ISSN No:-2456-2165

A Mathematical Scheme Defined On Strictly Rectangular Complex Matrix Spaces Involving Frobenius Norm Preservation under the Possibility of Matrix Rank Readjustment and Internal Redistribution of Variance using Spacer Component Matrices and a Completely Positive Trace Preserving Transformation

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Abstract:- The Research paper presents a mathematical framework on strictly rectangular complex matrix spaces that preserve the Frobenius norm but allow for possible readjustment of variance contribution per degrees of freedom as a result of allowed flexibility of the rank modification and alteration of the Principal axes. This is achieved through the utilization of Spacer component matrices and matrices generated from them and the use of a specific Completely Positive Trace preserving transformation determined solely based on the embedding dimension, which is the order of the embedded square matrix space associated with the strictly rectangular input matrix space.

Notations

- $M_{x \times y}(C)$ denotes the complex Matrix space of order 'x' by 'y'
- $M_{s \times s}(C)$ denotes the Embedded Matrix Space
- C^{w} denotes the complex coordinate space of order 'w'
- $c^{\bullet}$ denotes the complex conjugate of the complex number 'c'

- $|v\rangle \in C^{w}$, $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_w \end{bmatrix}_{w \times 1}$, $\langle v| = \begin{bmatrix} v_1^{\bullet} & v_2^{\bullet} & . & . & v_w^{\bullet} \end{bmatrix}_{1 \times w}$, $|m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{m \times 1}$, $|n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n \times 1}$,

$|s-m\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-m) \times 1}$, $|s-n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{(s-n) \times 1}$

- N denotes the set of all Natural numbers
- $I_{w\times w}$ denotes the Identity matrix of order 'w'
- A^H denotes the Hermitian conjugate of the matrix A
- $(X_{w\times w})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{w\times w}$, i.e. $(X_{w\times w})^{-1}X_{w\times w} = X_{w\times w}(X_{w\times w})^{-1} = I_{w\times w}$
- $X_{w\times w} \in M_{w\times w}(C)$, such that $X_{w\times w}$ is hermitian, positive definite , then $(X_{w\times w})^{-1/2} \in M_{w\times w}(C)$, $(X_{w\times w})^{-1/2}$ is hermitian, positive definite, such that: $(X_{w\times w})^{-1/2}(X_{w\times w})^{-1/2} = (X_{w\times w})^{-1}$, we also have: $(X_{w\times w})^{1/2} \in M_{w\times w}(C)$, $(X_{w\times w})^{1/2}$ is hermitian, positive definite, such that: $(X_{w\times w})^{1/2}(X_{w\times w})^{1/2} = X_{w\times w}$
- $\|\Omega_{x\times y}\|_F$ denotes the Frobenius norm of the matrix $\Omega_{x\times y}$
- $\max(a,b)$ denotes the maximum of the two inputs 'a' and 'b'
- $|a-b|$ denotes the absolute value of the difference between the two inputs 'a' and 'b'
- $diag(v_1, v_2, ..., v_w)$ denotes a diagonal matrix of order 'w', whose diagonal elements along the main diagonal, from top to bottom, are $v_1, v_2, ..., v_w$ respectively
- 'SVD' is the abbreviation for " Singular value decomposition "
- $\sigma(\Omega_{x\times y})$ denotes the Singular value spectrum of the matrix $\Omega_{x\times y}$

Keywords:- *Spacer matrix based generalized matrix multiplication, Spacer matrix components, embedding dimension, completely positive trace preserving transformations, Kraus operators*

I. INTRODUCTION

Matrix transformations play a very important role in several problems arising in data sciences, natural and social sciences and in solving diverse types of engineering problems. A particularly interesting study in this context is to transform matrices from one Matrix space to another, or to the same Matrix space, preserving one or more attributes of the numerical matrix array. There exist a diverse number of linear transformations in literature that can carry out some of such tasks. In context of problems pertaining to data analysis, a focus is on those transformations that preserve the total variation in the numerical data matrix (which is quantified as squared Frobenius norm of the numerical matrix) or capturing as much variation as possible while providing dimensionality reduction and ease of data visualization. It is known that the Frobenius norm is Unitarily Invariant, but under unitary transformations the variance-partitioning structure of the matrix remains unaltered owing to the fact that unitary transformations change the Principal axes but do not provide an opportunity for modification of the matrix rank.

