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# On Generalization Property of $I^s$-Open Sets in Ideal Topological Semigroups

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**Abstract:- In this paper, we introduce and investigate a new class of $I^s$−open sets, called generalized $I^s$−open sets in ideal topological semigroups a space. We study some properties on this class, such as product and relativity. Further more, the relationships between this class and other known classes are introduced and studied.**

***Keywords***:- *Open Set; Ideal Topological Space, Topological Semigroup. AMS Classification*: *Primary 54A05, 16W30.*

## I. INTRODUCTION

The notion of an ideal topological spaces is introduced by Kuratowski, [7]. Many researcher studied about the an ideal topological spaces. An ideal I on a topological space $(X,\tau)$ is a nonempty collection of subsets of $X$ which satisfies the following conditions:

1- if $A \in I$ and $B \subseteq A$ then $B \in I$, 2- if $A \in I$ and $B \in I$ then $A \cup B \in I$.

Applications to various fields were further investigated by Jankovic and Hamlett [2], Dontchev [6], and Arenas and et al. [4]. An ideal topological space is a topological space $(X,\tau)$ with an ideal I on $X$ and it is denoted by $(X,\tau,I)$.

This paper is organized as follows: Section 3 introduces the concept of generalized $I^s$− open sets in ideal topological semigroups with its relationship among other known sets. Section 4 introduces the properties of product and relativity of generalized $I^s$ -open sets.

## II. PRELIMINARIES

- *Theorem 2.1.*

  [5] For a topological space $(X,\tau)$ and $A,B \subseteq X$, if $B$ is an open set in $X$, then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

- *Theorem 2.2.*

  [5] For a topological space $(X,\tau)$,

  - $Cl(X-A) = X - Int(A)$ for all $A \subseteq X$;
  - $Int(X-A) = X - Cl(A)$ for all $A \subseteq X$.

- *Definition 2.3.*

  [8] A subset $A$ of a topological space $(X,\tau)$ is called a *generalized closed* (simply $g$−closed) *set*, if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an open subset of $(X,\tau)$. The complement of $g$−closed set is called a *generalized open* (simply $g$−open) *set*.

- *Theorem 2.4.*

  [8] Every closed set is a $g$−closed set.

- *Definition 2.5.*

  A topological space $(X,\tau)$ is called:

  - a $T_{1/2}$−space [8] if every $g$−closed set is a closed set.
  - a $T_1$−space [5] if for each disjoint point $x$ 6= $y \in X$, there are two open sets $G$ and $H$ in $X$ such that $x \in H$, $y \in G$, $x$ /$\in G$ and $y$ /$\in H$.

- *Theorem 2.6.*

  [8] A topological space $(X,\tau)$ is a $T_{1/2}$−space if and only if every singleton set is either open or a closed set.

- *Theorem 2.7.*

  [5] A topological space $(X,\tau)$ is a $T_1$−space if and only if every singleton set is a closed set.

In an ideal topological space $(X,\tau,I)$, $A^*(I)$ is

de- $(X_\pi,\tau,I)$ the ideal topological semigroup with

opined by: reation $\pi : X \times X \dashrightarrow X$,

where $\pi(x,y) = x$ or

$\pi(x,y) = y$ for all $x,y \in X$. $A^*(I) = \{x \in X : U \cap A \notin I$ for each open neighborhood $U$ of $x\}$

Is called the local function of $A$ with respect to I and $\tau$, [7]. When there is no chance for confusion $A^*(I)$ is denoted by $A^*$. For every ideal topological space $(X,\tau,I)$, there exists a topology $\tau^*$ finer than $\tau$, generated by the base

$$\beta(I,\tau) = \{U - I : U \in \tau \text{ and } I \in I\}.$$

Observe additionally that $Cl^*(A) = A \cup A^*$, [9] defines a Kuratowski closure operator for $\tau^*$. $Int^*(A)$ will denote the interior of $A$ in $(X,\tau^*)$. If I is an ideal on topological space $(X,\tau)$, then $(X,\tau,I)$ is called an ideal topological space.

