ISSN No:-2456-2165

‘’New Methods to Solve some Types of the First Order D.E.s .’’

Abdulhussein Kadhum Alsultani
Retired Mechanical Engineer
Iraq-Baghdad

Abstract:- There are six Rules to solve the non exact differential equations (D.E.s) so we must learn and keep them and choosethe proper Rule to find theIntegrating Factor [μ(x,y)] of the D.E. . so we must multiply I.F. by the equation to make it an exactand then integrate it , but sometimes this takes long time , more paper and more mistakes .There are two types of the first order differential equations either an exact or non exact and the exact D.E. can solved (integrated)directly but for non exact we must find the integrating factor μ(x,y). The exact D.E. usually is the dirivative of a function of one or many functions each one of them is linear and iindependent on the others.

But if one function or more are rational or logarthmic so the D.E. becomes non exact.

For me I discovred new methods to find the integrating factor in one prcedure to solve many types of the differential equations of the first orderin short time with simple ways (may be without integration) and this will increase the FLEXIBILITY tochoose the simple and short procedure.

There are two types of the non exact differential equations can not solved by my method :
1 . The homogeneous D.E. which contents the same term in M & N togerther .
2 . The editing D.E. which we can not find the right integrating factor of it from M and N .

Keywords:- *By checking most of the D.E.s so you may write at least one of its functions directly without integration then from that function you can calculate the integrating factor* $\boxed{Alsultani\ I.F.}$ *Simply .*

1. INTRODUCTION

Referring to my old article intitled (New way to solve the first order linear D.E.which consists of three terms)we find:
$ax^m y^n + bx^s = c_1$ **where** $(a, b, c_1, m\ and\ n)$ **are constants** [5]
when $differentiating it w.r.to x we find$
$anx^m y^{n-1}dy + amx^{m-1}y^n dx + bsx^{s-1}dx = 0$ **(1)**

then $(amx^{m-1}y^n + bsx^{s-1})dx + anx^m y^{n-1}dy = 0$ **(2)**

↑↑

Positive sign positive sign from ‘’Equation (2)’’ we find that :
M=(a$mx^{m-1}y^n + bsx^{s-1}$ $\quad and\quad N = anx^m y^{n-1}$

$\frac{\partial M}{\partial y} = amnx^{m-1}y^{n-1}$ **and** $\frac{\partial N}{\partial x} = amnx^{m-1}y^{n-1}$ **so** $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ **i.e.the D.E. is an exact**
so when dividing by the coefficient of (dy) we get
$\frac{dy}{dx} + \frac{amy}{anx} = -\frac{bsx^{s-m-1}}{any^{n-1}}$ **(3)**

then if we take the left side of ‘’Equation (3) we can write
$\frac{dy}{dx} + a\frac{my}{nx} \to a(\frac{ndy}{y} + \frac{mdx}{x}) \to a(n\int\frac{dy}{y} + m\int\frac{dx}{x}) \to a(nlny + mlnx) \to ae^{nlny+mlnx} \to a\,x^m y^n$ **so**
.f(x,y)=a$x^m y^n$ $\boxed{without integration}$ **(4)**

Then if in some D.E.s we can obtain $the expression \frac{dy}{dx} + a\frac{my}{nx}$ **that means we find separated function which it is a part of the all functions that differentiated previously.**

So any D.E. is an exactbut it is not if and only if there is one of the following causes :
1 . there is abbreviated(abstracted) variablefrom it .
2 . one of the functions or more is rational .
3 . one of the functions or more is logarithmic .

but if there is one function like ;
f(x,y)=a $x^m y^n = c$ **alone then when differentiate it w.r.to x**

$\mathrm{Df}(x,y) = \frac{df}{dx} + \frac{df}{dy}\frac{dy}{dx}$ so $\mathrm{Df}(x,y) = anx^m y^{n-1}dy + amx^{m-1}y^n dx = o$

Then it iseither the common factor $= ax^{m-1}y^{n-1} = o$ **or the rest of the D.E. i.e.**
nxdy +mydx=0
Now if for example the whole function F(x,y)=f(x,y) +g(x,y)
So f(x,y)=$ax^m y^n$ **and g(x,y)=** $\frac{b}{x^s}$
Then F(x,y)= $ax^m y^n + \frac{b}{x^s}$**=c where (a ,b and c)are arbitrary constants**
so by thedifferentiationwe get :
$F' = anx^m y^{n-1}dy + amx^{m-1}y^n dx - \frac{bsdx}{x^{s+1}} = 0$ **then :**
.$anx^{m+s+1}y^{n-1}dy + amx^{m+s}y^n dx - bsdx = o$ **by ordering**
$(amx^{m+s}y^n - bs)dx + anx^{m+s+1}y^{n-1}dy = 0$ **(5)**
M=$amx^{m+s}y^n - bs and N = anx^{m+s+1}y^{n-1}$

$$\frac{\partial M}{\partial y} = amnx^{m+s}y^{n-1} and \frac{\partial N}{\partial x} = an(m+s+1)x^{m+s}y^{n-1}$$

So$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ **and the D.E. becomes non exact**
But from ''Equation (5) wefind that ''
$\frac{dy}{dx} + \frac{amx^{m+s}y^n}{anx^{m+s+1}y^{n-1}} \rightarrow \frac{dy}{dx} + a\frac{my}{nx} so f(x,y) = ax^m y^n$ **as in '' (4),'' [here I separate the coefficient (a) alone]**

Now if we differentiate f(x,y) partially w.r. to (y) for example we get

$$\frac{\partial f}{\partial y} = \frac{\partial(ax^m y^n)}{\partial y} = anx^m y^{n-1} then this result must equal the derivative$$

