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The Density Matrix Description of Matrix Shell Systems Associated with the Matrix Shell Model formalism

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Abstract:- The article presents a mathematical framework to associate a Matrix Shell system (even type or of the odd type) with a unit trace, hermitian, positive definite or positive semi-definite matrix of order '2' (Density matrix representation of single qubit quantum states). This framework therefore, allows any evolution scheme defined on the fundamental matrix space associated with the Matrix Shell system and consistent with the Matrix shell model formalism to be mapped to an evolution on or inside of the unit solid sphere, centered at the origin, in the three dimensional space(Bloch sphere representation of single qubit quantum states)

Demonstration of the proposed mathematical framework is presented through its application on certain subsets of the Complex Matrix spaces $M_{2\times2}(C)$ and $M_{4\times4}(C)$ and on certain numerical examples from $M_{3\times3}(C)$.

***Keywords**:- Matrix Shell Model formalism, Composite pathways, Directional states, State-Interaction Matrix description of Matrix shells and of the Matrix Shell system, Density matrix representation of quantum states, Bloch sphere representation of single qubit quantum states.*

I. INTRODUCTION

The Matrix shell model framework [9,10,11,12] presents a visualization of a finite, complex square matrix as constituted of Identically structured, overlapping algebraic units termed as "Matrix Shells" , [9,10] studies and presents the diversity aspect associated with this framework by consideration of the concept of dual directionality at the level of the basic pathways comprising the matrix shells, resulting in multiple possible configuration states to be associated with a Matrix shell, in [10], the study focuses on determination of the preferential configurations of the individual matrix shells and the Matrix shell system, through the consideration of eigen-spectral proximity with respect to the ordered eigen-spectrum of matrix shell baseline matrices and the baseline matrix associated with the matrix shell system.

It can be observed that reduction in computational requirements and simplicity in the framework can be obtained by incorporating certain assumptions and additional restrictions on the model. [11] presents such a simplistic framework, termed as the 'Reduced framework', which removes the intricacies and computations contributed by matrix shell interaction elements by setting each of them to unity, the vast diversity aspect which resulted in enormous computational requirements is drastically reduced by considering an averaging method incorporated at the level of the basic pathways, this resulted in a single State-Interaction matrix description to be associated with a particular Matrix shell.

The present article introduces another simplified version of the matrix shell model framework; it considers the dual directionality aspect only at the level of Composite pathways α and β constituting each Matrix shell. The consideration of the directional duality at a higher hierarchical structure results in drastic simplification on the mathematical construction of the State-Interaction matrix descriptions, which can now be formulated as hermitian, positive definite or positive semi-definite matrices of order 2 , the fundamental requirement of Matrix shell model framework which restricts the numerical sample of the fundamental matrix associated with the Matrix shell system to be non-zero, ensures non-zero trace of the State-Interaction matrix description associated with the Matrix Shell system. This results in possibility of introducing an additional structural aspect; The Matrix Shell system, in each of its accessible numerical sample over the fundamental matrix space, can be associated with a unit trace, hermitian, positive definite or positive semi-definite matrix of order 2, these type of matrices have interpretation as 'Density matrix descriptions' of single Qubit quantum states [1,2,4,8,14,17,20,22,23,25,26,27] and hence, can be associated with three dimensional real vectors of magnitude less than or equal to unity. Therefore, in light of the framework proposed in this article, any matrix evolution scheme defined on the fundamental matrix space and consistent with the Matrix Shell model framework can be mapped into an evolution on 3 dimensional real space, more precisely, on the surface and inside of a solid sphere of unit radius in three dimensions, which has interpretation as the 'Bloch sphere representation' [1,2,4,8,14,17,20,22,23,25,26,27] of single Qubit quantum states.

The article presents the underlying mathematical framework and the relevant analytical expressions and results. The proposed framework is demonstrated through case studies and numerical Illustrations.