- *Theorem 2.8.*

[3] Let $(X,\tau,I)$ be an ideal topological space. Then for $A,B \subseteq X$, the following properties hold:

- $A \subseteq B$ implies that $A^* \subseteq B^*$;
- $G \in \tau$ implies that $G \cap A^* \subseteq (G \cap A)^*$;
- $A^* = Cl(A^*) \subseteq Cl(A)$;
- $(A \cup B)^* = A^* \cup B^*$;
- $(A^*)^* \subseteq A^*$.

By topological semigroup $(X_*,\tau)$, we mean a topological space $(X,\tau)$ which is space with associated multiplication $* : X \times X \to X$ such that $*$ is continuous function from the product space $X \times X$ into $X$. By an ideal topological semigroup $(X_*,\tau,I)$, we mean an ideal topological space $(X,\tau,I)$ with associated multiplication $* : X \times X \to X$ such that $*$ is continuous function from the product space $X \times X$ into $X$. A pair $(Y,\circ)$ is called I−*subspace* of an ideal topological semigroup $(X_\circ,\tau,I)$ if $Y$ is a subspace of $X$ and the continuous function $\circ$ takes the product $Y \times Y$ into $Y$ and $\circ(x,y) = *(x,y)$ for all $x,y \in Y$ . We denote the operation of any I−subspace with the same symbol used for the operation on the an ideal topological semigroup under consideration. For any ideal topological space $(X,\tau,I)$, we mean by

- *Definition 2.9.*

[1] A subset $A$ of an ideal topological semigroup $(X_*,\tau,I)$ is said to be an $I^s$−open set if $A \subseteq Cl[Int^*(Cl^*(A))]$. The complement of $I^s$−open set is said to be an $I^s$−closed set. For an ideal topological semigroup $(X_*,\tau,I)$, the set of all $I^s$-closed sets in $X$ denoted by $I^sC(X,\tau)$ and the set of all $I^s$-open sets in $X$ denoted by $I^sO(X,\tau)$.

- *Theorem 2.10.*

[1] For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*,\tau,I)$, ${}_IsCl(A) = A$ if and only if $A$ is an $I^s$−closed set.

- *Theorem 2.11.*

[1] For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*,\tau,I)$, ${}_IsInt(A) = A$ if and only if $A$ is an $I^s$−open set.

- *Theorem 2.12.*

[1] For a subsets $A,B \subseteq X$ of an ideal topological semigroup $(X_*,\tau,I)$, the following hold:

- If $A \subseteq B$ then ${}_IsCl(A) \subseteq {}_IsCl(B)$;
- IsCl(A) ∪ IsCl(B) ⊆ IsCl(A ∪ B);
- IsCl(A ∩ B) ⊆ IsCl(A) ∩ IsCl(B);
- IsCl(A) ⊆ Cl(A).

- *Theorem 2.13.*

[1] For a subsets $A,B \subseteq X$ of an ideal topological semigroup $(X_*,\tau,I)$, the following hold:

- If $A \subseteq B$ then ${}_IsInt(A) \subseteq {}_IsInt(B)$;
- ${}_IsInt(A) \cup {}_IsInt(B) \subseteq {}_IsInt(A \cup B)$;
- ${}_IsInt(A \cap B) = {}_IsInt(A) \cap {}_IsInt(B)$;
- $Int(A) \subseteq {}_IsInt(A)$ .

- *Theorem 2.14.*

[1] For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*,\tau,I)$, the following hold:

- ${}_IsInt(X - A) = (X - {}_IsCl(A)$;
- ${}_IsCl(X - A) = (X - I^sInt(A)$.

## III. GENERALIZED $I^S$-OPEN SETS

- *Definition 3.1.*

A subset $A$ of an ideal topological semigroup $(X_*,\tau,I)$ is called a generalized $I^s$closed set (simply$\mathcal{I}_g^s$-closed) if ${}_IsCl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open subset of $(X_*,\tau,I)$. The complement of$\mathcal{I}_g^s$-closed set is called a generalized $I^s$-open set (simply$\mathcal{I}_g^s$-open).

For an ideal topological semigroup $(X_*,\tau,I)$, the set of all $\mathcal{I}_g^s$-closed sets in $X$ denoted by $\mathcal{I}_g^s C(X,\tau)$and the set of all $\mathcal{I}_g^s$-open sets in $X$ denoted by$\mathcal{I}_g^s O(X,\tau)$.