The research paper presents a mathematical framework that allows for matrix transformation on strictly rectangular complex matrix spaces ($M_{m\times n}(C)$, where $m \neq n$) with preservation of the Frobenius norm and a possibility for matrix rank readjustment. The methodology presented operates through three distinct transformation blocks: mapping from the Input complex matrix space to the higher dimensional, embedded matrix space using the matrices generated from Spacer Component matrices[3,4,5,6,7] and associated symmetric transformation to obtain positive definiteness/semi-definiteness, this step is followed by a Completely Positive Trace Preserving transformation[1,8,11,12,13,15,16,18] (abbreviated as CPTP transformation) uniquely determined through the embedding dimension which is succeeded by a square root transformation preserving the positive definiteness/semi-definiteness. The last block of the transformation is the extraction step, it involves transformation back into the input strictly rectangular complex matrix space with a possible difference in the total variation, which is termed as "Unaccounted Variation" and is used to determine the efficiency of the extraction setup, using the Principal axes associated with the extracted matrix as the reference axes, the unaccounted variation is split among the available degrees of freedom (the rank of the extracted matrix $\bar{A}_{m\times n}$) and using this variance contribution and singular values associated with the degrees of freedom and the reference axes as the bases, the final matrix $\hat{A}_{m\times n}$ is thereby constructed, which possess the same Frobenius norm as the input matrix $A_{m\times n}$ but in general, possibly with a modified rank and redistributed variance per degrees of freedom. It is also possible that rank is preserved and there is only variance redistribution involved.

The paper presents the mathematical framework and numerical case studies, involving numerical samples from (m=2, n=3) and (m=3, n=4) complex matrix spaces for demonstration of the mathematical methodology. The article concludes with a discussion on the results of the numerical studies and insights obtained.

II. MATHEMATICAL FRAMEWORK

- The spacer component matrices $P_{n\times s}$ and $Q_{s\times m}$ are defined as follows:

$m\in N, n\in N$, We consider the case where $m\neq n$, the expression for the Embedding dimension, denoted by 's' is given as follows: $s=\max(m,n)+|m-n|$, under the imposed condition: $m\in N, n\in N$ and $m\neq n$, we have: $s\in N$, $s>m$ and $s>n$

$$P_{n\times s}=\begin{bmatrix} I_{n\times n} & (\frac{1}{n})|n\rangle\langle s-n| \end{bmatrix}_{n\times s}=\left[\begin{array}{ccccc|ccccc} 1 & 0 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ 0 & 1 & . & . & 0 & 1/n & 1/n & . & . & 1/n \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 1/n & 1/n & . & . & 1/n \end{array}\right]_{n\times s}$$

$$Q_{s\times m}=\begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})|s-m\rangle\langle m| \end{bmatrix}_{s\times m}=\left[\begin{array}{ccccc} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & 1 \\ \hline 1/m & 1/m & . & . & 1/m \\ 1/m & 1/m & . & . & 1/m \\ . & . & . & . & . \\ . & . & . & . & . \\ 1/m & 1/m & . & . & 1/m \end{array}\right]_{s\times m}$$

- $W_{s\times n}=(P_{n\times s})^H[(PP^H)^{-1/2}]_{n\times n}$, we then have: $W^HW=I_{n\times n}$
  $G_{s\times m}=(Q_{s\times m})[(Q^HQ)^{-1/2}]_{m\times m}$, we then have: $G^HG=I_{m\times m}$

- $\Gamma_{s\times s}=\begin{bmatrix} I_{(s-1)\times(s-1)} & |s-1\rangle_{(s-1)\times 1} \\ 0_{1\times(s-1)} & 1 \end{bmatrix}$, we define: $Z_{s\times s}=\Gamma_{s\times s}[(\Gamma^H\Gamma)^{-1/2}]_{s\times s}$, we then have:
  $Z^HZ=ZZ^H=I_{s\times s}$

- $A_{m\times n}\in M_{m\times n}(C)$, $A_{m\times n}\neq 0_{m\times n}$, rank($A_{m\times n}$) = a , $\sigma(A_{m\times n})=\{\sigma_1,\sigma_2,....,\sigma_a\}$, such that: $\sigma_1\geq\sigma_2\geq....\geq\sigma_a>0$