$(anx^{m+s-1}y^{n-1})$
But it is not equal so there must be integrating factor μ(x,y)
.$anx^m y^{n-1} = \mu(anx^{m+s-1}y^{n-1})$ **then**
$\mu = \frac{anx^m y^{n-1}}{anx^{m+s+1}y^{n-1}} = \frac{1}{x^{s+1}}$ **I called this** $\boxed{Alsultani I.F.}$
and to proof thatlet
$M_1 = \mu M = \frac{1}{x^{s+1}}(amx^{m+s}y^n - bs) = anx^{m-1}y^{n-1} + \frac{bs}{x^{s+1}}$
then $\frac{\partial M_1}{\partial y} = amnx^{m-1}y^{n-1}$
$N_1 = \mu N = \frac{1}{x^{s+1}}(anx^{m+s+1}y^{n-1}) = anx^m y^{n-1})$ **then** $\frac{\partial N_1}{\partial x} = amnx^{m-1}y^{n-1}$
$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ **now an exact D.E. .**
so F(x,y)=f(x,y)+g(x,y)= $ax^m y^n + \int \frac{-bsdx}{(x^{s+1})} + c = ax^m y^n + \frac{b}{x^s} + c$ **solution**
From now to the end of the research I will deal with the D.E.s which consist of four or more terms only .
types D.E.s

1. $\frac{dy}{dx} + \frac{my}{nx}$ *then* its solution (function) is $f = x^m y^n$
2. $\frac{dy}{dx} + \frac{-my}{nx}$ *then* its solution (function) is $f = \frac{y^n}{x^m}$
3. $\frac{dy}{dx} + \frac{amy}{anx} \rightarrow \frac{dy}{dx} + a\frac{my}{nx} then f = ax^m y^n$
4. $\frac{dy}{dx} + \frac{-amy}{anx} \rightarrow \frac{dy}{dx} + a\frac{-my}{nx} then f = a\frac{y^n}{x^m}$
5. $\frac{dy}{dx} + \frac{2xy}{2x^2} \rightarrow \frac{dy}{dx} + \frac{2y}{2x}$**so either f =** $x^2y^2 and its derivative = 2x^2ydy + 2xy^2dx$
.or f = 2xy and its derivative = 2xdy+2ydx and this must be checked in M and N. .

let us take the following cases :

- **CASE 1.**

F(x,y)=f(x,y)+g(x,y)+k(x,y)=c
Where f(x,y)= $ax^m y^n$ **, g(x,y)=**$\frac{b}{x}$ **and k(x,y)= rln|y| where (a ,b and r are constants)**

$$F' = f' + g' + k'$$

$F' = anx^m y^{n-1}dy + amx^{m-1}y^n dx - \frac{bdx}{x^2} + \frac{rdy}{y} = 0$ **(we see here that**$\mu = \frac{1}{x^2y}$**)**
$anx^{m+2}y^n dy + amx^{m+1}y^{n+1}dx - bydx + rx^2dy = 0$

$(amx^{m+1}y^{n+1} - \mathbf{b}y)dx + (\mathbf{a}nx^{m+2}y^{n} + \mathbf{r}x^{2})dy = 0$
↑ **positive sign** ↑ **positive sign**

.y$(amx^{m+1}y^{n} - \mathbf{b})$**dx+**$x^{2}(anx^{m}y^{n} + \mathbf{r})dy = 0$ **(outside the brackets y and** x^{2}**)**

And$\frac{dy}{dx} + \frac{ay(mx^{m+1}y^{n}}{ax^{2}(nx^{m}y^{n})}$ → $\frac{dy}{dx} + \mathbf{a}\frac{my}{nx}$ → $\int df = \mathbf{a}[\int \frac{ndy}{y} + \int \frac{mdx}{x}] = a[lny^{n} + lnx^{m}]$ **f=a**$[e^{lny^{n}+x^{m}}] = ax^{m}y^{n} then \frac{\partial f}{\partial y} = \frac{\partial \mathbf{a}(x^{m}y^{n})}{\partial y} = \mathbf{a}nx^{m}y^{n-1} = \mu[\mathbf{a}x^{2}(nx^{m}y^{n}]$ **so** $\mu\frac{1}{x^{2}y}$

Or $\frac{dy}{dx} + \frac{-\mathbf{b}y}{rx^{2}}$ → $\frac{rdy}{y} + \frac{-\mathbf{b}dx}{x^{2}}$ → $\mathbf{r}\int\frac{dy}{y} - \mathbf{b}\int\frac{dx}{x^{2}} = \mathbf{r}ln|y| + \frac{\mathbf{b}}{x} and\ \mu = \frac{1}{x^{2}y}$

$$\text{then } f = \frac{1}{x^{2}y}\int (amx^{m+1}y^{n+1})dx = a\,x^{m}y^{n}$$

F = $\mathbf{r}\ln|\mathbf{y}| + \frac{\mathbf{b}}{x} + \int \frac{1}{x^{2}y}(amx^{m+1}y^{n})dx + c = r\ln y + \frac{\mathbf{b}}{x} + ax^{m}y^{n}$**+c solution**

- **CASE 2.**

But when f(x,y)= $\frac{1}{m}x^{m}y^{m}$ **i.e. the multiplied factor = the powers of both ofx and y**

F=$\frac{1}{m}x^{m}y^{m} + \frac{\mathbf{b}}{x}$ **+r ln|y| then**

$F' = \frac{m}{m}x^{m}y^{m-1}dy + \frac{m}{m}x^{m-1}y^{m}dx - \frac{bdx}{x^{2}} + \frac{rdy}{y} = 0$

$(x^{m+1}y^{m+1} - \mathbf{b}y)dx + (x^{m+2}y^{m} + \mathbf{r}x^{2})dy = 0$

Or y$(x^{m+1}y^{m} - \mathbf{b})dx + x^{2}(x^{m}y^{m} + \mathbf{r})dy$**=0 (also here we see y and** x^{2} **outside the brackets) so although we say that :**

$\frac{dy}{dx} + \frac{x^{m+1}y^{m+1}}{x^{m+2}y^{m}}$ → $\frac{dy}{dx} + \frac{y}{x}$ **this result is not useful because this will tell us that f=xy**

And its derivative + xdy + ydx which are not exisisting in M and N but from the experience since x^{2} **is founded in the end of N and it is multiplied by dy then let**

.z=xy and f=$\frac{1}{\mathrm{m}}(z)^{\mathrm{m}}$ **[here** $(x^{m}{=}x^{2-1}forx^{2})dyand\ (x^{m} = x^{3-1} = x^{2}forx^{3})dy$