II. NOTATIONS

- N_0 denotes the set of all natural numbers
- C denotes the set of all complex numbers
- $N = 2n$ is used to denote the Even type Matrix Shell systems, n=1,2,3,.... , $n \in N_0$
- $N = 2n-1$ is used to denote the Odd type Matrix Shell systems, n=1,2,3,.... , $n \in N_0$
- $A(\lambda, N)$ denotes the Matrix Shells associated with the Matrix Shell system 'N' , where $\lambda = 0,1,...(n-1)$
- $A_{N\times N}$ denotes the Fundamental Matrix associated with the Matrix Shell system "N" , $A_{N\times N} = [a_{ij}], i = 1,2,....,N; j = 1,2,....,N$
- $M_{s\times s}(C)$ denotes the Complex Matrix space of order 's'
- $Trace(X_{s\times s})$ denotes the trace of the matrix $X_{s\times s}$
- X^{-1} denotes the inverse of the invertible matrix $X_{s\times s}$
- Y^H denotes the Hermitian conjugate of the matrix Y, $Y \in M_{s\times t}(C)$
- $R^3(R)$ denotes the Real Euclidean space of dimension 3
- C^s denotes the Complex coordinate space of dimension 's'
- $\theta^\bullet$ denotes the complex conjugate of the complex number θ, $i \in C$, such that: $i^2 = -1$
- $|\theta|$ denotes the modulus of the complex number θ
- $\|w\|_2$ denotes the Euclidean norm (vector 2-norm) of the vector $w_{s\times 1} \in C^s$
- $|W\rangle = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_q \end{bmatrix}_{q\times 1}, \langle V| = [v_1^\bullet \quad v_2^\bullet \quad . \quad . \quad v_p^\bullet]_{1\times p}, B = \begin{bmatrix} b_{ij} \end{bmatrix}_{p\times q}, \langle V|B|W\rangle = \sum_{i=1}^{p}\sum_{j=1}^{q} b_{ij} v_i^\bullet w_j$
- $\Delta(\lambda, N)$ denotes the Effective State-Interaction Matrix Description of the Matrix Shell $A(\lambda, N)$
- $\Delta(N)$ denotes the Effective State-Interaction Matrix Description of the Matrix Shell System 'N'
- $\rho(N)$ denotes the Density Matrix Description of the Matrix Shell System 'N'
- $\Delta(N \mid A_{N\times N} = B_{N\times N})$ denotes the Effective State-Interaction Matrix Description of the Matrix Shell System 'N' when the fundamental matrix $A_{N\times N}$ takes the numerical sample $B_{N\times N}$, $B_{N\times N} \in M_{N\times N}(C)$
- $\rho(N \mid A_{N\times N} = B_{N\times N})$ denotes the Density Matrix Description of the Matrix Shell System 'N' when the fundamental matrix $A_{N\times N}$ takes the numerical sample $B_{N\times N}$, $B_{N\times N} \in M_{N\times N}(C)$
- $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}, H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}_{2\times 2}, \Sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}, \Sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}_{2\times 2}, \Sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_{2\times 2}$
- $\vec{r}.\vec{\Sigma} = r_1\Sigma_1 + r_2\Sigma_2 + r_3\Sigma_3, \vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}_{3\times 1}, \vec{r} \in R^3(R)$, $|\vec{r}|$ denotes the magnitude of the vector $\vec{r}$
- $U \otimes V$ denotes the Kronecker product between the matrices 'U' and 'V'
- $[revdiag(1...1)]_{s\times s}$ denotes a square matrix of order 's', which has 1's along the reverse diagonal (the diagonal opposite to the main diagonal of the matrix) and 0's everywhere else.

III. MATHEMATICAL FRAMEWORK AND RELATED ANALYSIS

- The notations and relevant terminologies used in this article are in accordance to that used in [9,10,11,12] unless explicitly stated otherwise.

N=2n type Matrix Shell systems

Constituent Matrix Shells: $A(0,2n),....,A(n-1,2n)$

The Matrix Shell $A(\lambda,2n)$, where $\lambda \in \{0,1,....,(n-1)\}$, has the following associated properties:

- Directional states of the Composite Pathway α: $|\alpha_1\rangle,|\alpha_2\rangle$, Directional states of the Composite Pathway β: $|\beta_1\rangle,|\beta_2\rangle$, Where $|\alpha_1\rangle,|\alpha_2\rangle,|\beta_1\rangle,|\beta_2\rangle \in C^{6\lambda+4}$
- $|\alpha_1\rangle = \Sigma_{(6\lambda+4)\times(6\lambda+4)}|\alpha_2\rangle$, $|\alpha_2\rangle = \Sigma_{(6\lambda+4)\times(6\lambda+4)}|\alpha_1\rangle$, $|\beta_1\rangle = \Sigma_{(6\lambda+4)\times(6\lambda+4)}|\beta_2\rangle$, $|\beta_2\rangle = \Sigma_{(6\lambda+4)\times(6\lambda+4)}|\beta_1\rangle$
- $\Sigma_{(6\lambda+4)\times(6\lambda+4)} = \Sigma_1 \otimes [revdiag(1...1)]_{(3\lambda+2)\times(3\lambda+2)}$, thus : $\Sigma^H = \Sigma^{-1} = \Sigma$
- $\|\alpha_1\|_2^{\ 2} = \|\alpha_2\|_2^{\ 2} = a,\ \|\beta_1\|_2^{\ 2} = \|\beta_2\|_2^{\ 2} = b$ where $a \ge 0$, $b \ge 0$
- $\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle = c$, $\langle\alpha_1|\beta_2\rangle = \langle\alpha_2|\beta_1\rangle = d$ where $c,d \in C$

Therefore we have: $\Delta(\lambda, N=2n) = \begin{bmatrix} a(\lambda) & (\frac{1}{2})(c(\lambda)+d(\lambda)) \\ (\frac{1}{2})(c(\lambda)+d(\lambda))^{\bullet} & b(\lambda) \end{bmatrix}_{2\times2}$, where $\lambda = 0,1,....,(n-1)$

Properties of the N=2n Matrix Shell System

- $\Delta(N=2n) = \Delta(0,N=2n) + \sum_{\lambda=1}^{(n-1)} (\frac{1}{2})^{\lambda}\Delta(\lambda,N=2n)$

  $trace[\Delta(N=2n)] = \Omega(N=2n) = trace[\Delta(0,N=2n)] + \sum_{\lambda=1}^{(n-1)} (\frac{1}{2})^{\lambda} trace[\Delta(\lambda,N=2n)]$
- $\rho(N=2n) = (\frac{1}{\Omega(N=2n)})\Delta(N=2n) = (\frac{1}{2})[I_2 + (\vec{r}.\vec{\Sigma})]$, $\vec{r} \in R^3(R)$, $|\vec{r}| \le 1$

N=2n-1 type Matrix Shell systems

Constituent Matrix Shells: $A(0,2n-1),....,A(n-1,2n-1)$

The Matrix Shell $A(0,2n-1)$ has the following properties:

- $|\alpha_1\rangle = |\alpha_2\rangle = |\beta_1\rangle = |\beta_2\rangle = (\frac{1}{4})(a_{nn})\begin{bmatrix}1\\1\\1\\1\end{bmatrix}_{4\times1}$, Therefore: $\Delta(0,N=2n-1) = (\frac{1}{4})|a_{nn}|^2\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}_{2\times2}$

The Matrix Shell $A(\lambda,2n-1)$, where $\lambda \in \{1,....,(n-1)\}$, have the following properties:

- Directional states of the Composite Pathway α: $|\alpha_1\rangle,|\alpha_2\rangle$, Directional states of the Composite Pathway β: $|\beta_1\rangle,|\beta_2\rangle$, $|\alpha_1\rangle,|\alpha_2\rangle,|\beta_1\rangle,|\beta_2\rangle \in C^{6\lambda+1}$

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- $|\alpha_1\rangle = \Sigma_{(6\lambda+1)\times(6\lambda+1)}|\alpha_2\rangle, |\alpha_2\rangle = \Sigma_{(6\lambda+1)\times(6\lambda+1)}|\alpha_1\rangle, |\beta_1\rangle = \Sigma_{(6\lambda+1)\times(6\lambda+1)}|\beta_2\rangle, |\beta_2\rangle = \Sigma_{(6\lambda+1)\times(6\lambda+1)}|\beta_1\rangle,$
- $\Sigma_{(6\lambda+1)\times(6\lambda+1)} = \begin{bmatrix} 0_{3\lambda\times3\lambda} & 0_{3\lambda\times1} & \Sigma_{3\lambda\times3\lambda} \\ 0_{1\times3\lambda} & 1 & 0_{1\times3\lambda} \\ \Sigma_{3\lambda\times3\lambda} & 0_{3\lambda\times1} & 0_{3\lambda\times3\lambda} \end{bmatrix}_{(6\lambda+1)\times(6\lambda+1)}$, where $\Sigma_{3\lambda\times3\lambda} = [revdiag(1...1)]_{3\lambda\times3\lambda}$

  thus: $\Sigma^H = \Sigma^{-1} = \Sigma$

- $\|\alpha_1\|_2^2 = \|\alpha_2\|_2^2 = a$, $\|\beta_1\|_2^2 = \|\beta_2\|_2^2 = b$ where $a \ge 0, b \ge 0$
- $\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle = c$, $\langle\alpha_1|\beta_2\rangle = \langle\alpha_2|\beta_1\rangle = d$, $c, d \in C$