  Total Variation = $(\|A_{m\times n}\|_F)^2=\sum_{j=1}^{a}\sigma_j^{\,2}$, we have: $B_{s\times s}\in M_{s\times s}(C)$, where $B_{s\times s}=G_{s\times m}A_{m\times n}(W_{s\times n})^H$

- $\rho_{s\times s}=(\frac{1}{2})(BB^H)_{s\times s}+(\frac{1}{2})(B^HB)_{s\times s}$, therefore we have: $trace(\rho_{s\times s})=(\|B_{s\times s}\|_F)^2=(\|A_{m\times n}\|_F)^2$,
  $(\rho_{s\times s})^H=\rho_{s\times s}$, $\rho_{s\times s}$ is Positive semi-definite or Positive definite

ISSN No:-2456-2165

- The CPTP transformation step:

$$p_t = \frac{(\frac{1}{2})^t}{1-(\frac{1}{2})^s}$$, where: $t = 1,2,..,s$, therefore we have: $p_t > 0$ and $\sum_{t=1}^{s} p_t = 1$

$\hat{\rho}_{s\times s} = \sum_{t=1}^{s} p_t [Z^t{}_{s\times s} \rho_{s\times s} (Z^t{}_{s\times s})^H]$, where: $Z^t{}_{s\times s} = Z_{s\times s}........Z_{s\times s}$ ('t' times) , i.e., $Z^1{}_{s\times s} = Z_{s\times s}$, $Z^2{}_{s\times s} = Z_{s\times s} Z_{s\times s}$, $Z^3{}_{s\times s} = Z_{s\times s} Z_{s\times s} Z_{s\times s}$ and so on

Therefore, $(\hat{\rho}_{s\times s})^H = \hat{\rho}_{s\times s}$, $\hat{\rho}_{s\times s}$ is Positive semi-definite or Positive definite , $trace(\hat{\rho}_{s\times s}) = trace(\rho_{s\times s}) = (\|A_{m\times n}\|_F)^2$

- $\hat{\rho}_{s\times s} \in M_{s\times s}(C)$, rank($\hat{\rho}_{s\times s}$) = d , SVD of $\hat{\rho}_{s\times s}$: $\hat{\rho}_{s\times s} = R_{s\times d} \Lambda_{d\times d} (R_{s\times d})^H$, where we have: $R^H R = I_{d\times d}$, $\Lambda_{d\times d} = diag(\mu_1, \mu_2, ..., \mu_d)$, $\mu_1 \geq \mu_2 \geq .... \geq \mu_d > 0$
  $\hat{B}_{s\times s} = R_{s\times d} (\Lambda^{1/2})_{d\times d} (R_{s\times d})^H$, where: $(\Lambda^{1/2})_{d\times d} = diag(\sqrt{\mu_1}, \sqrt{\mu_2}, ..., \sqrt{\mu_d})$, therefore we have:
  $(\|\hat{B}_{s\times s}\|_F)^2 = trace(\hat{\rho}_{s\times s}) = trace(\rho_{s\times s}) = (\|B_{s\times s}\|_F)^2 = (\|A_{m\times n}\|_F)^2$

- $\bar{A}_{m\times n} \in M_{m\times n}(C)$, rank($\bar{A}_{m\times n}$) = b , $\bar{A}_{m\times n}$ is defined as follows: $\bar{A}_{m\times n} = (G_{s\times m})^H \hat{B}_{s\times s} W_{s\times n}$
  SVD of $\bar{A}_{m\times n}$: $\bar{A}_{m\times n} = U_{m\times b} \bar{\Delta}_{b\times b} (V_{n\times b})^H$, where we have: $U^H U = V^H V = I_{b\times b}$, $\bar{\Delta}_{b\times b} = diag(\bar{\sigma}_1, \bar{\sigma}_2, ..., \bar{\sigma}_b)$, $\bar{\sigma}_1 \geq \bar{\sigma}_2 \geq .... \geq \bar{\sigma}_b > 0$, $(\|\bar{A}_{m\times n}\|_F)^2 = \sum_{j=1}^{b} \bar{\sigma}_j{}^2$