$and\ x^{m} = x^{4-1} = x^{3}$**.for** x^{4}**)dy .**

But also we can take the right insides of the brackets

$\frac{dy}{dx} + \frac{-\mathbf{b}y}{rx^{2}} andbyseparatingthevariables$ **we get**

$$\frac{rdy}{y} + \frac{-\mathbf{b}dx}{x^{2}} then\ \mathbf{r}\int\frac{dy}{y} - b\int\frac{dx}{x^{2}} \rightarrow \mathbf{r}\ln|y| + \frac{\mathbf{b}}{x} whichareg(x,y) + k(x,y)$$

Also here$\mu = \frac{1}{x^{2}y}$

$$M_{1} = \mu M = \frac{y}{x^{2}y}(x^{m+1}y^{m+1} - \mathbf{b}) = x^{m-1}y^{m} - bx^{-2}$$

Then $\frac{\partial M_{1}}{\partial y} = x^{m-1}y^{m-1}$

N₁= $\mu N = \frac{1}{x^{2}y}a[x^{2}(x^{m}y^{m} + \mathbf{r})] = ax^{m}y^{m-1} + \frac{r}{y}then\frac{\partial N_{1}}{\partial x} = ax^{m-1}y^{m-1}$

Then $\frac{\partial M_{1}}{\partial Y} = \frac{\partial N_{1}}{\partial x} sobecomesanexact$

So F= rln $|y| + \frac{\mathbf{b}}{x} + \int\frac{1}{x^{2}y}(\ x^{m+1}y^{m+1})dx = \mathbf{r}lny + \frac{\mathbf{b}}{x} + \frac{1}{m}x^{m}y^{m} + c$**solution**

Notes

The same as in case 1 .

For example

F=$\frac{1}{3}x^{3}y^{3} + \frac{1}{x}$ **+ ln|y| =c**

So $F' = x^{3}y^{3-1}dy + x^{3-1}y^{3}dx - \frac{dx}{x^{2}} + \frac{dy}{y} = 0$

$(x^{4}y^{4} - y)dx + (x^{5}y^{3} + x^{2})dy = 0$ **and**

y$(x^{4}y^{3} - 1)dx + x^{2}(x^{3}y^{3} + 1)dy$**=0 also** $\mu = \frac{1}{x^{2}y}$

$$\frac{dy}{dx} + \frac{x^{4}y^{4}}{x^{5}y^{3}} \rightarrow \frac{dy}{dx} + \frac{y}{x}so z = xy \text{ and } \mathrm{m} = 4 - 1 = 3\ thenf = \frac{1}{m}z^{m} = \frac{1}{3}(xy)^{3}$$

$\frac{dy}{dx} + \frac{-y}{x^{2}} andbyseparatingthevariables$ **we get**

$$\frac{dy}{y} + \frac{-dx}{x^{2}} then \int\frac{dy}{y} - \int\frac{dx}{x^{2}} \rightarrow \ln|y| + \frac{1}{x} whichareg(x,y) + k(x,y)$$

F= ln $|y| + \frac{1}{x} + \int\frac{1}{x^{2}y}(\ x^{4}y^{4})dx = lny + \frac{1}{x} + \frac{1}{3}x^{3}y^{3} + c$**solution**

Another example
Solve $(x^4y - y^4)dx + (-x^5 + 2x^2y^3)dy = 0$
Solution
$\frac{dy}{dx} + \frac{x^4y}{-x^5} \rightarrow \frac{dy}{dx} + \frac{y}{-x} then z = \frac{x}{y} we know y^m = y^{4-1}$ so m=3 and $f = \frac{x^m}{my^m} = \frac{x^3}{3y^3}$

$\frac{\partial f}{\partial x} = \frac{\partial(\frac{x^3}{3y^3})}{\partial x} = \frac{x^2}{y^3} = \mu(x^4y) so \mu = \frac{1}{x^2y^4}$
Then F= $\frac{x^3}{3y^3} + \int \frac{1}{x^2y^4}(-y^4)dx + \int \frac{1}{x^2y^4}(2x^2y^3)dy$ +c $= \frac{x^3}{3y^3} + \frac{1}{x} + 2lny +$ c **solution**

- **CASE 3 .**

Now let the function f(x,y)=$\frac{y^n}{x^m}$
Then F= $\frac{y^n}{x^m}$ + ln|y| +$\frac{1}{x}$
$F' = \frac{nx^my^{n-1}dy - mx^{m-1}y^ndx}{x^{2m}} + \frac{dy}{y} - \frac{dx}{x^2}$ **then**
$nx^{-m+2}y^ndy - mx^{-m+1}y^{n+1}dx + x^{2m}dy - x^{2m-2}ydx = 0$,
$(-mx^{-m+1}y^{n+1} - x^{2m-2}y)dx + (nx^{-m+2}y^n + x^{2m})dy = 0$
↑ $negativsign$ ↑ **positive sign**

$$\frac{dy}{dx} + \frac{-mx^{-m+1}y^{n+1}}{nx^{-m+2}y^n} \rightarrow \frac{dy}{dx} + \frac{-my}{nx} \rightarrow n\int\frac{dy}{y} - m\int\frac{dx}{x} = nln|y| - mln|x| \rightarrow f = \frac{y^n}{x^m}$$

$$\frac{\partial f}{\partial y} = \frac{\partial(\frac{y^n}{x^m})}{\partial y} = \frac{ny^{n-1}}{x^m} = \mu(nx^{-m+2}y^n) \rightarrow \mu = \frac{ny^{n-1}}{x^m(nx^{-m+2}y^n)} = \frac{1}{x^2y}$$

$$F = \frac{y^n}{x^m} - \frac{1}{x^2y}\int ydx + \frac{1}{x^2y}\int x^2dy = \frac{y^n}{x^m} + \frac{1}{x} + \ln|y| + c$$

Notes
a) the opposite signs (negative and positive) means rational function .
b) the parts of $\mu are outside of M bracket and inside of N bracket$

- **CASE 4**

Sometimes the D.E. is homogeneous which causes the abbreviation of the number of the power of one of the variables so we must multiply it by the same number . .