Therefore we have: $\Delta(\lambda, N = 2n-1) = \begin{bmatrix} a(\lambda) & (\frac{1}{2})(c(\lambda)+d(\lambda)) \\ (\frac{1}{2})(c(\lambda)+d(\lambda))^{\bullet} & b(\lambda) \end{bmatrix}_{2\times2}$

Where $\lambda = 1, ..., (n-1)$

<u>Properties of the N=2n-1 Matrix Shell System</u>

- $\Delta(N = 2n-1) = \Delta(0, N = 2n-1) + \sum_{\lambda=1}^{(n-1)} (\frac{1}{2})^{\lambda} \Delta(\lambda, N = 2n-1)$

  $trace[\Delta(N = 2n-1)] = \Omega(N = 2n-1) = trace[\Delta(0, N = 2n-1)] + \sum_{\lambda=1}^{(n-1)} (\frac{1}{2})^{\lambda} trace[\Delta(\lambda, N = 2n-1)]$

- $\rho(N = 2n-1) = (\frac{1}{\Omega(N = 2n-1)})\Delta(N = 2n-1) = (\frac{1}{2})[I_2 + (\vec{r}.\vec{\Sigma})], \ \vec{r} \in R^3(R)$, $|\vec{r}| \le 1$

IV. CASE STUDIES

1. $U(2)$ Subset of the Matrix Space $M_{2\times2}(C)$:

$$U(2) = \{U(\gamma,\mu,\nu) \in M_{2\times2}(C) \,|\, U(\gamma,\mu,\nu) = e^{+i\gamma}\begin{bmatrix} \mu & -\nu^{\bullet} \\ \nu & \mu^{\bullet} \end{bmatrix}_{2\times2}, \gamma \in [0, 2\pi); \mu, \nu \in C, |\mu|^2 + |\nu|^2 = 1\}$$

We have: $|\alpha_1\rangle = e^{+i\gamma}\begin{bmatrix} -\nu^{\bullet} \\ \mu \\ \mu^{\bullet} \\ \nu \end{bmatrix}_{4\times1}$, $|\alpha_2\rangle = e^{+i\gamma}\begin{bmatrix} \nu \\ \mu^{\bullet} \\ \mu \\ -\nu^{\bullet} \end{bmatrix}_{4\times1}$, $|\beta_1\rangle = e^{+i\gamma}\begin{bmatrix} \mu \\ \nu \\ -\nu^{\bullet} \\ \mu^{\bullet} \end{bmatrix}_{4\times1}$, $|\beta_2\rangle = e^{+i\gamma}\begin{bmatrix} \mu^{\bullet} \\ -\nu^{\bullet} \\ \nu \\ \mu \end{bmatrix}_{4\times1}$

Therefore, we have the following set of results:

- $\|\alpha_1\|_2^2 = \|\alpha_2\|_2^2 = a = 2$, $\|\beta_1\|_2^2 = \|\beta_2\|_2^2 = b = 2$, therefore: $a = b = 2$
- $\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle = c = (\mu^{\bullet} - \mu)(\nu + \nu^{\bullet})$, $\langle\alpha_1|\beta_2\rangle = \langle\alpha_2|\beta_1\rangle = d = -(\mu^{\bullet} - \mu)(\nu + \nu^{\bullet})$, therefore: $c + d = 0$
- $\Delta(N = 2 \,|\, A = U(\gamma,\mu,\nu)) = \begin{bmatrix} a & (\frac{1}{2})(c+d) \\ (\frac{1}{2})(c+d)^{\bullet} & b \end{bmatrix}_{2\times2} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$, therefore:

$trace[\Delta(N=2 \mid A=U(\gamma,\mu,\nu))]=4$

- $\rho(N=2 \mid A=U(\gamma,\mu,\nu))=(\frac{1}{2})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{2\times 2}$, $\rho(N=2 \mid A=U(\gamma,\mu,\nu))=(\frac{1}{2})(I_2+\vec{r}.\vec{\Sigma})$, where $\vec{r}\in R^3(R),\ \vec{r}=0_{3\times 1}$

2. $U(2)\otimes U(2)$ Subset of the Matrix Space $M_{4\times 4}(C)$:

$$U(2)\otimes U(2)=\{U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)\in M_{4\times 4}(C) \mid U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)=e^{+i\gamma_1}\begin{bmatrix}\mu_1 & -\nu_1^{\bullet}\\ \nu_1 & \mu_1^{\bullet}\end{bmatrix}_{2\times 2}\otimes e^{+i\gamma_2}\begin{bmatrix}\mu_2 & -\nu_2^{\bullet}\\ \nu_2 & \mu_2^{\bullet}\end{bmatrix}_{2\times 2},$$
$$\gamma_1,\gamma_2\in[0,2\pi);\mu_1,\nu_1,\mu_2,\nu_2\in C,|\mu_1|^2+|\nu_1|^2=1,|\mu_2|^2+|\nu_2|^2=1\}$$

Constituent matrix shells associated with the $N=4$ Matrix Shell system: $A(0,4)$, $A(1,4)$

$$A(0,4):\quad |\alpha_1\rangle=e^{+i\delta}\begin{bmatrix}-\nu_1^{\bullet}\nu_2\\ \mu_1\mu_2^{\bullet}\\ \mu_1^{\bullet}\mu_2\\ -\nu_1\nu_2^{\bullet}\end{bmatrix}_{4\times 1},|\alpha_2\rangle=e^{+i\delta}\begin{bmatrix}-\nu_1\nu_2^{\bullet}\\ \mu_1^{\bullet}\mu_2\\ \mu_1\mu_2^{\bullet}\\ -\nu_1^{\bullet}\nu_2\end{bmatrix}_{4\times 1},|\beta_1\rangle=e^{+i\delta}\begin{bmatrix}\mu_1\mu_2^{\bullet}\\ -\nu_1\nu_2^{\bullet}\\ -\nu_1^{\bullet}\nu_2\\ \mu_1^{\bullet}\mu_2\end{bmatrix}_{4\times 1},|\beta_2\rangle=e^{+i\delta}\begin{bmatrix}\mu_1^{\bullet}\mu_2\\ -\nu_1^{\bullet}\nu_2\\ -\nu_1\nu_2^{\bullet}\\ \mu_1\mu_2^{\bullet}\end{bmatrix}_{4\times 1},$$ where

$\delta=\gamma_1+\gamma_2$

Therefore, we have the following set of results:

- $\|\alpha_1\|_2^{\ 2}=\|\alpha_2\|_2^{\ 2}=\|\beta_1\|_2^{\ 2}=\|\beta_2\|_2^{\ 2}=x_0=2(|\mu_1|^2|\mu_2|^2+|\nu_1|^2|\nu_2|^2)$
- $\langle\alpha_1|\beta_1\rangle=\langle\alpha_2|\beta_2\rangle=\langle\alpha_1|\beta_2\rangle=\langle\alpha_2|\beta_1\rangle=x_1=-(\mu_1^{\bullet}\mu_2+\mu_1\mu_2^{\bullet})(\nu_1^{\bullet}\nu_2+\nu_1\nu_2^{\bullet})$
- $\Delta(0,4)=\begin{bmatrix}x_0 & x_1\\ x_1^{\bullet} & x_0\end{bmatrix}_{2\times 2}$

$$A(1,4):\ |\alpha_1\rangle=\begin{bmatrix}|\theta_{+1}\rangle_{4\times 1}\\ |\theta_{01}\rangle_{2\times 1}\\ |\theta_{-1}\rangle_{4\times 1}\end{bmatrix}_{10\times 1},|\alpha_2\rangle=\begin{bmatrix}|\theta_{+2}\rangle_{4\times 1}\\ |\theta_{02}\rangle_{2\times 1}\\ |\theta_{-2}\rangle_{4\times 1}\end{bmatrix}_{10\times 1},|\beta_1\rangle=\begin{bmatrix}|\phi_{+1}\rangle_{4\times 1}\\ |\phi_{01}\rangle_{2\times 1}\\ |\phi_{-1}\rangle_{4\times 1}\end{bmatrix}_{10\times 1},|\beta_2\rangle=\begin{bmatrix}|\phi_{+2}\rangle_{4\times 1}\\ |\phi_{02}\rangle_{2\times 1}\\ |\phi_{-2}\rangle_{4\times 1}\end{bmatrix}_{10\times 1},$$ where :