- The percentage Efficiency (η) is defined as follows: $\eta = [1 - \frac{\bar{\varepsilon}}{(\|A_{m\times n}\|_F)^2 + (\|\bar{A}_{m\times n}\|_F)^2}] \times 100$

  We define Unaccounted Variation ($\bar{\varepsilon}$) and the Baseline Variation ($\hat{\varepsilon}$) as follows:

  $\bar{\varepsilon} = (\|A_{m\times n}\|_F)^2 - (\|\bar{A}_{m\times n}\|_F)^2$ When $(\|A_{m\times n}\|_F)^2 > (\|\bar{A}_{m\times n}\|_F)^2$ and we have:

  $\bar{\varepsilon} = (\|\bar{A}_{m\times n}\|_F)^2 - (\|A_{m\times n}\|_F)^2$ When $(\|\bar{A}_{m\times n}\|_F)^2 > (\|A_{m\times n}\|_F)^2$

  When: $(\|\bar{A}_{m\times n}\|_F)^2 = (\|A_{m\times n}\|_F)^2$ then $\bar{A}_{m\times n}$ itself as the desired output matrix, $\bar{A}_{m\times n} = \hat{A}_{m\times n}$ in this case.

  $\hat{\varepsilon} = \frac{\bar{\varepsilon}}{b} = (\bar{\varepsilon}$ / rank ($\bar{A}_{m\times n}$))

- Construction of $\hat{A}_{m\times n}$:

  Case: $(\|A_{m\times n}\|_F)^2 > (\|\bar{A}_{m\times n}\|_F)^2$
  $\hat{\sigma}_j = [\bar{\sigma}_j{}^2 + \hat{\varepsilon}]^{1/2}$ Where: $j = 1,2,....,b$

  Case: $(\|\bar{A}_{m\times n}\|_F)^2 > (\|A_{m\times n}\|_F)^2$
  $\hat{\sigma}_1 = [(\|\bar{A}_{m\times n}\|_F)^2 - \bar{\varepsilon}]^{1/2}$, $\hat{\sigma}_2 = ... = \hat{\sigma}_b = 0$

$\hat{\Delta}_{b\times b} = diag(\hat{\sigma}_1, \hat{\sigma}_2, ..., \hat{\sigma}_b)$, $\hat{A}_{m\times n}$ is defined as follows: $\hat{A}_{m\times n} = U_{m\times b}\hat{\Delta}_{b\times b}(V_{n\times b})^H$, therefore: $(\|\hat{A}_{m\times n}\|_F)^2 = \sum_{j=1}^{b} \hat{\sigma}_j^{\ 2}$

We have the following conservation relationship:

$$(\|A_{m\times n}\|_F)^2 = \sum_{j=1}^{a} \sigma_j^{\ 2} = (\|B_{s\times s}\|_F)^2 = trace(\rho_{s\times s}) = trace(\hat{\rho}_{s\times s}) = (\|\hat{B}_{s\times s}\|_F)^2 = (\|\hat{A}_{m\times n}\|_F)^2 = \sum_{j=1}^{b} \hat{\sigma}_j^{\ 2}$$

Numerical Case Studies

- The numerical calculations are performed using the Scilab 5.4.1 Computational software
- rank($X_{m\times n}$) refers to the ' numerical rank ' of the matrix $X_{m\times n}$, which is given as the number of singular values of the matrix $X_{m\times n}$ greater than the threshold parameter θ_{thd} , where :

$$\theta_{thd} = (\max(m,n)) \times (\sigma_{\max}(X_{m\times n})) \times ((\frac{1}{2})^{52})$$

- $m = 2, n = 3$, therefore $s = 4$

$$P_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} , \quad Q_{4\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} , \quad W_{4\times 3} = (\frac{1}{6}) \begin{bmatrix} 4+\sqrt{3} & -2+\sqrt{3} & -2+\sqrt{3} \\ -2+\sqrt{3} & 4+\sqrt{3} & -2+\sqrt{3} \\ -2+\sqrt{3} & -2+\sqrt{3} & 4+\sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix} ,$$

$$G_{4\times 2} = (\frac{1}{2\sqrt{2}}) \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1+\sqrt{2} \\ 1 & 1 \\ 1 & 1 \end{bmatrix} , \quad \Gamma_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad p_1 = \frac{8}{15}, p_2 = \frac{4}{15}, p_3 = \frac{2}{15}, p_4 = \frac{1}{15}$$

$$Z_{4\times 4} = \begin{bmatrix} 0.918643 & -0.081357 & -0.081357 & 0.377964 \\ -0.081357 & 0.918643 & -0.081357 & 0.377964 \\ -0.081357 & -0.081357 & 0.918643 & 0.377964 \\ -0.377964 & -0.377964 & -0.377964 & 0.755929 \end{bmatrix}$$