Example
Solvethe homomgeneous D.E.

$$(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0 \quad (6)$$

Solution

$$.y(y^2 - 2x^2)dx + x(2y^2 - x^2)dy = 0 \quad (7)$$

We see μ=xy and its parts (x & y) are outside the bracket but inside the brackets there are homogeneous variables of power (2). so multiply by 2 and also in (6) there is$(x^3) which means x^{2+1}$ **so the power is (2) .**

$$2\,y(y^2 - 2x^2)dx + 2x(2y^2 - x^2)dy = 0$$

From eq. (1) $\frac{dy}{dx} + \frac{2y^3}{4xy^2} \rightarrow \frac{dy}{dx} + \frac{2y}{4x}$ **so** $f = x^2y^4$ $i.e.$ $\boxed{without integration}$
$\frac{dy}{dx} + \frac{-4x^2y}{-2x^3} \rightarrow \frac{dy}{dx} + \frac{-3y}{-2x}$ **so** $g = -x^4y^2$ i.e. $\boxed{without integration}$

F= $x^2y^4 - x^4y^2 + c$ **solution**
Its better to find the I.F. from f(x,y) so $\frac{\partial(x^2y^4)}{\partial y} = 4x^2y^3$
$4x^2y^3 = \mu(4xy^2) then \boxed{\mu = xy}$ **so** g(x,y) = $\int xy(-4x^2y)dx = -x^4y^2$
Or $M_1 = \mu M = xy(2y^3 - 4x^2y) = 2xy^4 - 4x^3y^2 then \frac{\partial M_1}{\partial y} = 8xy^3 - 8x^3y$

$N_1=\mu N=xy(4xy^2-2x^3)=4x^2y^3-2x^4y$ then $\frac{\partial N_1}{\partial x}=8xy^3-8x^3y$
$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$ i.e. an exact
$F=\int(2xy^4-4x^3y^2)dx+c=x^2y^4-x^4y^2+c$

So $F(x,y)=f(x,y)+g(x,y)=x^2y^4-x^4y^2=x^2y^2(y^2-x^2)=c$
The problem may be solved as a homogeneous and
$I.F.=\mu=\frac{1}{Mx+Ny}=\frac{1}{xy[y^2-2x^2+2y^2-x^2]}=\frac{1}{3xy^3-3x^3y}$

Or y=vx and dy=vdx+xdv so

$$\frac{dx}{x}=\frac{2v^2-1}{3(v^2-v)}dv$$

The following differential equations can not solve by my new method ;
1) $(2x^2y^2+y)dx-(x^3y-3x)dx=0$ because of
$\frac{dy}{dx}+\frac{2x^2y^2}{-x^3y}\rightarrow\frac{dy}{dx}+\frac{2y}{-x}\rightarrow f=\frac{-x}{y^2}$ and $\frac{\partial(-\frac{x}{y^2})}{\partial x}=\frac{-1}{y^2}=\mu_1(2x^2y^2)$ so $\mu_1=\frac{1}{x^2y^4}$ and
$\frac{dy}{dx}+\frac{y}{3x}\rightarrow g=xy^3$ then $\frac{\partial g}{\partial x}=y^3=\mu_2$ i.e. $\mu_2=\mu_1$ so $\mu_1\neq\mu_2$
2)in some of the homogeneous D.E.s if there is one term or morein M and N in the same time like;
a) $(x^2+y^2-xy)dx-xydy=0$ $because we cannot say \frac{dy}{dx}+\frac{-xy}{-xy}$
b) $y^2dx+(x^2+3xy+4y^2)dx=0$ $because we cannot say \frac{dy}{dx}+\frac{y^2}{4y^2}$

We knew that the integration of the non exact differential equation is longer and harder than it's differentiation because of the absence of the integrating factor.

Then Itried to find some methods to solve many differential equations directly by separating two of it'stermsto find their orignal function by logarithms without integrating the D.E. completely (function after function) and solveing the other types by finding one function directly $\boxed{without integration}$ and then obtaining theI.F.from it to complete the solution by integrationsimply .

So I succeed

Main Results
from here to the end of the research I will deal with the D.E.s of four or more terms .

- **Example 1**
  Solve
  .y(xy+2y+2)dx+(2xy+5)dy=0

  Solution

$\frac{dy}{dx}+\frac{2y^2}{2xy}\rightarrow\frac{dy}{dx}+\frac{2y}{2x}$ so g(x,y)=2xy but not (x^2y^2)because we have no its direvatives $2y^2xdx+2x^2ydy in M and N respectively$
so $\frac{\partial g}{\partial x}=\frac{\partial(2xy)}{\partial x}=2x=\mu(2xy)$ and $\mu=\frac{1}{y}$
$M_1=\mu M=\frac{y(x+2y+2)}{y}=x+2y+2$ so $\frac{\partial M_1}{\partial y}=2$
$N_1=\mu N=\frac{1}{y}(2xy+5)=2x+\frac{5}{y}$ so $\frac{\partial N_1}{\partial x}=2$ then $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$ i.e. an exact
$F=2xy+\int\frac{1}{y}(xy^2+2y)dx+\int\frac{1}{y}(5dy)=\frac{x^2y}{2}+2xy+5\ln|y|+c$
By the old methods
$x^{\propto}y^{\beta}M and x^{\propto}y^{\beta}N then I.F.=x^{\propto}y^{\beta}$ so long method

- **Example 2**
  Intergrate $(-3y^3-x^2y)dx+(2xy^2+x^4)dy=0$
  Solution
  $(-3y^3-x^2y)dx+(2xy^2+x^4)dy=0$
  $y(-3y^2-x^2)dx+(2xy^2+x^4)dy=0$

we see above $\mu consists of y which is outside of M bracket and x^4$ **which is inside the N bracket** $x^{2m-2} = x^{2(3)-2} = x^4$

$$\frac{dy}{dx} + \frac{-3y^3}{2xy^2} \rightarrow \frac{dy}{dx} + \frac{-3y}{2x} \rightarrow f = \frac{y^2}{x^3}$$

$$\frac{\partial f}{\partial y} = \frac{\partial \frac{y^2}{x^3}}{\partial y} = \frac{2y}{x^3} = \mu(2xy^2) so \mu = \frac{2y}{x^3(2xy^2)} = \frac{1}{x^4y}$$

M₁=$\mu M = \frac{1}{x^4y}(-3y^3 - x^2y) = \frac{-3y^2}{x^4} - \frac{1}{x^2} so \frac{\partial M_1}{\partial y} = \frac{-6y}{x^4}$

