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The Concept of Global Baseline Matrix and of Baseline Matrices Associated with Elements of M(r,c) Subsets of Complex Matrix Spaces of Order m by n, Where m≠n

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Abstract:- The present article introduces the concept of Global Baseline Matrix associated with M(r,c) subsets of Complex Matrix spaces of order 'm' by 'n', where m≠n. It then presents a mathematical scheme to define subspaces of the corresponding Matrix space using elements of M(r,c) subset that are involved in computation of the Global Baseline matrix. The article next introduces the concept of "Local Baseline Matrix", i.e. the Baseline matrix associated with an element of the M(r, c) subset and finally develops the concept of Fundamental subset associated with an element of M(r, c) subset.

Keywords:- *M(r,c) subsets of Complex Matrix spaces, Global Baseline Matrix of M(r,c) subsets, Baseline Matrix of elements of M(r,c) subsets, Spacer Matrices associated with Complex Matrix spaces, Discrete dynamical systems, Markov Matrix, Hadamard Product of matrices.*

Notations

- N denotes the set of all Natural numbers
- C denotes the set of all Complex numbers
- $M_{m\times n}(C)$ denotes the Complex Matrix space of Matrices of order m by n
- $R(A)$ denotes the Global Mass Factor associated with the matrix $A_{m\times n}$
- $\hat{r}$ is the numerical realization of the $R(A)$ Factor
- $C(A)$ denotes the Global Alignment Factor associated with the matrix $A_{m\times n}$
- $\hat{c}$ is the numerical realization of the $C(A)$ Factor
- $|c|$ denotes the modulus of the complex number c
- $c^{\bullet}$ denotes the complex conjugate of the complex number c
- $\{|e_1\rangle,|e_2\rangle,....,|e_m\rangle\}$ denotes the standard Orthonormal basis in C^m and $\{|f_1\rangle,|f_2\rangle,....,|f_n\rangle\}$ denotes the standard Orthonormal basis in C^n
- $|m\rangle=\begin{bmatrix}1\\1\\.\\.\\1\end{bmatrix}_{m\times 1}, |n\rangle=\begin{bmatrix}1\\1\\.\\.\\1\end{bmatrix}_{n\times 1}, |V\rangle=\begin{bmatrix}v_1\\v_2\\.\\.\\v_s\end{bmatrix}_{s\times 1}, \langle V|=[v_1^{\bullet} \quad v_2^{\bullet} \quad . \quad . \quad v_s^{\bullet}]_{1\times s}$
- $B=\left[b_{ij}\right]_{m\times n}, \langle V|B|W\rangle=\sum_{i=1}^{m}\sum_{j=1}^{n}b_{ij}v_i^{\bullet}w_j$
- $M(r=\hat{r},c=\hat{c})$ is a subset of the Complex Matrix space $M_{m\times n}(C)$, characterized by the numerical values of the Global Mass factor and Global Alignment factor, $\hat{r}$ and $\hat{c}$, respectively.
- $'s'$ denotes the Embedding Dimension associated with ordered pairs (m,n) and (n,m)
- $X_{m\times n}$ denotes the Spacer Matrix associated with the Matrix space $M_{n\times m}(C)$

- $\hat{\Gamma}(\ )$ denotes the complete transformation associated with the "Phase readjustment Algorithm" presented in [27]
- W^H denotes the Hermitian conjugate of the matrix W
- W^T denotes the Transpose of the matrix W
- $U_{m\times n} \circ V_{m\times n}$ denotes the Hadamard Product[11,20,31] of the matrices U and V of order m by n
- $B_{m\times n} > 0$ implies $B = \sum_{x=1}^{m}\sum_{y=1}^{n} b_{xy} |e_x\rangle\langle f_y|$, B is real-valued and $b_{xy} > 0$, $\forall$ $x = 1,2,....,m$ and $y = 1,2,....,n$
- $B_{m\times n} \geq 0$ implies $B = \sum_{x=1}^{m}\sum_{y=1}^{n} b_{xy} |e_x\rangle\langle f_y|$, B is real-valued and $b_{xy} \geq 0$, $\forall$ $x = 1,2,....,m$ and $y = 1,2,....,n$
- $(R_{v\times v})^{\lambda} = (R_{v\times v})(R_{v\times v})....(R_{v\times v})$ (ordinary matrix multiplication 'λ' times)