- *Example 3.2.*

In an ideal topological semigroup $(X_\pi,\tau,I)$, where $X = \{a,b,c\}$,

$$\tau = \{\emptyset,X\}, I = \{\emptyset,\{a\}\}, \text{and } \tau^* = \{\emptyset,X,\{b,c\}\}$$

$$\mathcal{I}_g^s C(X,\tau) = P(X),$$

And

$$\mathcal{I}_g^s O(X,\tau) = P(X).$$

- *Theorem 3.3.*

Every $I^s$-open set is an $I_g{}^s$-open set.

*Proof.* Let $A$ be an $I^s$-open subset of an ideal topological semigroup $(X_*,\tau,I)$. Then $X{-}A$ is $I^s$-closed set. Hence $X - A = {}_IsCl(X - A) \subseteq U$ whenever $X - A \subseteq U$ and $U$ is open set. That is,$A$ is $\mathcal{I}_g^s$-open set□

The converse of Theorem (3.3), no need to be true. In example (3.2),$\{a\}$ is $\mathcal{I}_g^s$-open set but it is not $I^s$open set.

- *Corollary 3.4.*

Every $I^s$-closed set is an $I_g{}^s$-closed set.

- *Theorem 3.5.*

Let $(X_*,\tau,I)$ be an ideal topological semigroup. If $(X,\tau)$ is a $T_{1/2}$-space. Then every$\mathcal{I}_g^s$closed set in $X$ is $I^s$-closed.

*Proof.* Let $A$ be an$\mathcal{I}_g^s$-closed set in $(X_*,\tau,I)$. Suppose that $A$ is not $I^s$-closed set. Then there is at least $x \in {}_IsCl(A)$ such that $x \notin A$. Since $(X,\tau)$ is a $T_{1/2}$space then by Theorem (2.6), $\{x\}$ is an open or closed set in $X$. If $\{x\}$ is a closed set in $X$ then $X - \{x\}$ is an open set. Since $x \notin A$, we have $A \subseteq X - \{x\}$.

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Since $A$ is $\mathcal{I}_g^s$-closed set and $X - \{x\}$ is an open subset of $X$ containing $A$, we get ${}_IsCl(A) \subseteq X - \{x\}$. Hence $x \in X - {}_IsCl(A)$ and this contradiction, because $x \in {}_IsCl(A)$. If $\{x\}$ is an open set then it is $I^s$-open set. Since $x \in {}_IsCl(A)$ we have $\{x\} \cap A \neq \emptyset$. That is, $x \in A$ and this contradiction. Hence $A$ is $I^s$-closed set in $(X_*,\tau,I)$. □

➢ *Theorem 3.6.*

Let $(X_*,\tau,I)$ be an ideal topological semigroup. Every $g$-open set in $(X_*,\tau)$ is $I_g^s$-open set.

*Proof.* Let $A$ be a $g$-open set in $(X,\tau)$. Then $X-A$ is $g$-closed set. Hence $X-A = Cl(X-A) \subseteq U$ whenever $X - A \subset U$ and $U$ is open set. Since ${}_IsCl(X - A) \subseteq Cl(X - A)$, we get ${}_IsCl(X - A) \subseteq U$ whenever

$X - A \subseteq U$ and $U$ is open set. Therefore $X - A$ is $\mathcal{I}_g^s$-closed set. That is $A$ is $\mathcal{I}_g^s$-open set. □

The converse of Theorem (3.6,) no need to be true. for Example,

➢ *Example 3.7.*

In a ideal topological semigroup

$(X_\pi,\tau,I)$, where $X = \{a,b,c\}$, $\tau = \{\emptyset,X,\{a,b\}\}$, $I = \{\emptyset,\{a\}\}$ and $\tau^* = \{\emptyset,X,\{b\},\{a,b\},\{b,c\}\}$,

$\{b,c\}$ is an $\mathcal{I}_g^s$-open set and it is not $g$-open set, because $U = \{a,b\}$ is an open set in $(X,\tau)$ and $\{a\} \subseteq U$ but $Cl(\{a\}) = X * U$.