N₁=$\mu N = \frac{1}{x^4y}(2xy^2 + x^4) = \frac{2y}{x^3} + \frac{1}{y} so \frac{\partial N_1}{\partial x} = \frac{-6y}{x^4} then \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial X}$

F=$\frac{y^2}{x^3} - \frac{1}{x^4y}\int x^2y dx + \frac{1}{x^4y}\int x^4 dy + c = \frac{y^2}{x^3} + \frac{1}{x} + \ln|y| + c$ **solution**

Or $\frac{dy}{dx} + \frac{-yx^2}{x^4} \rightarrow \frac{dy}{dx} + \frac{-yx^2}{x^4} \rightarrow \frac{dy}{y} + \frac{-x^2 dx}{x^4}$ **so** $\mu = \frac{1}{x^4y}$

$$\text{and } \frac{dy}{y} + \frac{-dx}{x^2} \rightarrow \ln|y| + \frac{1}{x}$$

or the problem can solved as I.F= $x^{\propto}y^{\beta}$

- **Example 3**

Solve $(y^4 + 2xy)dx + (xy^3 + 2y^2 - 2x^2)dy = 0$

Solution

$$(y^4 + 2xy)dx + (xy^3 + 2y^2 - 2x^2)dy = 0 \quad eq.(1)$$

$$\mu = \frac{1}{y^3}$$

$$\uparrow\uparrow$$

$y(y^3 + 2x)dx + (xy^3 + 2y^2 - 2x^2)dy = 0$**eq .(2)**

From eq .(2) we see that (y^3) **exists inside the two brackets so** $\mu = \frac{1}{y^3}$

$\frac{dy}{dx} + \frac{y^4}{xy^3} \rightarrow \frac{dy}{dx} + \frac{y}{x}$ **then f(x,y) = xy**

$\frac{\partial f(x,y)}{\partial y} = \frac{\partial(xy)}{\partial y} = x$ **then** $x = \mu(xy^3)$ **so** $\boxed{\mu = \frac{1}{y^3}}$

M₁=μM= $\mu = \frac{1}{y^3}[y(y^3 + 2xy)] = y - \frac{2x}{y^2}$

$$so \frac{\partial}{\partial y} M_1 = 1 + \frac{4x}{y^3}$$

N₁=μN=$\frac{1}{y^3}(xy^3 + 2y^2 - 2x^2) = x - \frac{2}{y} - \frac{2x^2}{y^3}$

$$\frac{\partial N_1}{\partial x} = 1 - \frac{4x}{y^3}$$

F(x,y) = xy+$\int \frac{1}{y^3}(2xy)dx + \int_{(\ y only)} \frac{1}{y^3}(xy^3 + 2y^2 - 2x^2)dy = xy + \frac{x^2}{y^2} + 2\ln|y| = c$ **End**

Another solution

$(y^4 + 2xy)dx + (xy^3 + 2y^2 - 2x^2)dy = 0$

Solution

$$\frac{dy}{dx} + \frac{2xy}{-2x^2} \rightarrow \frac{dy}{dx} + \frac{2y}{-2x} \rightarrow g(x,y) = \frac{x^2}{y^2}$$

$$\frac{\partial g}{\partial x} = \frac{\partial \frac{x^2}{y^2}}{\partial x} = \frac{2x}{y^2} = \mu(2xy) so \mu = \frac{2x}{2xy^3} = \frac{1}{y^3}$$

$$F = \int \frac{1}{y^3}(y^4)dx + \frac{x^2}{y^2} + \int \frac{1}{y^3}(2y^2)dy + c$$

F= $xy + \frac{x^2}{y^2} + 2\ln|y|$

The old solution

M= $y^4 + 2xy$ **and** $\frac{\partial M}{\partial y} = 4y^3 + 2x$

N= $xy^3 + 2y^2 - 2x^2$ **and** $\frac{\partial N}{\partial x} = y^3 - 4x$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y(y^3+2x)}(y^3-4x-4y^3-2x)=\frac{-3}{y} then I.F.=e^{\int\frac{-3dy}{y}}=\frac{1}{y^3}$$

- **Example 4**

Solve $(3y+4xy^2)dx+(2x+3x^2y)dy=0$
Solution

$$\frac{dy}{dx}+\frac{3y}{2x} \text{ then } f(x,y)= x^3y^2 \boxed{without integration}$$

$$\frac{dy}{dx}+\frac{4xy^2}{3x^2y} \rightarrow \frac{dy}{dx}+\frac{3y}{2x} \text{ and } g(x,y)=x^4y^3 \boxed{without integration}$$

F(x,y) = f(x,y)+g(x,y) = $x^3y^2+x^4y^3=x^3y^2(1+xy)+c\boxed{without integration}$ **End .**
Another solution

$$\frac{dy}{dx}+\frac{3y}{2x} \text{ then } f(x,y)= x^3y^2$$

Then $\frac{\partial f}{\partial y}=\frac{x^3y^2}{\partial y}=2x^3y=\mu(2x) su \mu=x^2y$
F=$x^3y^2+\int x^2y(4xy^2)dx=x^3y^2+x^4y^3=x^3y^2(1+xy)+c$ **End**
The old method of solution
I.F.=$\frac{1}{xM-yN}=\frac{1}{xy+x^2y^2} where xy+x^2y^2\neq 0$

- **Example 5**

Solve $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$
Solution
$(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$

$$\mu=\frac{1}{y^3}$$

$.y(y^3+2)dx+(xy^3+2y^4-4x)dy=0$

So $\frac{dy}{dx}+\frac{y^4}{xy^3}\rightarrow \frac{dy}{dx}+\frac{y}{x}$ **then f=xy**

$$\frac{\partial(xy)}{\partial y}=x \quad \text{so } x=\mu(xy^3) \text{ then } \mu=\frac{1}{y^3}$$

F(x,y)= f(x,y)+$\int\frac{1}{y^3}(2y)dx+\int_{(\ y only)}\frac{1}{y^3}(2y^4)dy=xy+\frac{2x}{y^2}+y^2=$ **c End**
The old solution