I. INTRODUCTION

M(r,c) subsets of complex matrix spaces $M_{m\times n}(C)$, where m≠n , are characterized by the Global mass factor 'r' and the Global alignment factor 'c'. In this article the concept of Global Baseline Matrix $\langle G\rangle_{m\times n}$ associated with a M(r,c) subset is presented. The mathematical framework used in computation of the Global Baseline Matrix is then utilized to formulate subspaces of the complex matrix space $M_{m\times n}(C)$ using elements of the M(r,c) subset to form appropriate spanning sets. The article introduces the concept of local baseline in context of the framework and defines the Baseline Matrices associated with elements of M(r,c) subset, this is followed by presenting the concept of the Fundamental subset $FS(A_{m\times n})$ associated with an element $A_{m\times n}$ of the M(r,c) subset.

The formulation of the Global Baseline matrix, spanning sets for the subspaces of $M_{m\times n}(C)$ and the Local Baseline matrices involves Markov type matrices [5, 9, 13, 30] and their powers, created from the singular values of the Spacer Matrix $X_{m\times n}$ corresponding to the Matrix space $M_{n\times m}(C)$ and accompanying Phase terms readjustment to ensure the compatibility criterion ($a_{ij} = 0 \mapsto r_{ij} = 0, c_{ij} = 1$)

The article presents numerical examples at appropriate places to illustrate the introduced concepts and concludes with a discussion on the presented framework and the features of the numerical illustrative examples.

II. MATHEMATICAL FRAMEWORK AND ASSOCIATED ANALYSIS

The following results, stated in [21], [22], [23], [24], [25], [26], [27], [28] provide the mathematical groundwork for the concepts introduced in this article:

- $A \in M_{m\times n}(C)$, $A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij} |e_i\rangle\langle f_j|$, $a_{ij} = r_{ij}c_{ij}$, such that: $a_{ij} \neq 0 \mapsto r_{ij} = |a_{ij}|, c_{ij} \in C, |c_{ij}| = 1$ and $a_{ij} = 0 \mapsto r_{ij} = 0, c_{ij} = 1$
- $R(A) = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij}$, $C(A) = \prod_{i=1}^{m}\prod_{j=1}^{n} c_{ij}$, we therefore have the following:

$R(A) \geq 0, C(A) \in C, |C(A)| = 1, \forall A \in M_{m\times n}(C)$

- $M(r = \hat{r}, c = \hat{c}) \subset M_{m\times n}(C), M(r = \hat{r}, c = \hat{c}) = \{A \in M_{m\times n}(C) \mid A \neq 0_{m\times n}, R(A) = \hat{r}, C(A) = \hat{c}\}$, where we have the condition: $\hat{r} > 0, \hat{c} \in C, |\hat{c}| = 1$
- $\Sigma_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} r_{ij} |e_i\rangle\langle f_j|$, $\Phi_{m\times n} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} |e_i\rangle\langle f_j|$, we have: $A_{m\times n} = \Sigma_{m\times n} \circ \Phi_{m\times n}$
- $s = \max(m,n) + |m - n|$, we have : $m \neq n \ \mapsto \ s^2 > m.n$

The Analytical Expressions of the Markov type matrices [5, 9, 13, 30] $P(X)$ and $Q(X)$ generated from the Spacer Matrix[24,25,26,28] $X_{m\times n}$ associated with the Matrix space $M_{n\times m}(C)$:

- case ($m<n$):

- rank($X_{m\times n}$) = m
- singular values (In non-increasing order) of $X_{m\times n}$: $\sigma_1=\dfrac{s}{\sqrt{m.n}}$, $\sigma_2=....=\sigma_m=1$
- $\sigma_T=\sum_{y=1}^{m}\sigma_y$
- $|p_X\rangle_{m\times 1}=\left[(\dfrac{\sigma_1}{\sigma_T})\quad(\dfrac{\sigma_2}{\sigma_T})\quad .\quad .\quad(\dfrac{\sigma_m}{\sigma_T})\right]^T$
- $|q_X\rangle_{n\times 1}=\left[(\dfrac{n\sigma_1}{\sigma_T[2n-m]})\quad(\dfrac{n\sigma_2}{\sigma_T[2n-m]})\quad .\quad(\dfrac{n\sigma_m}{\sigma_T[2n-m]})\quad\dfrac{1}{[2n-m]}\quad\dfrac{1}{[2n-m]}\quad .\quad\dfrac{1}{[2n-m]}\right]^T$

