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On Coefficient Estimates for New Subclasses of q -Bi-Spirallike Functions

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Abstract:- In this paper, we introduce and investigate two new subclasses of the function class $\sum @$ of λ-q-bi-spirallike functions defined in the open unit disc. Furthermore, We find estimates on the coefficients $|a_2|, |a_3|$ and $|a_4|$ for functions in these two new subclasses for functions.

***Keywords*:-** *Univalent Functions, **Bi**-Univalent Functions, **q-λ**-Spirallike, Subordination, Coefficients Bounds.*

I. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

Which are analytic in the open disc $E = \{z: z \in C \text{ and } |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of function in $\mathcal{A}$ which are univalent in E and indeed normalized by $f(0) = f'(0) - 1 = 0$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \ \ (z \in E),$$

and

$$f(f^{-1}(\omega)) = \omega, (|\omega| < r_0(f), r_0(f) \geq \tfrac{1}{4})$$

A function $f \in \mathcal{A}$ is said to bi-univalent function in E if f and f^{-1} are together univalent functions in E. Let $\sum @$ denote the class of bi-univalent functions defined in E. The inverse function $f^{-1}(\omega)$ is given by

$$h(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_3^2 - 5a_2a_3 + a_4)\omega^4 + \ldots \qquad (2)$$

Spacek [22] introduced the concept of spirallikeness which is a natural generalization of starlikeness. Spirallike functions can be characterized by the following analytic condition:

A function f in $\mathcal{A}$ is λ-spirallik if and only if,

$$\Re\left\{e^{i\lambda} \ \frac{zf'(z)}{f(z)}\right\} > 0, \ \ z \in E, \qquad (3)$$

Where $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. In [11], Jackson introduced and studied the concept of the q-derivative operator ∂_q as follows :

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z\,(1-q)}, \ \ (z \neq 0, 0 < q < 1, \ \ \partial_q f(0) = f'(0)). \ \ (4)$$

Equivalently (4), may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} \ [n]_q \ a_n \ z^{n-1}, \ \ z \neq 0, \qquad (5)$$

Where $[n]_q = \frac{1-q^n}{1-q}$, note that as $q \to 1^-$, $[n]_q \to n$.

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- *Definition 1.1 Let λ-q-$\mathcal{SP}_{\sum @}(\sigma)$ denote the class of λ-q-bi-spirallike functions of order σ, $(|\lambda| \leq \pi/2, 0 \leq \sigma < 1)$. The function $f(z)$, given by (1), is said it is in λ-q-$SP_{\sum @}(\sigma)$ if it satisfies:*

$$f \in \Sigma \ @and \ \Re\left(e^{i\lambda}\frac{z\partial_q f(z)}{f(z)}\right) > \sigma\cos\lambda \ (z \in E), \ (6)$$

and

$$\Re\left(e^{i\lambda}\frac{\omega\partial_q h(\omega)}{h(\omega)}\right) > \sigma\cos\lambda \ (\omega \in E). \qquad (7)$$

- *2 Main Results*

- *Theorem 2.1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$*
  Be in λ-q-$\mathcal{SP}^{\beta}_{\sum @}$, $(|\lambda| \leq \frac{\pi}{2}, 0 \leq \beta < 1$. Then

$$|a_2| \leq \frac{2\beta}{\sqrt{((2[3]_q - [2]_q^2 - 1)\beta + ([2]_q - 1)^2)}}\sqrt{cos\left(\frac{\lambda}{\beta}\right)}$$

$$|a_3| \leq \begin{cases} \frac{2\beta}{[3]_q - 1} cos\left(\frac{\lambda}{\beta}\right), & 0 \leq \beta \leq \frac{[2]_q - 1}{[2]_q + 1}, \\ \frac{4\beta^2}{([2]_q - 1)^2 + (2[3]_q - [2]_q([2]_q - 1) - 3)\beta} cos\left(\frac{\lambda}{\beta}\right), & \frac{[2]_q - 1}{[2]_q + 1} \leq \beta \leq 1. \end{cases}$$

$$|a_4| \leq \begin{cases} \frac{2\beta}{[4]_q - 1}\left[1 - \frac{2}{3}A_1\sqrt{cos\left(\frac{\lambda}{\beta}\right)}\, cos\left(\frac{\lambda}{\beta}\right)\right], & 0 < \beta < A, \\ \frac{2\beta}{[4]_q - 1}\left[1 + \frac{2}{3}A_1\sqrt{cos\left(\frac{\lambda}{\beta}\right)}\, cos\left(\frac{\lambda}{\beta}\right)\right], & A \leq \beta < A_4, \\ \frac{2\beta}{[4]_q - 1}\left[\frac{([2]_q - 1)^3(5[4]_q + 2[3]_q[2]_q - 4[3]_q - 2[2]_q - 1) + L)}{([2]_q - 1)^3(5[4]_q - 4[3]_q - 2[2]_q + 1)\beta + 2([3]_q - 1)} + \frac{2}{3}A_1\sqrt{cos\left(\frac{\lambda}{\beta}\right)}\, cos\left(\frac{\lambda}{\beta}\right)\right], & A_4 \leq \beta \leq 1, \end{cases}$$