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{-3}{y} then I.F.=e^{\int\frac{-3dy}{y}}=\frac{1}{y^3}$$

- **Example 6**

Solve $(x^2+y^2+2x)dx+2ydy=0$
Solution
$(x^2+y^2+2x)dx+2ydy=0$
Since M= x^2+y^2+2x **which contents 3 terms and N = 2y which contents1 term and there is a common derivative between them so we expect that** $\mu=e^x$ **.**

$\frac{dy}{dx}+\frac{y^2}{2y}\rightarrow \frac{dy}{dx}+\frac{y}{2}$ **then** $f(x,y)= y^2e^{\int dx}=y^2e^x$

$\frac{\partial f(x,y)}{\partial y}=\frac{\partial(y^2e^x)}{\partial y}=2ye^x$ **then** $2ye^x=\mu(2y)$ **so** $\mu=e^x$

F(x,y) = $y^2e^x + \int e^x(x^2 + 2x)dx = e^x(y^2 + x^2) + c$ End

The old way of the solution is

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 1\ sol.F. = e^{\int 1dx} = e^x$$

- **Example 7**

Solve (-2y+$x^2y^3)dx - (2x + 2x^3)dy = 0$

Solution

(-2y+$x^2y^3)dx + (-2x - 2x^3)dy = 0$

$$\mu = \frac{1}{x^3y^3}$$

(-2y+$x^2y^3)dx + (-2x - 2x^3)dy = 0$

$$\frac{dy}{dx} + \frac{-2y}{-2x} \quad \text{then} \quad f(x,y) = x^{-2}y^{-2} = \frac{1}{x^2y^2}$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial \frac{1}{x^2y^2}}{\partial y} = \frac{-2}{x^2y^3} \text{ then } \frac{-2}{x^2y^3} = \mu(-2x) \text{ so } \mu = \frac{1}{x^3y^3}$$

F(x,y) = $\frac{1}{x^2y^2} + \int \frac{1}{x^3y^3}(x^2y^3)dx + \int_{(y only)} \frac{1}{x^3y^3}(-2x^3)dy = \frac{1}{x^2y^2} + ln|x| + \frac{1}{y^2}$ +c End .

Another solution

(-2y+$x^2y^3)dx + (-2x - 2x^3)dy = 0$

Then

$$\frac{dy}{dx} + \frac{x^2y^3}{-2x^3} \to \frac{dy}{dx} + \frac{y^3}{-2x} \to \frac{-2dy}{y^3} + \frac{dx}{x} \to \int \frac{-2dy}{y^3} + \int \frac{dx}{x} = \frac{1}{y^2} + \ln|x|$$

F=$\frac{1}{x^2y^2} + \frac{1}{y^2} + \ln|x| + cEnd$

The old solution is

I.F. = $x^{\alpha}y^{\beta}$

- **Example 8**

Solve y(8x-9y)dx+2x(x-3y)dy = 0

Solution

$\frac{dy}{dx} + \frac{8xy}{2x^2} \to \frac{dy}{dx} + \frac{8y}{2x} = 2(\frac{4y}{x})$ then $f(x,y) = 2x^4y$ without integration

$\frac{\partial(2x^4y)}{\partial y} = 2x^4 = \mu(2x^2)$ then $\mu = x^2$

$$\frac{dy}{dx} + \frac{-9y^2}{-6xy} \to \frac{dy}{dx} - 3\left(\frac{3y}{2x}\right) \quad \text{so } g(x,y) = -3x^3y^2 \boxed{\text{without integration}}$$

F(x,y) = $2x^4y - 3x^3y^2 = xy(2x^3 - 3x^2) + c$ withoutintegration

Also $\frac{\partial(-3x^3y^2)}{\partial y}$ = -6 $x^3y = \mu(-6xy)$

$$\mu = x^2$$

The old solution is

I.F. = $x^{\alpha}y^{\beta}$

- **Example 9**

Solve -y(y+x)dx+(xy+$x^2)dy = 0$

Solution

Homogeneous and its power is (2) so multiply the D.E. by (2)

-2y(y+x)dx+ 2(xy+$x^2)dy = 0$

$$\frac{dy}{dx}+\frac{-2y^2}{2xy} \rightarrow \frac{dy}{dx}+\frac{-2y}{2x} \quad \text{so } f(x,y)=\frac{y^2}{x^2} \boxed{\text{without integration}}$$

$\frac{\partial f(x,y)}{\partial y}=\frac{\partial(\frac{y^2}{x^2})}{\partial y}=\frac{2y}{x^2} so \frac{2y}{x^2}=\mu(2xy) then \mu=\frac{1}{x^3}$ **i.e. not exact**

F(x,y) = $\frac{y^2}{x^2}+\int\frac{1}{x^3}(-xy)dx=\frac{y^2}{2x^2}+\frac{y}{x}=c$ **End.**

The same results we can obtain if

$\frac{dy}{dx}+\frac{-2xy}{2x^2}\rightarrow\frac{dy}{dx}+\frac{-2y}{2x}$ **and g(x,y) =** $\frac{y^2}{x^2}$ $\boxed{\text{without integration}}$

$$\frac{\partial g(x,y)}{\partial y}=\frac{\partial(\frac{y^2}{x^2})}{\partial y}=\frac{2y}{x^2} so \frac{2y}{x^2}=\mu(2xy) then \mu=\frac{1}{x^3}$$

F(x,y) = $\int\frac{1}{x^3}(-xy)dx+\frac{y^2}{x^2}=\frac{y}{x}+\frac{y^2}{x^2}=c$ **End**

The old solution is

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{-3}{x}$$

- **Example 10**

Solve $y(x^3-y)dx-x(x^3+y)dy=0$

Solution

$$\frac{dy}{dx}+\frac{-y^2}{-xy}\rightarrow\frac{dy}{dx}+\frac{-y}{-x} \quad \text{then } g(x,y)=\frac{1}{xy}$$

$$\frac{\partial g(x,y)}{\partial x}=\frac{\partial\frac{1}{xy}}{\partial x}=\frac{-1}{x^2y}=\mu(-y^2) \quad then \mu=\frac{1}{x^2y^3}$$

F(x,y) = $\frac{1}{xy}+\int\frac{1}{x^2y^3}(x^3y)dx=\frac{1}{xy}+\frac{x^2}{2y^2}=c$