- case ($m>n$):

- rank($X_{m\times n}$) = n
- singular values (In non-increasing order) of $X_{m\times n}$: $\sigma_1=\dfrac{s}{\sqrt{m.n}}$, $\sigma_2=....=\sigma_n=1$
- $\sigma_T=\sum_{y=1}^{n}\sigma_y$
- $|p_X\rangle_{m\times 1}=\left[(\dfrac{m\sigma_1}{\sigma_T[2m-n]})\quad(\dfrac{m\sigma_2}{\sigma_T[2m-n]})\quad .\quad(\dfrac{m\sigma_n}{\sigma_T[2m-n]})\quad\dfrac{1}{[2m-n]}\quad\dfrac{1}{[2m-n]}\quad .\quad\dfrac{1}{[2m-n]}\right]^T$
- $|q_X\rangle_{n\times 1}=\left[(\dfrac{\sigma_1}{\sigma_T})\quad(\dfrac{\sigma_2}{\sigma_T})\quad .\quad .\quad(\dfrac{\sigma_n}{\sigma_T})\right]^T$

Therefore, we have the following for both cases ($m<n$ and $m>n$):

- $|p_X\rangle\geq 0_{m\times 1}$, $|q_X\rangle\geq 0_{n\times 1}$, $\langle m|p_X\rangle=\langle n|q_X\rangle=1$
- $[P(X)]_{m\times m}=|p_X\rangle\langle m|+(\dfrac{1}{m.n})[I_{m\times m}-|p_X\rangle\langle m|]$
- $[Q(X)]_{n\times n}=|q_X\rangle\langle n|+(\dfrac{1}{m.n})[I_{n\times n}-|q_X\rangle\langle n|]$

The Analytical Expression of the Global Baseline matrix $\langle G\rangle_{m\times n}$ associated with $M(r=\hat{r},c=\hat{c})$:

We consider $M(r=\hat{r},c=\hat{c})\subset M_{m\times n}(C)$, $\hat{r}>0$, $\hat{c}\in C$, $|\hat{c}|=1$

- $\Sigma(0)_{m\times n}=\sum_{x=1}^{m}\sum_{y=1}^{n}(\dfrac{\hat{r}}{m.n})|e_x\rangle\langle f_y|$, $\Phi(0)_{m\times n}=\sum_{x=1}^{m}\sum_{y=1}^{n}\exp(+\dfrac{i\varepsilon}{m.n})|e_x\rangle\langle f_y|$ here 'i' is the imaginary unit, i.e. $i^2=-1$, we have $\varepsilon\in[0,2\pi)$ such that : $\exp(+i\varepsilon)=\hat{c}$
- We define the matrices $S_{m\times m}$ and $T_{n\times n}$ as follows:

$$S_{m\times m} = diag[1, \exp(+i(\frac{2\pi}{n})), ..., \exp(+i(m-1)(\frac{2\pi}{n}))]$$

$$T_{n\times n} = diag[1, \exp(+i(\frac{2\pi}{m})), ..., \exp(+i(n-1)(\frac{2\pi}{m}))]$$

- $\langle\Sigma\rangle_{m\times n} = (\frac{1}{m.n})\sum_{\lambda=1}^{(m.n)}[P(X)_{m\times m}]^{\lambda}(\Sigma(0)_{m\times n})[Q(X)^{T}{}_{n\times n}]^{\lambda}$
- If $\langle\Sigma\rangle_{m\times n} > 0$, then $\langle\Phi\rangle_{m\times n} = [S_{m\times m}]^{(m.n)}(\Phi(0)_{m\times n})[T_{n\times n}]^{(m.n)}$
- If $\langle\Sigma\rangle_{m\times n} \geq 0$, then $\langle\Phi\rangle_{m\times n} = \hat{\Gamma}([S_{m\times m}]^{(m.n)}(\Phi(0)_{m\times n})[T_{n\times n}]^{(m.n)})$
- $\langle G\rangle_{m\times n} = \langle\Sigma\rangle_{m\times n} \circ \langle\Phi\rangle_{m\times n}$, clearly $\langle G\rangle_{m\times n} \in M(r=\hat{r}, c=\hat{c})$

Numerical Example: (Determination of the Global Baseline Matrix associated with $M(r=2, c=1)$, Where $M(r=2, c=1) \subset M_{2\times 3}(C)$

