c END

Example 6

Solve $x(1 - x^3)y' = x^2 + y - 2xy^2$

Solution Divide by $x(1 - x^3)$

$y' = \frac{x^2}{x(1-x^3)} + \frac{y}{x(1-x^3)} - \frac{2xy^2}{x(1-x^3)}$

$= \frac{x}{1-x^3} + \frac{y}{x - x^4} - \frac{2y^2}{1-x^3}$

A= $-\frac{2}{1-x^3}$, B=$\frac{1}{x-x^4}$ and C=$\frac{x}{1-x^3}$

$$\frac{dy}{dx} + (2Ay_1 + B)y = -A$$

$\frac{dy}{dx} + \left(\frac{-2(2)x^2}{1-x^3} + \frac{1}{x-x^4}\right)y = -\frac{-2}{1-x^3}$

$$\frac{dy}{dx} + \frac{1 - 4x^3}{x - x^4} y = \frac{2}{1 - x^3}$$

Let $u = x - x^4$ then $du = 1 - 4x^3$

Ix= $e^{\int \frac{du}{u}} = e^{\ln u}$=u= $x - x^4$

$(x - x^4)\frac{dy}{dx} + (x - x^4)\frac{1-4x^3}{x-x^4} y = \frac{x - x^4}{1 - x^3}$=x

$y(x - x^4) = \int x dx + c = \frac{x^2}{2}$+c

.y= $\frac{x^2 + 2c}{2(x - x^4)}$ so $\frac{1}{y} = \frac{2(x - x^4)}{x^2 + 2c}$

.$y_2 = x^2 + \frac{2(x - x^4)}{x^2 + N}$ where N=2c

II. CONCLUSIONS

As it is clear that $\boxed{Alsultani\ D.E.2}$ is the fast and direct way to solve Ricatti equation

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