➢ *Theorem 3.8.*

If $A$ is an $\mathcal{I}_g^s$-closed set in an ideal topological semigroup $(X_*,\tau,I)$ and $B$ is a closed set in $(X,\tau)$ then $A \cap B$ is $\mathcal{I}_g^s$-closed set.

*Proof.* Let $U$ be an open subset of $(X,\tau)$ such that $A \cap B \subseteq U$. Since $B$ is closed set in $(X,\tau)$ we obtain $U \cup (X - B)$ is an open set in $(X,\tau)$. Since $A$ is an $\mathcal{I}_g^s$-closed set in $X$ and $A \subseteq U \cup (X - B)$ so on $IsCl(A) \subseteq U \cup (X - B)$. Hence

$$\begin{aligned} IsCl(A \cap B) &\subseteq IsCl(A) \cap {}_IsCl(B) \subseteq {}_IsCl(A) \cap Cl(B) \\ &= IsCl(A) \cap B \subseteq [U \cup (X - B)] \cap B \\ &\subseteq U \cap B \subseteq U. \end{aligned}$$

Thus, $A \cap B$ is $\mathcal{I}_g^s$-closed set. □

➢ *Theorem 3.9.*

For any $x \in X$ in an ideal topological semigroup $(X_*,\tau,I)$, either the set $\{x\}$ is $I^s$-closed or the set $X - \{x\}$ is $\mathcal{I}_g^s$-closed in $(X_*,\tau,I)$.

*Proof.* If $\{x\}$ is not $I^s$-closed set in $(X_*,\tau,I)$ then $\{x\}$ is not closed set in $X$ and so $X - \{x\}$ is not open set in $X$. Then the set $X$ is only open set in itself containing $\{x\}$ and hence ${}_IsCl(X - \{x\}) \subseteq X$. That is, $X - \{x\}$ is $\mathcal{I}_g^s$-closed in $(X_*,\tau,I)$. □

➢ *Theorem 3.10.*

A subset $A$ of an ideal topological semigroup $(X_*,\tau,I)$ is $I_g^s$-closed if and only if for each $x \in {}_IsCl(A)$, $Cl(\{x\}) \cap A \neq \emptyset$.

*Proof.* Suppose that $A$ is an $\mathcal{I}_g^s$-closed set in $(X_*,\tau,I)$ and $x \in {}_IsCl(A)$ be any point. Let $Cl(\{x\}) \cap A = \emptyset$. Since $Cl(\{x\})$ is a closed set in $X$ we obtain $X - Cl(\{x\})$ is an open set in $X$. Since $A \subseteq X - Cl(\{x\})$ and $A$ is $I_g^s$-closed set we get ${}_IsCl(A) \subseteq X - Cl(\{x\})$ but this contradicts with $x \notin X - Cl(\{x\})$. Hence $Cl(\{x\}) \cap A \neq \emptyset$.

Conversely, suppose that for each $x \in {}_IsCl(A)$, $Cl(\{x\}) \cap A \neq \emptyset$ and $U$ be any open set in $X$ such that $A \subseteq U$. Let $x \in {}_IsCl(A)$. Then $Cl(\{x\}) \cap A \neq \emptyset$. Then there is at least $z \in Cl(\{x\})$ and $z \in A$. Then $z \in Cl(\{x\})$ and $z \in U$. Since $U$ is an open set in $X$ we get $\{x\} \cap U \neq \emptyset$. Hence $x \in U$ and so $IsCl(A) \subseteq U$. Hence, $A$ a $\mathcal{I}_g^s$-closed set. □

➢ *Theorem 3.11.*

A subset $A$ of an ideal topological semigroup $(X_*,\tau,I)$ is an $\mathcal{I}_g^s$-open set if and only if $F \subseteq {}_IsInt(A)$ whenever $F \subseteq A$ and $F$ is closed subset of $(X,\tau)$.

*Proof.* Let $A$ be an $\mathcal{I}_g^s$-open subset of $X$ and $F$ be a closed subset of $(X,\tau)$ such that $F \subseteq A$. Then

$X - A$ is $\mathcal{I}_g^s$-closed set, $X - A \subseteq X - F$ and $X - F$ is an open subset of $(X,\tau)$. By Theorem(2.14), we get $X - {}_IsInt(A) = {}_IsCl(X - A) \subseteq X - F$, that is, $F \subseteq {}_IsInt(A)$.