- $s=4$, $X_{2\times 3} = \begin{bmatrix} \frac{7}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{7}{6} & \frac{2}{3} \end{bmatrix}_{2\times 3}$, $\sigma_1 = \frac{4}{\sqrt{6}}$, $\sigma_2 = 1$
- $P(X) = \begin{bmatrix} 0.683503 & 0.516837 \\ 0.316497 & 0.483163 \end{bmatrix}_{2\times 2}$, $Q(X) = \begin{bmatrix} 0.554294 & 0.387628 & 0.387628 \\ 0.237372 & 0.404039 & 0.237372 \\ 0.208333 & 0.208333 & 0.375 \end{bmatrix}_{3\times 3}$ ...(upto 6 decimal places)
- $S_{2\times 2} = diag[1, (-1/2)+(\sqrt{3}/2)i]$, $T_{3\times 3} = diag[1,-1,1]$, here 'i' is the imaginary unit, i.e. $i^2 = -1$
- $\Sigma(0)_{2\times 3} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}_{2\times 3}$, $\Phi(0)_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$
- $\langle\Phi\rangle_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$, $\langle\Sigma\rangle_{2\times 3} = \begin{bmatrix} 0.567953 & 0.352993 & 0.311449 \\ 0.353565 & 0.219933 & 0.194107 \end{bmatrix}_{2\times 3}$ ...(upto 6 decimal places)
- $\langle G\rangle_{2\times 3} = \begin{bmatrix} 0.567953 & 0.352993 & 0.311449 \\ 0.353565 & 0.219933 & 0.194107 \end{bmatrix}_{2\times 3}$ .....(upto 6 decimal places)

Subspaces of $M_{m\times n}(C)$ generated from elements of $M(r=\hat{r}, c=\hat{c})$ Subset:

We have: $M(r=\hat{r}, c=\hat{c}) \subset M_{m\times n}(C)$, $\hat{r} > 0$, $\hat{c} \in C$, $|\hat{c}| = 1$

$t = 0,1,2,....$, $t \in \{0\} \cup N$, The Recursion Model is as follows:

- $\Sigma(t)_{m\times n} = [P(X)_{m\times m}]^{t}(\Sigma(0)_{m\times n})[Q(X)^{T}{}_{n\times n}]^{t}$, $t = 1,2,3,.....$

- If $\Sigma(t)_{m\times n} > 0$, Then $\Phi(t)_{m\times n} = [S_{m\times m}]^t (\Phi(0)_{m\times n})[T_{n\times n}]^t$,

If $\Sigma(t)_{m\times n} \geq 0$, Then $\Phi(t)_{m\times n} = \hat{\Gamma}([S_{m\times m}]^t (\Phi(0)_{m\times n})[T_{n\times n}]^t)$, $t = 1, 2, 3, ...$

- $G(t)_{m\times n} = \Sigma(t)_{m\times n} \circ \Phi(t)_{m\times n}$, $t = 0, 1, 2, ...,$ $t \in \{0\} \cup N$, clearly $G(t)_{m\times n} \in M(r = \hat{r}, c = \hat{c})$ $\forall t \in \{0\} \cup N$

- $V(t) = span(G(0)_{m\times n}, ..., G(t)_{m\times n})$, $V(t) \subseteq M_{m\times n}(C)$, $t = 0, 1, 2, ...,$ $t \in \{0\} \cup N$

Numerical Example:

We consider: $M(r = 2, c = 1) \subset M_{2\times 3}(C)$, then we have the following:

- $G(0)_{2\times 3} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}_{2\times 3}$, $V(0) = span(G(0)_{2\times 3})$

- $G(1)_{2\times 3} = \begin{bmatrix} 0.531971 & -0.351613 & 0.316756 \\ -0.177198 + 0.306916i & 0.117121 - 0.202860i & -0.105511 + 0.182750i \end{bmatrix}_{2\times 3}$ ... (upto 6 decimal places)

  $V(1) = span(G(0)_{2\times 3}, G(1)_{2\times 3})$

- $G(2)_{2\times 3} = \begin{bmatrix} 0.569356 & 0.353086 & 0.311288 \\ -0.176813 - 0.306250i & -0.109651 - 0.189921i & -0.096671 - 0.167438i \end{bmatrix}_{2\times 3}$ ....(upto 6 decimal places)

$V(2) = span(G(0)_{2\times 3}, G(1)_{2\times 3}, G(2)_{2\times 3})$ and so on.