Where

$$L = (2([3]_q + [2]_q) - ([3]_q[2]_q + 4))$$

$$A = \frac{3([2]_q - 1)(2[3]_q - [2]_q^2 - 1) + \sqrt{(36[3]_q^2 + [2]_q^4 + 42[2]_q^2 + 8[2]_q^3 + 73) - A_5}}{4(3[3]_q[2]_q + 3[2]_q - [2]_q^3 - 8)}$$

$$A_1 = \frac{2(2(3[3]_q[2]_q + 3[2]_q - [2]_q^3 - 8)\beta^2 - 3([2]_q - 1)(2[3]_q - [2]_q^2 - 1)\beta - ([2]_q - 1)^3)}{3([2]_q - 1)^3\sqrt{(([2]_q - 1)^2 + (2[3]_q - [2]_q^2 - 1))}}$$

$$A_4 = \frac{([3]_q[2]_q - [2]_q - [3]_q + 1)}{[3]_q[2]_q}$$

$$A_5 = 12[3]_q[2]_q^2 - 36[3]_q - 88[2]_q - 24[3]_q[2]_q).$$

Proof. Let

$$e^{i\lambda}\frac{z\partial_q f(z)}{f(z)} = g(z) \ (z \in E, \frac{-\pi}{2}\beta < \lambda < \frac{\pi}{2}\beta), \qquad (9)$$

$g(z)$ is analytic in E and satisfies $g(0) = e^{i\lambda}$ and $|\arg g(z)| < \frac{\pi}{2}\beta \ (z \in E)$. It can be checked that the function $\varphi(z)$ defined by:

$$g(z)^{\frac{1}{\beta}} = \cos(\frac{\lambda}{\beta})\varphi(z) + i\sin(\frac{\lambda}{\beta}), (z \in E),$$

Is a member of the class $\mathcal{P}$.

Let $\varphi(z) = 1 + d_1 z + d_2 z^2 + \ldots, (z \in E)$.

By comparing coefficient in (9), we have

$$a_2 = \frac{\beta d_1 e^{-i\left(\frac{\lambda}{\beta}\right)}}{[2]_q - 1} \cos\left(\frac{\lambda}{\beta}\right), \quad (10)$$

$$([3]_q - 1)a_3 - ([2]_q - 1)a_2^2 = \beta d_2 e^{-i\left(\frac{\lambda}{\beta}\right)} \cos\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)}{2} d_1^2 e^{-2i\left(\frac{\lambda}{\beta}\right)} \cos^2\left(\frac{\lambda}{\beta}\right). \quad (11)$$

$$([4]_q - 1)a_4 - D a_2 a_3 + ([2]_q - 1)a_2^3 = \beta d_3 e^{-i\left(\frac{\lambda}{\beta}\right)} \cos\left(\frac{\lambda}{\beta}\right) + \beta(\beta - 1) d_1 d_2 e^{-2i\left(\frac{\lambda}{\beta}\right)} \cos^2\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)(\beta-2)}{6} d_1^3 e^{-3i\left(\frac{\lambda}{\beta}\right)} \cos^3\left(\frac{\lambda}{\beta}\right), \quad (12)$$

Where

$D = \left([3]_q + [2]_q - 2\right)$.

Similarly we take

$$e^{i\lambda} \frac{\omega \partial_q h(\omega)}{h(\omega)} = G(\omega) \ (z \in E, \frac{-\pi}{2}\beta < \lambda < \frac{\pi}{2}\beta), \quad (13)$$

Where $G(\omega)$ is Analytic in E and Satisfies

$G(0) = e^{i\lambda}$ *and* $|\arg G(\omega)| < \frac{\pi}{2}\beta, (\omega \in E)$.

The function $q(\omega)$ defined by

$G(\omega)^{\frac{1}{\beta}} = \cos(\frac{\lambda}{\beta}) q(\omega) + i\sin(\frac{\lambda}{\beta}), (\omega \in E)$.

Is a Member of the class $\mathcal{P}$. Let $q(\omega) = 1 + c_1\omega + c_2\omega^2 + \ldots, (\omega \in E)$.