Conversely, suppose that $F \subseteq {}_IsInt(A)$ where $F$ is a closed subset of $(X,\tau)$ such that $F \subseteq A$. Then for any open subset $U$ of $(X,\tau)$ such that $X - A \subseteq U$, we have $X - U \subseteq A$ and $X - U \subseteq {}_IsInt(A)$. Then by Theorem(2.14), $X - {}_IsInt(A) = {}_IsCl(X - A) \subseteq U$.

Hence $X - A$ is $\mathcal{I}_g^s$-closed. That is, $A$ is $\mathcal{I}_g^s$-open set.□

➢ *Theorem 3.12.*

If $A$ is $\mathcal{I}_g^s$-closed subset of an ideal topological semigroup $(X_*,\tau,I)$ then ${}_IsCl(A) - A$ contains no nonempty closed set in $(X,\tau)$.

*Proof.* Suppose that ${}_IsCl(A) - A$ contains a nonempty closed set $F$ in $(X,\tau)$. Then

$F \subseteq {}_IsCl(A) - A \subseteq {}_IsCl(A)$.

Since $A \subseteq {}_IsCl(A)$ we have $F \subseteq X - A$ and so $A \subseteq$ - closed set and $X-F$ is an open subset of $(X,\tau)$, we conclude ${}_IsCl(A) \subseteq X - F$ and so $F \subseteq X - {}_IsCl(A)$. Therefore

$F \subseteq {}_IsCl(A) \cap (X - {}_IsCl(A)) = \emptyset$

$X - F$. Since $A$ is $\mathcal{I}_g^s$

And so $F = \emptyset$. Hence ${}_IsCl(A) - A$ contains no nonempty closed set in $(X,\tau)$. □

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➢ *Corollary 3.13.*

If $A$ is an $\mathcal{I}_g^s$-closed subset of an ideal topological semigroup $(X_*,\tau,\mathrm{I})$ then ${}_{\mathrm{I}}sCl(A)-A$ is an $\mathcal{I}_g^s$-open set.

*Proof.* By Theorem (3.12), ${}_{\mathrm{I}}sCl(A)-A$ contains no nonempty closed set in $(X_*,\tau)$ and it is clear that $\emptyset \subseteq {}_{\mathrm{I}}sInt({}_{\mathrm{I}}sCl(A)-A)$ then by Theorem (3.11), $\mathrm{I}sCl(A)-A$ is an $\mathcal{I}_g^s$-open set in $(X_*,\tau,\mathrm{I})$. □

➢ *Theorem 3.14.*

If $A$ is an $\mathcal{I}_g^s$-closed subset of an ideal topological semigroup $(X_*,\tau,\mathrm{I})$ and $B \subseteq X$. If

$A \subseteq B \subseteq {}_{\mathrm{I}}sCl(A)$ we obtain $B$ is an $\mathcal{I}_g^s$-closed set.

*Proof.* Let $U$ be an open set in $(X,\tau)$ such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since $A$ is an $\mathrm{I}_g{}^s$-closed set then ${}_{\mathrm{I}}sCl(A) \subseteq U$. Since $B \subseteq {}_{\mathrm{I}}sCl(A)$ then

$$\mathrm{I}sCl(B) \subseteq {}_{\mathrm{I}}sCl[{}_{\mathrm{I}}sCl(A)] = {}_{\mathrm{I}}sCl(A) \subseteq U.$$

That is, $B$ is an $\mathrm{I}_g{}^s$-closed set in $(X_*,\tau,\mathrm{I})$.□

➢ *Theorem 3.15.*

Let $A$ be an $\mathcal{I}_g^s$-closed subset of an ideal topological semigroup $(X_*,\tau,\mathrm{I})$. Then $A = \mathrm{I}sCl({}_{\mathrm{I}}sInt(A))$ if and only if ${}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A$ is a closed set in $(X,\tau)$.