$$|\theta_{+1}\rangle=e^{+i\delta}\begin{bmatrix}\nu_1^{\bullet}\nu_2^{\bullet}\\ -\mu_2\nu_1^{\bullet}\\ -\mu_1\nu_2^{\bullet}\\ \mu_1\mu_2\end{bmatrix}_{4\times 1},|\theta_{-1}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1^{\bullet}\mu_2^{\bullet}\\ \mu_1^{\bullet}\nu_2\\ \mu_2^{\bullet}\nu_1\\ \nu_1\nu_2\end{bmatrix}_{4\times 1},|\theta_{+2}\rangle=e^{+i\delta}\begin{bmatrix}\nu_1\nu_2\\ \mu_2^{\bullet}\nu_1\\ \mu_1^{\bullet}\nu_2\\ \mu_1^{\bullet}\mu_2^{\bullet}\end{bmatrix}_{4\times 1},|\theta_{-2}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1\mu_2\\ -\mu_1\nu_2^{\bullet}\\ -\mu_2\nu_1^{\bullet}\\ \nu_1^{\bullet}\nu_2^{\bullet}\end{bmatrix}_{4\times 1},$$

$$|\theta_{01}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1\mu_2^{\bullet}\\ \mu_1^{\bullet}\mu_2\end{bmatrix}_{2\times 1},|\theta_{02}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1^{\bullet}\mu_2\\ \mu_1\mu_2^{\bullet}\end{bmatrix}_{2\times 1},|\phi_{01}\rangle=e^{+i\delta}\begin{bmatrix}-\nu_1\nu_2^{\bullet}\\ -\nu_1^{\bullet}\nu_2\end{bmatrix}_{2\times 1},|\phi_{02}\rangle=e^{+i\delta}\begin{bmatrix}-\nu_1^{\bullet}\nu_2\\ -\nu_1\nu_2^{\bullet}\end{bmatrix}_{2\times 1},$$

$$|\phi_{+1}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1\mu_2\\ \mu_1\nu_2\\ \mu_2\nu_1\\ \nu_1\nu_2\end{bmatrix}_{4\times 1},|\phi_{-1}\rangle=e^{+i\delta}\begin{bmatrix}\nu_1^{\bullet}\nu_2^{\bullet}\\ -\mu_2^{\bullet}\nu_1^{\bullet}\\ -\mu_1^{\bullet}\nu_2^{\bullet}\\ \mu_1^{\bullet}\mu_2^{\bullet}\end{bmatrix}_{4\times 1},|\phi_{+2}\rangle=e^{+i\delta}\begin{bmatrix}\mu_1^{\bullet}\mu_2^{\bullet}\\ -\mu_1^{\bullet}\nu_2^{\bullet}\\ -\mu_2^{\bullet}\nu_1^{\bullet}\\ \nu_1^{\bullet}\nu_2^{\bullet}\end{bmatrix}_{4\times 1},|\phi_{-2}\rangle=e^{+i\delta}\begin{bmatrix}\nu_1\nu_2\\ \mu_2\nu_1\\ \mu_1\nu_2\\ \mu_1\mu_2\end{bmatrix}_{4\times 1}$$

Therefore, we have the following set of results:

- $\|\alpha_1\|_2^{\;2} = \|\alpha_2\|_2^{\;2} = a = 2(1+|\mu_1|^2|\mu_2|^2)$, $\|\beta_1\|_2^{\;2} = \|\beta_2\|_2^{\;2} = b = 2(1+|\nu_1|^2|\nu_2|^2)$
- $\langle\alpha_1|\beta_1\rangle = \langle\alpha_2|\beta_2\rangle = c = (\mu_1-\mu_1^\bullet)(\mu_2-\mu_2^\bullet)(\nu_1\nu_2+\nu_1^\bullet\nu_2^\bullet)-(\mu_1^\bullet\mu_2)(\nu_1\nu_2^\bullet)-(\mu_1\mu_2^\bullet)(\nu_1^\bullet\nu_2)$
- $\langle\alpha_1|\beta_2\rangle = \langle\alpha_2|\beta_1\rangle = d = (\nu_1+\nu_1^\bullet)(\nu_2+\nu_2^\bullet)(\mu_1\mu_2+\mu_1^\bullet\mu_2^\bullet)-(\mu_1^\bullet\mu_2)(\nu_1^\bullet\nu_2)-(\mu_1\mu_2^\bullet)(\nu_1\nu_2^\bullet)$
- $$\Delta(1,4) = \begin{bmatrix} a & (\frac{1}{2})(c+d) \\ (\frac{1}{2})(c+d)^\bullet & b \end{bmatrix}_{2\times 2}$$
- $\Delta(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)) = \Delta(0,4)+(\frac{1}{2})\Delta(1,4)$

$$\Delta(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)) = \begin{bmatrix} x_0+(\frac{1}{2})a & x_1+(\frac{1}{4})(c+d) \\ x_1^\bullet+(\frac{1}{4})(c+d)^\bullet & x_0+(\frac{1}{2})b \end{bmatrix}_{2\times 2}$$

- $trace[\Delta(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2))] = \Omega = 2x_0+(\frac{1}{2})(a+b) = 2+5(|\mu_1|^2|\mu_2|^2+|\nu_1|^2|\nu_2|^2)$
- $\rho(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)) = (\frac{1}{\Omega})\Delta(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2))$

$$\rho(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)) = \left(\frac{1}{2x_0+(\frac{1}{2})(a+b)}\right)\begin{bmatrix} x_0+(\frac{1}{2})a & x_1+(\frac{1}{4})(c+d) \\ x_1^\bullet+(\frac{1}{4})(c+d)^\bullet & x_0+(\frac{1}{2})b \end{bmatrix}_{2\times 2}$$

- $\rho(N=4\,|\,A=U(\gamma_1,\mu_1,\nu_1,\gamma_2,\mu_2,\nu_2)) = (\frac{1}{2})(I_2+\vec{r}.\vec{\Sigma})$, $\vec{r}\in R^3(R)$, $\vec{r} = \begin{bmatrix} r_1 \\ 0 \\ r_3 \end{bmatrix}_{3\times 1}$, $r_1^{\;2}+r_3^{\;2}\le 1$

<u>Numerical Illustrations</u>

- $\{I_2,\Sigma_1,\Sigma_2,\Sigma_3\}\otimes\{I_2,\Sigma_1,\Sigma_2,\Sigma_3\}$ Basis set of the Matrix Space $M_{4\times 4}(C)$:

- $\rho(N=4\,|\,A=I_2\otimes I_2) = \rho(N=4\,|\,A=I_2\otimes\Sigma_3) = \rho(N=4\,|\,A=\Sigma_3\otimes I_2) = \rho(N=4\,|\,A=\Sigma_3\otimes\Sigma_3) = \rho_1$

$$\rho_1 = (\frac{1}{7})\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2}, \quad \text{therefore}: \ \vec{r} = \begin{bmatrix} 0 \\ 0 \\ (1/7) \end{bmatrix}_{3\times 1},$$

- $\rho(N=4\,|\,A=\Sigma_1\otimes\Sigma_1) = \rho(N=4\,|\,A=\Sigma_1\otimes\Sigma_2) = \rho(N=4\,|\,A=\Sigma_2\otimes\Sigma_1) = \rho(N=4\,|\,A=\Sigma_2\otimes\Sigma_2) = \rho_2$

$$\rho_2 = (\frac{1}{7})\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}_{2\times 2}, \vec{r} = \begin{bmatrix} 0 \\ 0 \\ -(1/7) \end{bmatrix}_{3\times 1}$$