By comparing coefficient in (13), we have

$$-a_2 = \frac{\beta c_1 e^{-i\left(\frac{\lambda}{\beta}\right)}}{[2]_q - 1} \cos\left(\frac{\lambda}{\beta}\right), \quad (14)$$

$$(2[3]_q - [2]_q - 1)a_2^2 - ([3]_q - 1)a_3 = \beta c_2 e^{-i\left(\frac{\lambda}{\beta}\right)} \cos\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)}{2} c_1^2 e^{-i2\left(\frac{\lambda}{\beta}\right)} \cos^2\left(\frac{\lambda}{\beta}\right). \quad (15)$$

$$-(T a_2 a_3 + ([4]_q - 1)a_4) = \beta c_3 e^{-i\left(\frac{\lambda}{\beta}\right)} \cos\left(\frac{\lambda}{\beta}\right) + \beta(\beta - 1) c_1 c_2 e^{-2i\left(\frac{\lambda}{\beta}\right)} \cos^2\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)(\beta-2)}{6} c_1^3 e^{-3i\left(\frac{\lambda}{\beta}\right)} \cos^3\left(\frac{\lambda}{\beta}\right). \quad (16)$$

Where

$T = (5[4]_q - 2[3]_q - [2]_q - 2)a_2^3 - (5[4]_q - [3]_q - [2]_q - 3)$.

From (10) and (14) we have

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$$c_1 = -d_1. \qquad (17)$$

We shall obtain a refined estimate on $|d_1|$ for use in the estimates of $|a_3|$ and $|a_4|$. For this purpose we first add (11) with (15), then use the relations (17) and get the following:

$$(2([3]_q - [2]_q)a_2^2 = \beta(d_2 + c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)}{2}(d_1^2 + c_1^2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right).$$

Putting $a_2 = \frac{\beta d_1 e^{-i\left(\frac{\lambda}{\beta}\right)}}{[2]_q - 1}\cos\left(\frac{\lambda}{\beta}\right)$ from (10) we have after simplification:

$$d_1^2 = \frac{([2]_q-1)^2(d_2+c_2)}{(([2]_q-1)^2+(2[3]_q-[2]_q^2-1)\beta)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}.$$

By applying the familiar inequalities $|d_2| \le 2$ and $|c_2| \le 2$ we get:

$$|d_1| \le \sqrt{\frac{4([2]_q-1)^2}{((2([3]_q-[2]_q^2-1)\beta+([2]_q-1)^2)\cos\left(\frac{\lambda}{\beta}\right)}} = \frac{2([2]_q-1)}{\sqrt{((2[3]_q-[2]_q^2-1)\beta+([2]_q-1)^2)\cos\left(\frac{\lambda}{\beta}\right)}} \qquad (19)$$

and

$$|a_2| \le \frac{\beta|d_1|\cos\left(\frac{\lambda}{\beta}\right)}{[2]_q-1} = \frac{2\beta}{\sqrt{((2[3]_q-[2]_q^2-1)\beta+([2]_q-1)^2)}}\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}.$$

We next find a bound on $|a_3|$. For this we substract (15) from (11) and get

$$2([3]_q - 1)a_3 = 2([3]_q - 1)a_2^2 + \beta(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)}{2}(d_1^2 - c_1^2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right).$$

The relation $d_1^2 = c_1^2$ from (17), reduces the above expression to

$$2([3]_q - 1)a_3 = 2([3]_q - 1)a_2^2 + \beta(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right). \qquad (20)$$

Using $a_2 = \frac{\beta c_1 e^{-i\left(\frac{\lambda}{\beta}\right)}}{[2]_q - 1}\cos\left(\frac{\lambda}{\beta}\right)$ and (18), we get

$$2([3]_q - 1)a_3 = \frac{2([3]_q-1)}{([2]_q-1)^2}\beta^2 d_1^2 e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right) + \beta(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$$

$$\frac{2([3]_q-1)}{([2]_q-1)^2}\beta^2\left(\frac{([2]_q-1)^2(d_2+c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}}{(([2]_q-1)^2+(2[3]_q-[2]_q^2-1)\beta)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)\right) +$$

$$= K,$$

$$K = \frac{\beta}{(([2]_q-1)^2+(2[3]_q-[2]_q^2-1)\beta)}\left[Md_2 + (([2]_q^2 - 1)\beta - ([2]_q - 1)^2)c_2\right]e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right).$$

Where

$$K = \beta(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right),$$

$$M = \left((4[3]_q - [2]_q^2 - 3)\beta + ([2]_q - 1)^2\right)$$

Therefore, the inequalities $|d_2| \leq 2$ and $|c_2| \leq 2$ give the following: $2([3]_q - 1)|a_3|$

$$\leq \begin{cases} \frac{2\beta}{(([2]_q-1)^2 + (2[3]_q - [2]_q^2 - 1)\beta)}\left((4[3]_q - [2]_q^2 - 3)\beta + 2([2]_q - 1)^2 - ([2]_q^2 - 1)\beta\right)\cos\left(\frac{\lambda}{\beta}\right), & 0 \leq \beta \leq \frac{[2]_q - 1}{[2]_q + 1}, \\ \frac{2\beta}{(([2]_q-1)^2 + (2[3]_q - [2]_q^2 - 1)\beta)}\left((4[3]_q - [2]_q^2 - 3)\beta + ([2]_q^2 - 1)\beta\right)\cos\left(\frac{\lambda}{\beta}\right), & \frac{[2]_q - 1}{[2]_q + 1} \leq \beta \leq 1. \end{cases}$$