$x^3+2y=cxy^2$ **End**

the old solution is by inspectiion

- **Example 11**

A – solve $-(3y^4+x^3y)dx+(3xy^3+x^4)dy=0$

B – solve $-(y^4+x^3y)dx+(xy^3+x^4)dy=0$

Solution A

$(-3y^4-x^3y)dx+(3xy^3+x^4)dy=0$

$$\frac{dy}{dx}+\frac{-3y^4}{3xy^3}\rightarrow\frac{dy}{dx}+\frac{-3y}{3x} \quad \text{then } f(x,y)=x^{-3}y^3=\frac{y^3}{x^3}\boxed{\text{without integration}}$$

$$\frac{\partial f(x,y)}{\partial y}=\frac{\partial\frac{y^3}{x^3}}{\partial y}=\frac{3y^2}{x^3} \quad \text{so} \quad \frac{3y^2}{x^3}=\mu(3xy^3) \quad \text{then} \quad \mu=\frac{1}{x^4y}$$

F(x,y) = $\frac{y^3}{x^3}+\int\frac{1}{x^4y}(-x^3y)dx+\int_{yonly}\frac{1}{x^4y}(x^4)dy=\frac{y^3}{x^3}-ln|x|+ln|y|+c$

Another solution

$(-3y^4-x^3y)dx+(3xy^3+x^4)dy=0$

Solution

$$(y^4-x^3y)dx+(3xy^3+x^4)dy=0$$

so $y(y^3-x^3)dx+(3xy^3+x^4)dy=0$ **i.e.** $\mu=\frac{1}{x^4y}$

$\frac{dy}{dx}+\frac{-x^3}{x^4} \rightarrow \frac{dy}{dx}+\frac{-y}{x}$ so $\frac{dy}{y}-\frac{dx}{x} \rightarrow \boldsymbol{lny - lnx then g = ln\frac{y}{x}}$

$$\frac{\partial g}{\partial y}=\frac{\partial ln\frac{y}{x}}{\partial y}=\frac{1}{y}=\mu(x^4) so \mu=\frac{1}{x^4y}$$

F= $\frac{y^3}{x^3}-ln|x|+\ln|y|+c$ without integration

Solution (B)

$$-(y^4+x^3y)dx+(xy^3+x^4)dy=0$$

Then $-y(y^3+x^3)dx+(xy^3+x^4)dy=0$

Solution
$\frac{dy}{dx}+\frac{-x^3y}{x^4} \rightarrow \frac{dy}{dx}+\frac{-y}{x}so\frac{dy}{y}-\frac{dx}{x} \rightarrow g(x,y)=\ln y-\ln x$ without integration $\frac{\partial g}{\partial y}=\frac{\partial ln\frac{y}{x}}{\partial y}=\frac{1}{y}=\mu(x^4) so \mu=\frac{1}{x^4y}$

note ; $\mu=\frac{1}{x^4y}$ ($y outside M and x^4$ **is inside N)**

F(x,y) = $\frac{1}{x^4y}\int -y^4dx - lnx + lny + c = \frac{y^3}{3x^3} - ln|x| + ln|y| + c$ **End .**
Another solution
-$(y^4+x^3y)dx+(xy^3+x^4)dy=0 \rightarrow -y(y^3+x^3)dx+x(y^3+x^3)dy=0$
We see that there is homogeneous D.E. inside the brackets of power 3
So multiply it by 3
-$3(y^4+x^3y)dx+3(xy^3+x^4)dy$
Then $\frac{dy}{dx}+\frac{-3y^4}{3xy^3} \rightarrow \frac{dy}{dx}+\frac{-3y}{3x} \rightarrow f=\frac{y^3}{x^3} so \frac{\partial f}{\partial y}=\frac{\partial(\frac{y^3}{x^3})}{\partial y}=\frac{3y^2}{x^3}=\mu(3xy^3)$
So $\mu=\frac{1}{x^4y}$
F=$\frac{y^3}{x^3}-\int\frac{1}{x^4y}(3x^3y)dx+\int\frac{1}{x^4y}(3x^4)dy+c=\frac{y^3}{x^3}-3\,ln|x|+3\,ln|y|+c$ **END**
The old solution is
Let y= vx so dy=vdx+xdv

- **Example 12**

Solve
$(3x^2+y^2)dx+(2xy+x^3+xy^2-y)dy=0$
.
Solution
Since there are (2) parameters in M(x,y) and (4) in N(x,y) so we expect that I,F. is equal (e^y**)**

$$\frac{dy}{dx}+\frac{3x^2}{x^3} \rightarrow \frac{dy}{dx}+\frac{(3)}{x} \quad so \quad f(x,y)=x^3e^{\int dy}=x^3e^y$$

$\frac{\partial(x^3e^y)}{\partial y}=x^3e^y then x^3e^y = \mu(x^3)so\mu=e^y$

F(x,y)=$x^3e^y+\int y^2e^ydx+\int e^y(-y)dy$=$e^y(x^3+xy^2-y+1)$**=c End .**
The old solution

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=1 \quad soI.F.= e^{\int 1dy}=e^y$$

- **Example 13**

Solve $(x^2+y^2+x)dx+xydy=0$

Solution
Semi homogeneous because d(xy) =xdy+ydx which are not foundin the D.E. so multiply by 2 then $2(x^2+y^2+x)dx+2xydy=0$

$$\frac{dy}{dx}+\frac{2y^2}{2xy}\rightarrow \quad \frac{dy}{dx}+\frac{2y}{2x} \quad \text{so } f = x^2y^2$$

$$\frac{\partial(x^2y^2)}{\partial y}=2x^2y then\ 2x^2y=\mu(2xy) so \mu = x$$

F(x,y) = $x^2y^2+\int x(x^2+x)dx = x^2y^2 + \frac{x^4}{4}+\frac{x^3}{3}$+c **End .**

The old solution

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{x} sol. F.=e^{\int\frac{1dx}{x}}=x$$