- $\rho(N=4\,|\,A=I_2\otimes\Sigma_1)=\rho(N=4\,|\,A=\Sigma_1\otimes I_2)=\rho(N=4\,|\,A=I_2\otimes\Sigma_2)=\rho(N=4\,|\,A=\Sigma_2\otimes I_2)=\rho_3$
  $\rho(N=4\,|\,A=\Sigma_1\otimes\Sigma_3)=\rho(N=4\,|\,A=\Sigma_3\otimes\Sigma_1)=\rho(N=4\,|\,A=\Sigma_2\otimes\Sigma_3)=\rho(N=4\,|\,A=\Sigma_3\otimes\Sigma_2)=\rho_3$
  $\rho_3=(\frac{1}{2})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{2\times 2},\ \vec{r}=0_{3\times 1}$

- $\rho(N=4\,|\,A=H_2\otimes H_2)=(\frac{1}{6})\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}_{2\times 2},\quad \vec{r}=\begin{bmatrix}-(1/3)\\ 0\\ 0\end{bmatrix}_{3\times 1}$

- $N=3$ Matrix Shell system:

$$G_1=\begin{bmatrix}1 & 0 & 1\\ 0 & 4 & 0\\ 1 & 0 & 1\end{bmatrix}_{3\times 3},\quad G_2=\begin{bmatrix}1 & 0 & -i\\ 0 & 4 & 0\\ i & 0 & 1\end{bmatrix}_{3\times 3},\quad G_3=\begin{bmatrix}1 & 0 & 1\\ 0 & 4 & 0\\ 1 & 0 & -1\end{bmatrix}_{3\times 3}$$

We have the following results:

- $\Delta(0,3\,|\,A=G_1)=\Delta(0,3\,|\,A=G_2)=\Delta(0,3\,|\,A=G_3)=\begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix}_{2\times 2}$

- $\rho(N=3\,|\,A=G_1)=(\frac{1}{2})\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}_{2\times 2},\quad \vec{r}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}_{3\times 1}$

- $\rho(N=3\,|\,A=G_2)=\rho(N=3\,|\,A=G_3)=(\frac{1}{14})\begin{bmatrix}7 & 6\\ 6 & 7\end{bmatrix}_{2\times 2},\quad \vec{r}=\begin{bmatrix}(6/7)\\ 0\\ 0\end{bmatrix}_{3\times 1}$

V. DISCUSSION AND CONCLUSION

The objective of this article is to analyze and understand the correspondence between the properties of the matrices belonging to the fundamental matrix space and their mapped counterpart; the unit-trace hermitian positive definite/positive semi-definite matrices of order 2, i.e. the density matrix descriptions, in light of the proposed mathematical framework.

The case studies considered in this article refer to subsets of the complex matrix spaces of order 2 and of order 4 which can be associated with N=2 and N=4 Matrix Shell systems respectively. The subset U(2) is the set of all unitary matrices of order 2, the interrelationships existing among the matrix elements of a U(2) matrix results in each such matrix being mapped to the same density matrix description: $(1/2)I_{2\times 2}$ and hence, are all mapped to the origin point of the Bloch sphere representation. The subset $U(2)\otimes U(2)$ of the complex matrix space $M_{4\times 4}(C)$ is observed to have such a degeneracy lifted owing to more intricate interrelationships among its matrix elements and differential overlap features with respect to the presented mathematical framework. These subset of matrices are found to be mapped to Bloch sphere points on or inside of the sphere that are confined in the XZ plane (under standard conventions of coordinate axes assignment pertaining to the Bloch sphere representation). The numerical illustration presented alongside this discussion provides a numerical demonstration of this feature; the 16 elements of the basis set of the matrix space $M_{4\times 4}(C)$ splits into three categories consisting of 4, 4 and 8 elements, which are mapped to points inside of the Bloch sphere on positive and the negative side of the Z axis and to the origin point of the Bloch sphere, respectively.

It can thus be observed that the interplay between the existing interrelationships among the elements of the fundamental matrix and their behavior under the presented mathematical framework determine the mapped point on the associated Bloch sphere representation. The three numerical samples of the N=3 Matrix Shell system considered in the article demonstrates that the mapped points can lie both on and inside of the sphere under the mathematical conditions imposed by the framework, it can be observed here that one of the three is mapped to the unit vector on positive X axis while the other two are mapped to the same interior point.

Therefore, devising appropriate evolution methodologies on the fundamental matrix space consistent with the Matrix shell model framework, it is possible to map and analyze this evolution behavior as trajectories/sequence of points in real three dimensional space, confined on and inside of the Bloch sphere mathematical construct.

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