The Analytical Expression of the Baseline Matrix $\bar{A}_{m\times n}$ associated with matrix $A_{m\times n} \in M(r = \hat{r}, c = \hat{c})$:

We have: $A_{m\times n} \in M(r = \hat{r}, c = \hat{c})$, $\hat{r} > 0$, $\hat{c} \in C$, $|\hat{c}| = 1$, $A_{m\times n} = \Sigma_{m\times n} \circ \Phi_{m\times n}$

- $\bar{\Sigma}_{m\times n} = [P(X)_{m\times m}](\Sigma_{m\times n})[Q(X)^T{}_{n\times n}]$

- If $\bar{\Sigma}_{m\times n} > 0$, Then $\bar{\Phi}_{m\times n} = S_{m\times m} \Phi_{m\times n} T_{n\times n}$

If $\bar{\Sigma}_{m\times n} \geq 0$, Then $\bar{\Phi}_{m\times n} = \hat{\Gamma}(S_{m\times m} \Phi_{m\times n} T_{n\times n})$

- $\bar{A}_{m\times n} = \bar{\Sigma}_{m\times n} \circ \bar{\Phi}_{m\times n}$, clearly $\bar{A}_{m\times n} \in M(r = \hat{r}, c = \hat{c})$

The Fundamental set $FS(A_{m\times n})$ associated with matrix $A_{m\times n}$, $A_{m\times n} \in M(r = \hat{r}, c = \hat{c})$:

We consider the following:

- $(\hat{\Sigma} \,|\, \lambda, \bar{\lambda}, \langle \lambda \rangle)_{m\times n} = \lambda(\Sigma_{m\times n}) + \bar{\lambda}(\bar{\Sigma}_{m\times n}) + \langle \lambda \rangle(\langle \Sigma \rangle_{m\times n})$, $\lambda \geq 0$, $\bar{\lambda} \geq 0$, $\langle \lambda \rangle \geq 0$, $\lambda + \bar{\lambda} + \langle \lambda \rangle = 1$

- If $(\hat{\Sigma} \,|\, \lambda, \bar{\lambda}, \langle \lambda \rangle)_{m\times n} > 0$, Then $\langle \hat{\Phi} \rangle_{m\times n} = \Phi_{m\times n}$

If $(\hat{\Sigma} \mid \lambda, \bar{\lambda}, \langle \lambda \rangle)_{m\times n} \geq 0$, Then $\langle \hat{\Phi} \rangle_{m\times n} = \hat{\Gamma}(\Phi_{m\times n})$

- $FS(A_{m\times n}) = \{\hat{A}_{m\times n} \in M_{m\times n}(C) \mid \hat{A}_{m\times n} = (\hat{\Sigma} \mid \lambda, \bar{\lambda}, \langle \lambda \rangle)_{m\times n} \circ \langle \hat{\Phi} \rangle_{m\times n}, \lambda \geq 0, \bar{\lambda} \geq 0, \langle \lambda \rangle \geq 0, \lambda + \bar{\lambda} + \langle \lambda \rangle = 1\}$ clearly , $FS(A_{m\times n}) \subseteq M(r = \hat{r}, c = \hat{c})$

Numerical Examples:

1. $A_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}$, $A_{2\times 3} \in M(r = 2, c = 1)$, We have the following:

$$\Sigma_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} , \quad \Phi_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$\bar{\Sigma}_{2\times 3} = \begin{bmatrix} 0.579202 & 0.371067 & 0.250071 \\ 0.362720 & 0.270344 & 0.166596 \end{bmatrix}_{2\times 3}$$ ... (upto 6 decimal places)

$$\bar{\Phi}_{2\times 3} = \begin{bmatrix} 1 & -1 & 1 \\ \theta & -\theta & \theta \end{bmatrix}_{2\times 3}$$, Where $\theta = \exp(+i\frac{2\pi}{3})$, here ‘i’ is the imaginary unit, i.e. $i^2 = -1$

$$\bar{A}_{2\times 3} = \begin{bmatrix} 0.579202 & -0.371067 & 0.250071 \\ -0.181360 + 0.314124i & 0.135172 - 0.234125i & -0.083298 + 0.144276i \end{bmatrix}_{2\times 3}$$ ... (upto 6 decimal places)