*Proof.* Let ${}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A$ be closed set in $(X,\tau)$. Since ${}_{\mathrm{I}}sInt(A) \subseteq A$ and $A \subseteq {}_{\mathrm{I}}sCl(A)$, we conclude $\mathrm{I}sCl({}_{\mathrm{I}}sInt(A)) \subseteq {}_{\mathrm{I}}sCl(A)$. Then ${}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A \subseteq {}_{\mathrm{I}}sCl(A)-A$, this implies $\mathrm{I}sCl({}_{\mathrm{I}}sInt(A))-A \subseteq X-A \Rightarrow A \subseteq X-({}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A)$.

Since $A$ is an $\mathcal{I}_g^s$-closed set and $X-({}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A)$ is an open set in $(X,\tau)$ containing $A$,we have $\mathrm{I}sCl(A) \subseteq X-({}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A)$, this implies

$$\mathrm{I}sCl({}_{\mathrm{I}}sInt(A))-A \subseteq X-{}_{\mathrm{I}}sCl(A).$$

Therefore,

$$\mathrm{I}sCl({}_{\mathrm{I}}sInt(A))-A \subseteq {}_{\mathrm{I}}sCl(A)\cap(X-{}_{\mathrm{I}}sCl(A)) = \emptyset.$$

Hence ${}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A=\emptyset$, that is, $\mathrm{I}sCl({}_{\mathrm{I}}sInt(A)) = A$.

Conversely, if $A = {}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))$ then $\mathrm{I}sCl({}_{\mathrm{I}}sInt(A))-A = \emptyset$ and hence ${}_{\mathrm{I}}sCl({}_{\mathrm{I}}sInt(A))-A$ is a closed set in $(X,\tau)$. □

## IV. PRODUCT AND RELATIVELY

For a bitopological semigroup $(X_*,\tau,\rho)$ and a subset $A$ of $X$, the $\tau\rho$-closure set of $A$ is defined as the intersection of all $\tau\rho$-closed sets containing $A$ and it is denoted by ${}_{\tau\rho}Cl(A)$. The $\tau\rho$-interior set of $A$ is defined as the union of all $\tau\rho$-open sets of $X$ contained in $A$ and it is denoted by ${}_{\tau\rho}Int(A)$.

➢ *Definition 4.1.*

A subset $A \subseteq X$ is said to be $I_{\tau\rho}$*closed set* in a bitopological semigroup $(X_*,\tau,\rho)$ if $\tau\rho Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open subset in $(X,\tau)$. The complement of $I_{\tau\rho}$-closed set is said to be $I_{\tau\rho}$*-open set.*

➢ *Lemma 4.2.*

For a subset of an ideal topological semigroup $(X_*,\tau,\mathrm{I})$,

- ${}_{\tau\tau\mathrm{I}}Cl(A) = {}_{\mathrm{I}}sCl(A)$.
- ${}_{\tau\tau\mathrm{I}}Int(A) = {}_{\mathrm{I}}sInt(A)$.

*Proof.* It is clear from the definitions.

➢ *Theorem 4.3.*

A subset $A \subseteq X$ is an $\mathcal{I}_g^s$-closed set in an ideal topological semigroup $(X_*,\tau,\mathrm{I})$ if and only if it is $I_{\tau\tau\mathrm{I}}$-closed set in bitopological semigroup $(X_*,\tau,\tau_{\mathrm{I}})$.

*Proof.* It is clear from the definitions and Lemma (4.2).

➢ *Theorem 4.4.*

Let $Y$ be an open subspace of an ideal topological semigroup $(X_*,\tau,\mathrm{I})$ and $A \subseteq Y$. If

$A$ is an $\mathcal{I}_g^s$-closed set in $(X_*,\tau,\mathrm{I})$ then $A$ is $I_{\tau|Y\tau\mathrm{I}|Y}$ closed set in bitopological semigroup $(Y_\circ,\tau|_Y,\tau_{\mathrm{I}}|_Y)$.

*Proof.* Let $O$ be an open subset in $(Y,\tau|_Y)$ such that $A \subseteq O$. Then $O = U \cap Y$ for some open set $U$ in $(X,\tau)$ and so $A \subseteq U$. Since $A$ is an $\mathrm{I}_g{}^s$-closed set in $(X_*,\tau,\mathrm{I})$, we get ${}_{\mathrm{I}}sCl(A) \subseteq U$. By Theorem (4.3) and Lemma (4.2), ${}_{\tau\tau\mathrm{I}}Cl|_Y(A) = {}_{\mathrm{I}}sCl(A)|_Y(A) = {}_{\mathrm{I}}sCl(A)\cap Y \subseteq U\cap Y = O$.