- **Example 14**

Solve $(3x^2+y^2+4)dx+x(x-2y)dy=0$

Solution

$\frac{dy}{dx}+\frac{y^2}{-2xy}\rightarrow \frac{dy}{dx}+\frac{y}{-2x} so g=\frac{-y^2}{x}$ **then** $\frac{\partial g}{\partial y}=\frac{-2y}{x}=\mu(-2xy) so \mu=\frac{1}{x^2}$

$$F=\frac{-y^2}{x}+\int\frac{1}{x^2}(3x^2+4)dx+\int\frac{x^2}{x^2}dy+c=\frac{-y^2}{x}+3x+y-\frac{4}{x}+c End$$

The old solution is

$\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=\frac{-2}{X} sol. F.= e^{\int\frac{-2dx}{x}} = e^{-2lnx}=\frac{1}{x^2}$

- **Example 15**

Solve y(xy+2)dx+($\frac{5x^2y}{2}$**+8x - 4)**$dy=0$

Solution

$$\frac{dy}{dx}+\frac{xy^2}{\frac{5x^2y}{2}} \rightarrow \frac{dy}{dx}+\frac{2y}{5x} so f=x^2y^5 then \frac{\partial f}{\partial y}=5x^2y^4=\mu\left(\frac{5x^2y}{2}\right) so \mu=2y^3$$

F=$x^2y^5+\int 2y^3(2y)dx-\int 2y^3(4y^2)dy+c=x^2y^5+4xy^4-2y^4+c$ **END**

The old solution is $F.=x^{\propto}y^{\beta}$

- **Example 16**

Solve $(5x^4-2x^2y+2y^2$**- 4xy)dx – 2(**$x^2-2y)dy=0$

Solution

$$\frac{dy}{dx}+\frac{-2x^2y}{-2x^2} \rightarrow f=-2x^2ye^x then \frac{\partial f}{\partial y}=-2x^2e^x=\mu(-2x^2) so \mu=e^x$$

F= $\int e^x(5x^4-2x^2y+2y^2-4xy)dx+c$

The old solution is

$$I.F.=x^{\propto}y^{\beta}$$

- **Example 17**

solve (xy-4lnx+$x^2\sin x^2)dx+x^2dy=0$

solution

$\frac{dy}{dx}+\frac{xy}{x^2} \rightarrow \frac{dy}{dx}+\frac{y}{x} \rightarrow f=xy so \frac{\partial f}{\partial y}$= x =$\mu(x^2)$ $then \mu=\frac{1}{x}$

F=xy+$\int\frac{1}{x}(-4lnx+x^2\sin x^2)dx= xy-2(lnx)^2$ - $\frac{cosx^2}{2}$+c

The old solution is

$\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=\frac{-1}{X}$**and I,F.** = $\frac{1}{x}$

- **Example 18**

Solve y(1- xy)dx+x(xy-1)dy=0

Solution

Since we have yf_1**(xy)+x**f_2**(xy) =0 then we can multiply the D.E. by (-1)**

- y(1- xy)dx-x(xy-1)dy=0 so $\frac{dy}{dx}+\frac{-y}{-x} then f=\frac{-1}{xy} and \frac{\partial f}{\partial y}=\frac{1}{xy^2}=\mu(x)$

$$so\mu = \frac{1}{x^2y^2}$$

F= $\frac{-1}{xy} - \int \frac{1}{x^2y^2}(xy^2)dx + \int \frac{1}{x^2y^2}(x^2y)dy + c = \frac{-1}{xy} - lnx + lny + c$ **END**

The old solution is

I.F. = $\frac{1}{Mx=Ny} = \frac{1}{xy(1-xy-xy+1)} = \frac{1}{2x^2y^2}$

- **Example 19**

Solve $(2y^4 - 2x^3y)dx - (4xy^3 - x^4)dy = 0$

Solution

$\frac{dy}{dx} + \frac{2y^4}{-4xy^3} \rightarrow \frac{dy}{dx} + 2\frac{y}{-2x} \rightarrow f = -2\frac{y^2}{x} then \frac{\partial f}{\partial y} = \frac{-4y}{x}$=μ(-4x$y^3$)

μ= $\frac{1}{x^2y^2}$

F= $\int \frac{1}{X^2Y^2}(2y^4 - 2x^3y)dx$+C=$-2\frac{y^2}{x} - \frac{x^2}{y} + C$ **END**

The old solution is

I.F. = $\frac{1}{Mx+Ny} = \frac{1}{xy(2y^3-2x^3-4y^3+x^3)} = \frac{1}{-(2xy^4+x^4y)}$

- **Example 20**

$$solve\ (-y^3 + x^2y)dx + (xy^2 - x^3)dy = 0$$

Solution

$\frac{dy}{dx} + \frac{-y^3}{xy^2} \rightarrow \frac{dy}{dx} + \frac{-y}{x} \rightarrow f = \frac{y}{x}\ so\ \frac{\partial f}{\partial y} = \frac{1}{x}$=μ(x$y^2$) **then** μ=$\frac{1}{x^2y^2}$

F=$\frac{y}{x} + \int \frac{1}{x^2y^2}(x^2y)dx + c = \frac{y}{x} + \frac{x}{y} + c$ ***solution***

II. CONCLUSIONS

- The new methods are innovative and by them we facflitate the solutions of the problems either by finding the answers directjy i.e. without integeation or by finding the integratibg factor direcrectly or simply .
- I see that these mothods can be new subjects of the CALCULUS book .
- I see that these methods can share in the Intelligent Artificial (AI)because of the short time , less papers and efforts (may be decreas the human mistakes) .

REFERENCES

[1.]Paul Blanchard, Robert L. Devaney. Glenn R. Hall. Differential Equation (Preliminary Edition).PWS Publishing. Boston. 1996.
[2.]William H. Boyce and Richard C. Diprima. Elimintary Differential Equations and Boundary Value Problems (6th Edition). Wiley. New York. 1996.
[3.]C.H. Edwards, Jr., David E. Penney, Elementary Differential Equations with Applications (Third Edition), Prentice-Hall,Englewood Cliffs, NJ, 1996.
[4.]R. Kent Nagle and Edward B. Saff, Fundamentals of Differential Equations (Third Edition), Addison Wesley, Reading, MA, 1993.
[5.]https://ijisrt.com/new-way-to-solve-the-first-order-linear-de-which-consist-of-three-terms.