Which Simplifies

$$|a_3| \leq \begin{cases} \frac{2\beta}{[3]_q - 1)}\cos\left(\frac{\lambda}{\beta}\right), & 0 \leq \beta \leq \frac{[2]_q - 1}{[2]_q + 1}; \\ \frac{4\beta^2}{(([2]_q-1)^2 + (2[3]_q - [2]_q([2]_q - 1) - 3))\beta}\cos\left(\frac{\lambda}{\beta}\right), & \frac{[2]_q - 1}{[2]_q + 1} \leq \beta \leq 1. \end{cases}$$

Now we find an estimate on $|a_4|$. At first we shall derive a relation connecting d_1, d_2, d_3, c_2 and c_3. To this end, Now we collect (12) and (16) we get

$$M(-a_2^3 + a_2 a_3) = \beta(d_3 + c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}cos\left(\frac{\lambda}{\beta}\right) + \beta(\beta - 1)(d_1 d_2 + c_1 c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}cos^2\left(\frac{\lambda}{\beta}\right) + \frac{\beta(\beta-1)}{6}(d_1^3 + c_1^3)e^{-i3\left(\frac{\lambda}{\beta}\right)}cos^3\left(\frac{\lambda}{\beta}\right), \quad (21)$$

Where

$$M = (5[4]_q - 2([3]_q + [2]_q) - 1).$$

Now we are putting $c_1 = -d_1$ in (21) we get

$$M(-a_2^3 + a_2 a_3) = \beta(d_3 + c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right) + \beta(\beta - 1)d_1(d_2 - c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right), \quad (22)$$

Where

$$M = (5[4]_q - 2([3]_q + [2]_q) - 1)$$

Substituting $a_3 = a_2^2 + \frac{\beta}{2([3]_q - 1)}(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$from(20) in (21) we get after simplification:

$$(5[4]_q - 2([3]_q + [2]_q) - 1)\frac{\beta}{2([3]_q - 1)}(d_2 - c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right) = \beta(d_3 + c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right) + \beta(\beta - 1)d_1(d_2 - c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)$$

Since $a_2 = \frac{\beta d_1}{[2]_q - 1}e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$, we have

SINCE $a_2 = \frac{\beta d_1}{[2]_q - 1}e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$, WE HAVE

$$\frac{M\beta^2}{2([3]_q-1)}d_1(d_2-c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right) = \beta(d_3+c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$$
$$+\beta(\beta-1)d_1(d_2-c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right),$$

Where

$M = (5[4]_q - 2([3]_q + [2]_q) - 1).$

$$d_1(d_2-c_2) = \frac{2([3]_q-1)(d_3+c_3)e^{i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}{(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}. \quad (23)$$

Or

$$2([4]_q-1)a_4 = -(5[4]_q-2[3]_q-3)a_2^3+5([4]_q-1)a_2a_3+\beta(d_3-c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$$
$$+ \quad \beta(\beta-1)(d_1d_2-c_1c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)+\frac{\beta(\beta-1)(\beta-2)}{6}(d_1^3-c_1^3)e^{-i3\left(\frac{\lambda}{\beta}\right)}\cos^3\left(\frac{\lambda}{\beta}\right).$$

Observing that $c_1 = -d_1$ we have $d_1^3 - c_1^3 = 2d_1^3$ and therefore

$$2([4]_q-1)a_4 = -(5[4]_q-2([3]_q+[2]_q)-1)a_2^3+(5[4]_q-2([3]_q+[2]_q)-1)a_2a_3-(2[2]_q-2)a_2^3$$
$$+ \quad 2([3]_q+[2]_q-2)a_2a_3+\beta(d_3-c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$$
$$+ \quad \beta(\beta-1)d_1(d_2+c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)+\frac{\beta(\beta-1)(\beta-2)}{3}d_1^3e^{-i3\left(\frac{\lambda}{\beta}\right)}\cos^3\left(\frac{\lambda}{\beta}\right).$$

We Replace

$$-(5[4]_q-2([3]_q+[2]_q)-1)a_2^3+(5[4]_q-2([3]_q+[2]_q)-1)a_2a_3$$

By the right hand side of (22) ,

put $a_3 = a_2^2 + \frac{\beta}{2([3]_q-1)}(d_2-c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)$

and $a_2 = \frac{\beta d_1 e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}{([2]_q-1)}.$