Hence $A$ is $I_{\tau|Y\tau\mathrm{I}|Y}$-closed set in $(Y_\circ,\tau|_Y,\tau_{\mathrm{I}}|_Y)$. . □

➢ *Theorem 4.5.*

Let $Y$ be an open subspace of an ideal topological space $(X_*,\tau,\mathrm{I})$ and $A \subseteq Y$. If $A$ is $I_{\tau|Y\tau\mathrm{I}|Y}$-closed set in bitopological semigroup $(Y_\circ,\tau|_Y,\tau_{\mathrm{I}}|_Y)$ and $Y$ is $\mathrm{I}^s$-closed set in $(X_*,\tau,\mathrm{I})$ then $A$ is $\mathcal{I}_g^s$-closed set in $(X_*,\tau,\mathrm{I})$.

*Proof.* Let $U$ be an open subset in $(X,\tau)$ such that $A \subseteq U$. Then $A \subseteq U\cap Y$ and $U\cap Y$ is open set in $(Y,\tau|_Y)$. Since $A$ is a $I_{\tau|Y\tau\mathrm{I}|Y}$-closed set in bitopological semigroup $(Y_\circ,\tau|_Y,\tau_{\mathrm{I}}|_Y)$, we get ${}_{\tau\mathrm{I}}Cl|_Y(A) \subseteq U\cap Y$. Since $Y$ is an open set in $(X,\tau)$ and $Y$ is an $\mathrm{I}^s$-closed set in $(X_*,\tau,\mathrm{I})$ we have By Theorem (4.4) and Lemma (4.2),

$$\begin{aligned}\mathrm{I}sCl(A) &= \mathrm{I}sCl(A\cap Y) \subseteq {}_{\mathrm{I}}sCl(A)\cap{}_{\mathrm{I}}sCl(Y)\\ &= \mathrm{I}sCl(A)\cap Y = {}_{\mathrm{I}}sCl|_Y(A) = {}_{\tau\tau\mathrm{I}}Cl|_Y(A)\\ &\subseteq U\cap Y \subseteq U.\end{aligned}$$

Hence $A$ is $\mathcal{I}_g^s$-closed set in $(X_*,\tau,\mathrm{I})$. □

➢ *Lemma 4.6.*

Let $(X_*,\tau,\mathrm{I})$ and $(Y_\circ,\rho,\mathrm{I}^0)$ be two ideals topological semigroup. If $A \subseteq X$ and $B \subseteq Y$ then the following hold:

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- $_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Int(A\times B) = {}_{I}sInt(A)\times{}_{I}0sInt(B)$.
- $_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Cl(A\times B) = {}_{I}sCl(A)\times{}_{I}0sCl(B)$.

*Proof.* 1. Let $(x,y)\in{}_{(\tau\times\rho)(\tau I\times\rho I}0_{)}Int(A\times B)$. Then $(A\times B)\cap(U\times Y)\subseteq(A\times Y)\cap(U\times Y)=(A\cap U)\times Y=\emptyset\times Y=\emptyset$. by the definition there is at least one a $(\tau\times\rho)(\tau_I\times\rho_I0)$-open set $U\times I$ in bitopological semigroup $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$ such that $(x,y)\in U\times I\subseteq A\times B$. Then by Theorem (2.9) $A$ is an $I^s$-open set in $(X_*,\tau,I)$ and $B$ is a $I^s$-open set in $(Y_\circ,\rho,I^0)$. So $x\in U\subseteq A$ and $y\in I\subseteq B$. Then $x\in{}_IsInt(A)$ and $y\in{}_I0_sInt(B)$. This implies $(x,y)\in{}_IsInt(A)\times I0_sInt(B)$. Therefore

$_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Int(A\times B)\subseteq{}_IsInt(A)\times{}_I0sInt(B)$. Similar, $IsInt(A)\times{}_I0sInt(B)\subseteq{}_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Int(A\times B)$.