2. $B_{2\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$, $B_{2\times 3} \in M(r = 2, c = 1)$, We have the following:

$$\Sigma_{2\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} , \quad \Phi_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$$

$$\bar{\Sigma}_{2\times 3} = \begin{bmatrix} 0.757724 & 0.324490 & 0.284793 \\ 0.350864 & 0.150255 & 0.131874 \end{bmatrix}_{2\times 3}$$ ... (upto 6 decimal places)

$$\bar{\Phi}_{2\times 3} = \begin{bmatrix} 1 & -1 & 1 \\ \theta & -\theta & \theta \end{bmatrix}_{2\times 3}$$, Where $\theta = \exp(+i\frac{2\pi}{3})$, here ‘i’ is the imaginary unit, i.e. $i^2 = -1$

$$\bar{B}_{2\times 3} = \begin{bmatrix} 0.757724 & -0.324490 & 0.284793 \\ -0.175432 + 0.303858i & 0.075128 - 0.130125i & -0.065937 + 0.114206i \end{bmatrix}_{2\times 3}$$ ... (upto 6 decimal places)

3. $D_{2\times 3} = \begin{bmatrix} -i & 0 & 0 \\ 0 & +i & 0 \end{bmatrix}_{2\times 3}$, $D_{2\times 3} \in M(r = 2, c = 1)$, We have the following:

$$\Sigma_{2\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} , \quad \Phi_{2\times 3} = \begin{bmatrix} -i & 1 & 1 \\ 1 & +i & 1 \end{bmatrix}_{2\times 3}$$

$$\bar{\Sigma}_{2\times3} = \begin{bmatrix} 0.579202 & 0.371067 & 0.250071 \\ 0.362720 & 0.270344 & 0.166596 \end{bmatrix}_{2\times3}$$ ... (upto 6 decimal places)

$$\bar{\Phi}_{2\times3} = \begin{bmatrix} -i & -1 & 1 \\ \theta & -i\theta & \theta \end{bmatrix}_{2\times3}$$, Where $\theta = \exp(+i\frac{2\pi}{3})$, here 'i' is the imaginary unit, i.e. $i^2 = -1$

$$\bar{D}_{2\times3} = \begin{bmatrix} -0.579202i & -0.371067 & 0.250071 \\ -0.181360 + 0.314124i & 0.234125 + 0.135172i & -0.083298 + 0.144276i \end{bmatrix}_{2\times3}$$ ... (upto 6 decimal places)

III. DISCUSSION AND CONCLUSION

The present article introduces the concept of Local and Global Baseline in context of M(r,c) subsets of Complex Matrix spaces $M_{m\times n}(C)$. Each element $A_{m\times n}$ belonging to the M(r,c) subset is associated with an M(r,c) element $\bar{A}_{m\times n}$, which is defined as the "Local Baseline Matrix" or the Baseline Matrix of $A_{m\times n}$. The M(r,c) subset is associated with a set element $\langle G \rangle_{m\times n}$, which is defined as its " Global Baseline Matrix" , it can be observed that starting with the initiator matrix pairs $(\Sigma_{m\times n}, \Phi_{m\times n})$ and $(\Sigma(0)_{m\times n}, \Phi(0)_{m\times n})$, with matrix pair $(S_{m\times m}, T_{n\times n})$ Markov matrices $P(X)_{m\times m}$, $Q(X)_{n\times n}$ and their powers forming the appropriate propagator matrices various interesting mathematical structures can be generated from and out of the M(r,c) subsets, in this category the article defines the iterated subspaces of $M_{m\times n}(C)$ generated using the $\{G(t)_{m\times n} \mid t = 0,1,2,..., \mid t \in \{0\} \cup N\}$ matrix sequence defined on the M(r,c) subset and introduces the concept of "Fundamental subset" of elements belonging to the M(r,c) subset.

The Numerical Illustrations demonstrate the above mathematical formulations using the $M(r=2, c=1)$ subset of the matrix space $M_{2\times3}(C)$. It can be observed that the three different numerical realization $A_{2\times3}$, $B_{2\times3}$ and $D_{2\times3}$ are associated with numerically different local baseline matrices which are also numerically different from the global baseline matrix associated with the $M(r=2, c=1)$ subset. Under appropriately defined proximity measures, these numerical separations can be quantified and utilized in numerical/computational studies involving the M(r,c) subsets and its associated mathematical structures.

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