This gives

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$$
\begin{aligned}
2([4]_q-1)a_4 = \;& \beta(d_3+c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)+\beta(\beta-1)d_1(d_2-c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)\\
+\;& H\frac{\beta d_1 e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}{([2]_q-1)}\left(\frac{\beta^2 d_1^2 e^{-2i\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)}{([2]_q-1)^2}+\frac{\beta}{2([3]_q-1)}(d_2-c_2)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right)\\
+\;& \beta(d_3-c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)+\beta(\beta-1)d_1(d_2+c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)\\
-\;& \frac{2\beta^3 d_1^3 e^{-3i\left(\frac{\lambda}{\beta}\right)}\cos^3\left(\frac{\lambda}{\beta}\right)}{([2]_q-1)^3}+\frac{\beta(\beta-1)(\beta-2)}{3}d_1^3 e^{-i3\left(\frac{\lambda}{\beta}\right)}\cos^3\left(\frac{\lambda}{\beta}\right)\\
=\;& 2\beta d_3 e^{-i\frac{\lambda}{\beta}}\cos\frac{\lambda}{\beta}+\frac{\beta([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1)}{([3]_q-1)([2]_q-1)^3}d_1(d_2-c_2)e^{-2i\frac{\lambda}{\beta}}\cos^2\frac{\lambda}{\beta}\\
+\;& \beta(\beta-1)d_1(d_2+c_2)e^{-2i\frac{\lambda}{\beta}}\cos^2\frac{\lambda}{\beta}+\frac{N\beta^3-B\beta^2+2([2]_q-1)^3\beta}{3([2]_q-1)^3}d_1^3 e^{-3i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right),
\end{aligned}
$$

Where

$H=2([3]_q+[2]_q-2)$
$N=\left(6([3]_q+[2]_q-3)+([2]_q-1)^3\right)$
$B=3([2]_q-1)^3$

Next, replacing $d_1(d_2-c_2)$ by the expression in the right hand side of (23) and c_1^2 by (18) we finally get

$$
\begin{aligned}
2([4]_q-1)a_4 = \;& \frac{\beta([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1)}{([3]_q-1)([2]_q-1)^3}\frac{2([3]_q-1)(d_3+c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)}{(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}\\
+\;& 2\beta d_3 e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)++\beta(\beta-1)d_1(d_2+c_2)e^{-2i\frac{\lambda}{\beta}}\cos^2\frac{\lambda}{\beta}\\
+\;& \frac{\left(6([3]_q+[2]_q-3)+([2]_q-1)^3\right)\beta^3-3([2]_q-1)^3\beta^2+2([2]_q-1)^3\beta}{3([2]_q-1)^3}\frac{([2]_q-1)^3(d_2+c_2)}{([2]_q-1)^2+(2[3]_q-[2]_q^2-1)\beta}d_1 e^{-2i\frac{\lambda}{\beta}}\cos^2\frac{\lambda}{\beta}=2\beta d_3 e^{-i\frac{\lambda}{\beta}}\cos\frac{\lambda}{\beta}+\frac{2\beta([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1)}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}(d_3+c_3)e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)+\frac{2\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)\beta^3-3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)\beta^2-([2]_q-1)^3\beta}{3([2]_q-1)(([2]_q-1)^2+(2[3]_q-[2]_q^2-1))}d_1(d_2+c_2)e^{-i2\left(\frac{\lambda}{\beta}\right)}\cos^2\left(\frac{\lambda}{\beta}\right)\\
=\;& \beta\left[\frac{2([2]_q-1)^3\left((5[4]_q+[2]_q[3]_q-4[3]_q-2[2]_q)\beta+(3[3]_q-[3]_q[2]_q+[2]_q-3)\right)}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}d_3\right.\\
&\left.+\frac{2\beta([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1)}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}c_3+\frac{2\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)\beta^2-H}{3([2]_q-1)(([2]_q-1)^2+(2[3]_q-[2]_q^2-1))}d_1(d_2+d_3)\cos\left(\frac{\lambda}{\beta}\right)\right]e^{-i\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right),
\end{aligned}
$$

Where $H=3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)\beta-([2]_q-1)^3$.

This Gives

$$
\begin{aligned}
|a_4|\le \frac{\beta}{2([4]_q-1)}&\left[\left|\frac{2\left((5[4]_q+[2]_q[3]_q-4[3]_q-2[2]_q)\beta+(3[3]_q-[3]_q[2]_q+[2]_q-3)\right)}{((5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1))}\right||d_3|\right.\\
&+\left|\frac{2(([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1))}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}\right||c_3|\\
&\left.+\left|\frac{2\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)\beta^2-H}{3([2]_q-1)(([2]_q-1)^2+(2[3]_q-[2]_q^2-1))}\right||d_1||d_2+d_3|\cos\left(\frac{\lambda}{\beta}\right)\right]\cos\left(\frac{\lambda}{\beta}\right)
\end{aligned}
$$

Where $H=3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)\beta-([2]_q-1)^3$.