2. Let $(x,y)\notin{}_IsCl(A)\times{}_I0_sCl(B)$. Then $x\notin IsCl(A)$ or $y\notin{}_I0_sCl(B)$. If $x\notin{}_IsCl(A)$ then by Theorem (2.9) there is an $I^s$-open set $U$ in $(X_*,\tau,I)$ containing $x$ such that $A\cap U=\emptyset$. Then by Theorem (4.3) $U$ is a $\tau\tau_I$-open set in bitopological semigroup $(X_*,\tau,\tau_I)$ containing $x$ and $Y$ is a $\rho\rho_I0$-open set in bitopological semigroup $(Y_\circ,\rho,\rho_I0)$ containing $y$. Hence $U\times Y$ is a $(\tau\tau_I)(\rho\rho_I0)$-open set in bitopological semigroup $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$ containing $(x,y)$ and Then by Lemma (4.6), $IsCl(A)\times{}_I0sCl(B)={}_{(\tau}\times_{\rho)(\tau I\times\rho}I0_{)}Cl(A\times B)\subseteq U_1\times U_2$.

This implies, ${}_IsCl(A)\subseteq U_1$ and ${}_IsCl(B)\subseteq U_2$. Hence $A$ is an $\mathcal{I}_g^s$-closed set in $(X_*,\tau,I)$ and $B$ is an $\mathcal{I}_g^s$-closed set in $(Y_\circ,\rho,I^0)$.

Conversely, suppose that $A$ is an $I_g^s$-closed set in $(X_*,\tau,I)$ and $B$ is $I_g^s$-closed set in $(Y_\circ,\rho,I^0)$. Let $U_1\times U_2$ be an open set in $(X\times Y,\tau\times\rho)$ and $U_1\times U_2\subseteq A\times B$. Then $U_1$ is an open set in $(X,\tau)$ and $U_2$ is an open set in $(Y,\rho)$ such that $U_1\subseteq A$ and $U_2\subseteq B$. By the hypothesis, ${}_IsCl(A)\subseteq U_1$ and ${}_IsCl(B)\subseteq U_2$. Hence by Lemmas (4.6) and (4.2),

$_{(\tau}\times_{\rho)(\tau I\times\rho}I0_{)}Cl(A\times B)={}_IsCl(A)\times{}_I0sCl(B)\subseteq U_1\times U_2$.

Hence $A\times B$ is $I_{(\tau\times\rho)(\tau I\times\rho I}0_{)}$-closed set in bitopological semigroup $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$. □

This implies $(x,y)\notin{}_{(\tau\times\rho)(\tau I\times\rho I}0_{)}Cl(A\times B)$. Therefore

$_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Cl(A\times B)\subseteq{}_IsCl(A)\times{}_I0sCl(B)$.

Similar,

${}_IsCl(A)\times{}_I0sCl(B)\subseteq{}_{(\tau}\times_{\rho)(\tau I}\times_{\rho}I0_{)}Cl(A\times B)$.

- *Theorem 4.7.*

Let $(X_*,\tau,I)$ and $(Y_\circ,\rho,I^0)$ be two ideals topological semigroups. A subset $A\times B\subseteq X\times Y$ is $I_{(\tau\times\rho)(\tau I\times\rho I}0_{)}$-closed set in bitopological semigroup $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$ if and only if $A$ is an $I_g^s$-closed set in $(X_*,\tau,I)$ and $B$ is an $I_g^s$-closed set in $(Y_\circ,\rho,I^0)$.

*Proof.* Suppose that $A\times B$ is $I_{(\tau\times\rho)(\tau I\times\rho I}0_{)}$-closed set in bitopological semigroup $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$. Let $U_1$ be an open set in $(X,\tau)$ and $U_2$ be an open set in $(Y,\rho)$ such that $U_1\subseteq A$ and $U_2\subseteq B$. Then $U_1\times U_2$ be an open set in $(X\times Y,\tau\times\rho)$ and $U_1\times U_2\subseteq A\times B$. Since $A\times B$ is $I_{(\tau\times\rho)(\tau I\times\rho I}0_{)}$-closed set in $(X_*\times Y_\circ,\tau\times\rho,\tau_I\times\rho_I0)$ we obtain $(\tau\times\rho)(\tau I\times\rho I0)Cl(A\times B)\subseteq U1\times U2$.

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