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$$|a_4| \leq \begin{cases} \frac{2\beta}{[4]_q-1}\left[\frac{2\left((5[4]_q+[2]_q[3]_q-4[3]_q-2[2]_q)\beta+(3[3]_q-[3]_q[2]_q+[2]_q-3)\right)}{((5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1))}+A_2+A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\right], & 0<\beta<A; \\ \frac{2\beta}{[4]_q-1}\left[\frac{2\left((5[4]_q+[2]_q[3]_q-4[3]_q-2[2]_q)\beta+(3[3]_q-[3]_q[2]_q+[2]_q-3)\right)}{((5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1))}+A_2-A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\right], & A\leq\beta<A_4; \\ \frac{2\beta}{[4]_q-1}\left[\frac{2\left((5[4]_q+[2]_q[3]_q-4[3]_q-2[2]_q)\beta+(3[3]_q-[3]_q[2]_q+[2]_q-3)\right)}{((5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1))}+A_3+A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\right], & A_4\leq\beta\leq 1. \end{cases} \quad (24)$$

$$A = \frac{3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)+\sqrt{(36[3]_q^2+[2]_q^4+42[2]_q^2+8[2]_q^3+73)-(12[3]_q[2]_q^2-36[3]_q-88[2]_q-24[3]_q[2]_q)}}{4\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)}$$

Where

$$A_1 = \frac{2\left(2\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)\beta^2-3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)\beta-([2]_q-1)^3\right)}{3([2]_q-1)\sqrt[3]{(([2]_q-1)^2+(2[3]_q-[2]_q^2-1))}}$$

$$A_2 = \frac{([3]_q[2]_q-[2]_q-[3]_q+1)-2([3]_q[2]_q-1)\beta}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}$$

$$A_3 = \frac{2(([3]_q[2]_q-1)\beta-([3]_q[2]_q-[2]_q-[3]_q+1))}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}$$

$$A_4 = \frac{([3]_q[2]_q-[2]_q-[3]_q+1)}{[3]_q[2]_q}$$

By applying the inequalities $|d_n| \leq 2, |c_n| \leq 2 \ (n = 2,3)$ we get

$$|a_4| \leq \begin{cases} \frac{2\beta}{[4]_q-1}\left[1-\frac{2}{3}A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & 0<\beta<A; \\ \frac{2\beta}{[4]_q-1}\left[1+\frac{2}{3}A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & A\leq\beta<A_4; \\ \frac{2\beta}{[4]_q-1}\left[\frac{([2]_q-1)^3(5[4]_q+2[3]_q[2]_q-4[3]_q-2[2]_q-1)+L}{([2]_q-1)^3(5[4]_q-4[3]_q-2[2]_q+1)\beta+2([3]_q-1)}+\frac{2}{3}A_1\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & A_4\leq\beta\leq 1, \end{cases}$$

Where

$$L = (2([3]_q + [2]_q) - ([3]_q[2]_q + 4))$$

$$A = \frac{3([2]_q-1)\left(2[3]_q-[2]_q^2-1\right)+\sqrt{(36[3]_q^2+[2]_q^4+42[2]_q^2+8[2]_q^3+73)-(12[3]_q[2]_q^2-36[3]_q-88[2]_q-24[3]_q[2]_q)}}{4\left(3[3]_q[2]_q+3[2]_q-[2]_q^3-8\right)}$$

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$$A_1 = \frac{2\left(2\left(3[3]_q[2]_q + 3[2]_q - [2]_q^3 - 8\right)\beta^2 - 3([2]_q - 1)\left(2[3]_q - [2]_q^2 - 1\right)\beta - ([2]_q - 1)^3\right)}{3([2]_q - 1)\sqrt[3]{(([2]_q - 1)^2 + (2[3]_q - [2]_q^2 - 1))}} A_4$$
$$= \frac{([3]_q[2]_q - [2]_q - [3]_q + 1)}{[3]_q[2]_q}.$$

As $q \to 1^-$ in the above Theorem we get the following:

- *Corollary 2.1 [21] Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* $\lambda\text{-}\mathcal{SP}_{\Sigma @}^{\beta}, (|\lambda| \leq \frac{\pi}{2}, 0 \leq \beta < 1$. *Then*

$$|a_2| \leq \frac{2\beta}{\sqrt{\beta+1}}\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}. \quad (25)$$

$$|a_3| \leq \begin{cases} \beta\cos\left(\frac{\lambda}{\beta}\right), & 0 \leq \beta \leq \frac{1}{3}, \\ \frac{4\beta^2}{1+\beta}\cos\left(\frac{\lambda}{\beta}\right), & \frac{1}{3} \leq \beta \leq 1. \end{cases} \quad (26)$$

$$|a_4| \leq \begin{cases} \frac{2\beta}{3}\left[1 - \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & 0 < \beta < \frac{3+\sqrt{73}}{32}; \\ \frac{2\beta}{3}\left[1 + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & \frac{3+\sqrt{73}}{32} \leq \beta < 1; \\ \frac{2\beta}{3}\left[\frac{15\beta}{5\beta} + \frac{2}{3}\frac{16\beta^2-3\beta-1}{\sqrt[3]{1+\beta}}\sqrt{\cos\left(\frac{\lambda}{\beta}\right)}\cos\left(\frac{\lambda}{\beta}\right)\right], & \frac{2}{5} \leq \beta \leq 1. \end{cases} \quad (27)$$

- *Theorem 2.2 Let* $f(z)$*, given by (1) in the class* $SP_{\Sigma @}(\sigma, q), (|\lambda| \leq \frac{\pi}{2}, 0 \leq \sigma < 1)$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\sigma)\cos\lambda}{([3]_q - [2]_q)}} \quad (28)$$

$$|a_3| \leq \frac{2(1-\sigma)\cos\lambda}{[3]_q - [2]_q} \quad (29)$$

and

$$|a_4| \leq \frac{2(1-\sigma)\cos\lambda}{[4]_q - 1}\left[1 + \frac{5[4]_q - [3]_q - 4}{([3]_q - [2]_q)\cos\lambda}\sqrt{\frac{2(1-\sigma)}{([3]_q - [2]_q)\cos\lambda}}\right]. \quad (30)$$

Proof. Let $f \in SP_{\Sigma @}(\sigma, q)$, then by Definition 1.1 we have

$$e^{-i\lambda}\frac{z\partial_q f(z)}{f(z)} = Q_1(z)\cos\lambda + i\sin\lambda \quad (31)$$

and

$$e^{-i\lambda}\frac{\omega\partial_q h(\omega)}{h(\omega)} = P_1(\omega)\cos\lambda + i\sin\lambda \quad (32)$$

Where $\Re(Q_1(z)) > \sigma$,

$$Q_1(z) = 1 + d_1 z + d_2 z^2 + \ldots (z \in E)$$

and $\Re(P_1(\omega)) > \sigma$,

$$P_1(\omega) = 1 + c_1\omega + c_2\omega^2 + \ldots (\omega \in E).$$

As in the proof of Theorem 2.1, by suitably comparing coefficient in (31) and (32) we have

$$a_2 e^{-i\lambda} = \frac{d_1 \cos\lambda}{[2]_q - 1} \quad (33)$$

$$(([3]_q - 1)a_3 - ([2]_q - 1)a_2^2)e^{-i\lambda} = d_2\cos\lambda, \quad (34)$$

$$(([4]_q - 1)a_4 - Ha_2a_3 + ([2]_q - 1)a_2^3)e^{-i\lambda} = d_3\cos\lambda. \quad (35)$$

Where $H = \left(([3]_q + [2]_q - 2)\right)$and

$$-a_2 e^{-i\lambda} = \frac{c_1 \cos\lambda}{[2]_q - 1} \quad (36)$$

$$(2[3]_-[2]_q - 1)a_2^2 - ([3]_q - 1)a_3 e^{-i\lambda} = c_2\cos\lambda \quad (37)$$

$$-\left(R + ([4]_q - 1)a_4\right)e^{-i\lambda} = c_3\cos\lambda, \quad (38)$$

Where

$$R = (5[4]_q - 2[3]_q - [2]_q - 2)a_2^3 - (5[4]_q - [3]_q - [2]_q - 3)a_2a_3.$$

In order to express d_1 interms of d_2 and c_2 we first add (34) and (37) and get

$$2([3]_q - [2]_q)a_2^2 = (d_2 + c_2)\frac{\cos\lambda}{e^{i\lambda}}. \quad (39)$$

Again putting $a_2 e^{i\lambda} = \frac{d_1 \cos\lambda}{[2]_q - 1}$ from (33) we have

$$\frac{2([3]_q - [2]_q)}{([2]_q - 1)^2}\frac{d_1^2 \cos\lambda}{e^{2i\lambda}} = (d_2 + c_2)\frac{\cos\lambda}{e^{i\lambda}}$$

Or equivalently

$$d_1^2 = (d_2 + c_2)\frac{([2]_q - 1)^2 e^{i\lambda}}{2([3]_q - [2]_q)\cos\lambda} \quad (40)$$

The familiar inequalities $|d_2| \le 2(1 - \sigma), |c_2| \le 2(1 - \sigma)$ yield

$$|d_1|^2 = \frac{2([2]_q - 1)^2(1 - \sigma)}{([3]_q - [2]_q)\cos\lambda}$$

Which Implies that

$$|d_1| = \sqrt{\frac{2([2]_q-1)^2(1-\sigma)}{([3]_q-[2]_q)\cos\lambda}}$$

$$\text{and} \quad |a_2| \le \frac{|d_2|\cos\lambda}{[2]_q-1} \le \sqrt{\frac{2(1-\sigma)}{([3]_q-[2]_q)\cos\lambda}}\cos\lambda = \sqrt{\frac{2(1-\sigma)\cos\lambda}{([3]_q-[2]_q)}}. \quad (41)$$

Following the Lines of Proof of Theorem 2.1, with Appropriate Changes, we Get that

$$2([3]_q-1)a_3 = \left(\frac{[3]_q}{[3]_q-[2]_q)}d_2 + \frac{([3]_q-2)}{([3]_q-[2]_q)}c_2\right)\frac{\cos\lambda}{e^{i\lambda}}$$

The inequalities $|d_2| \le 2(1-\sigma), |c_2| \le 2(1-\sigma)$ yield

$$|a_3| \le \frac{2(1-\sigma)\cos\lambda}{[3]_q-[2]_q} \quad (42)$$

We shall next find an estimate on $|a_4|$, By substracting (38) from (35) we get

$$2([4]_q-1)a_4 = -(5[4]_q-2[3]_q-3)a_2^3 + 5([4]_q-1)a_2a_3 + (d_3-c_3)\frac{\cos\lambda}{e^{i\lambda}}$$

A substitution of the value of a_2 from the relation (33) gives

$$2([4]_q-1)a_4 = -(5[4]_q-2[3]_q-3)d_1^3\frac{\cos^3\lambda}{([2]_q-1)^3e^{3i\lambda}} + 5([4]_q-1)d_1\frac{\cos\lambda}{([2]_q-1)e^{i\lambda}}a_3 + (d_3-c_3)\frac{\cos\lambda}{e^{i\lambda}}.$$

Therefore, using the inequalities $|d_3| \le 2(1-\sigma)$, $|c_3| \le 2(1-\sigma)$, the estimate for $|d_1|$ from (41)and the estimate for $|a_3|$ from (42), we get

$$\begin{aligned}2([4]_q-1)|a_4| &\le (5[4]_q-2[3]_q-3)|d_1^3|\frac{\cos^3\lambda}{([2]_q-1)^3} + 5([4]_q-1)|d_1|\frac{\cos\lambda}{([2]_q-1)}|a_3| + |d_3-c_3|\cos\lambda \\ &\le (5[4]_q-2[3]_q-3)\sqrt{\frac{2(1-\sigma)}{([3]_q-[2]_q)\cos\lambda}}\frac{2(1-\sigma)}{[3]_q-[2]_q}\cos^2\lambda \\ &+5([4]_q-1)\sqrt{\frac{2(1-\sigma)}{([3]_q-[2]_q)\cos\lambda}}\frac{2(1-\sigma)}{([3]_q-[2]_q)}\cos^2\lambda + 4(1-\sigma)\cos\lambda \\ &\le 4(1-\sigma)\cos\lambda\left[1+\frac{5[4]_q-[3]_q-4}{([3]_q-[2]_q)}\sqrt{\frac{2(1-\sigma)\cos\lambda}{([3]_q-[2]_q)}}\right].\end{aligned}$$

Or equivalently,

$$|a_4| \le \frac{2(1-\sigma)\cos\lambda}{[4]_q-1}\left[1+\frac{5[4]_q-[3]_q-4}{([3]_q-[2]_q)}\sqrt{\frac{2(1-\sigma)\cos\lambda}{([3]_q-[2]_q)}}\right].$$

As $q \to 1^-$ in the above Theorem we get the following:

- *Corollary 2.2 [21] Let $f(z)$, given by (1) in the class $\mathcal{SP}_{\Sigma@}(\sigma)$, $(|\lambda| \le \frac{\pi}{2}, 0 \le \sigma < 1)$. then*

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$$|a_2| \leq \sqrt{2(1-\sigma)\cos\lambda}$$
$$|a_3| \leq 2(1-\sigma)\cos\lambda$$
$$|a_4| \leq \frac{2(1-\sigma)\cos\lambda}{3}\left[1+13\sqrt{2(1-\sigma)\cos\lambda}\right].$$

II. CONCLUSIONS

In this paper, we introduced and investigated two new subclasses of the function class $\sum @$ of λ-q-bi-spirallike functions defined in the open unit disc. Furthermore, We find estimates on the coefficients $|a_2|, |a_3|$ *and* $|a_4|$for functions in these two new subclasses for functions. Future work making use of the values of a2 a3 and a4 we can caluculate Hankel determinant coefficient for the bi- spirallike function classes.

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