<u>Example 1:</u>

$A_{2\times 3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, therefore: rank($A_{2\times 3}$) = 1 , $(\|A_{2\times 3}\|_F)^2 = 6$

$\sigma(A_{2\times 3}) : \sigma = \sqrt{6} = 2.44949$

We have the following set of numerical results:

$$\rho_{4\times4} = (\frac{3}{2})\begin{bmatrix}1\\1\\1\\1\end{bmatrix}_{4\times1} \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}_{1\times4} \text{ , rank}(\rho_{4\times4}) = 1 \text{, trace}(\rho_{4\times4}) = 6$$

$$\hat{\rho}_{4\times4} = \begin{bmatrix}1.296585 & 1.296585 & 1.296585 & -0.654935\\1.296585 & 1.296585 & 1.296585 & -0.654935\\1.296585 & 1.296585 & 1.296585 & -0.654935\\-0.654935 & -0.654935 & -0.654935 & 2.110246\end{bmatrix} \text{, rank}(\hat{\rho}_{4\times4}) = 2 \text{, trace}(\hat{\rho}_{4\times4}) = 6$$

$$\bar{A}_{2\times3} = \begin{bmatrix}0.619619 & 0.619619 & 0.619619\\0.619619 & 0.619619 & 0.619619\end{bmatrix} \text{, rank}(\bar{A}_{2\times3}) = 1 \text{, } (\|\bar{A}_{2\times3}\|_F)^2 = 2.303569$$

$$\sigma(\bar{A}_{2\times3}) : \bar{\sigma} = 1.51775 \text{, } \eta = 55.48\%$$

$$\hat{A}_{2\times3} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\end{bmatrix} \text{, rank}(\hat{A}_{2\times3}) = 1 \text{, } (\|\hat{A}_{2\times3}\|_F)^2 = 6 \text{ , } \sigma(\hat{A}_{2\times3}) : \hat{\sigma} = 2.44949$$

<u>Example 2:</u>

$$A_{2\times3} = \begin{bmatrix}1 & 1 & 1\\-1 & -1 & -1\end{bmatrix} \text{, rank}(A_{2\times3}) = 1 \text{, } (\|A_{2\times3}\|_F)^2 = 6$$

$$\sigma(A_{2\times3}) : \sigma = \sqrt{6} = 2.44949$$

We have the following set of numerical results:

$$\rho_{4\times4} = \begin{bmatrix}2.25 & -0.75 & 0.75 & 0.75\\-0.75 & 2.25 & 0.75 & 0.75\\0.75 & 0.75 & 0.75 & 0.75\\0.75 & 0.75 & 0.75 & 0.75\end{bmatrix} \text{, rank}(\rho_{4\times4}) = 2 \text{, trace}(\rho_{4\times4}) = 6$$

$$\hat{\rho}_{4\times4} = \begin{bmatrix}2.148292 & -0.851708 & 0.648292 & -0.327468\\-0.851708 & 2.148292 & 0.648292 & -0.327468\\0.648292 & 0.648292 & 0.648292 & -0.327468\\-0.327468 & -0.327468 & -0.327468 & 1.055123\end{bmatrix} \text{, rank}(\hat{\rho}_{4\times4}) = 3 \text{, trace}(\hat{\rho}_{4\times4}) = 6$$

$$\bar{A}_{2\times3} = \begin{bmatrix}1.304162 & -0.427888 & 0.438137\\-0.427888 & 1.304162 & 0.438137\end{bmatrix} \text{, rank}(\bar{A}_{2\times3}) = 2 \text{, } (\|\bar{A}_{2\times3}\|_F)^2 = 4.151784$$

$$\sigma(\bar{A}_{2\times3}) : \bar{\sigma}_1 = 1.732051, \bar{\sigma}_2 = 1.073212 \text{ , } \eta = 81.79\ \%$$

$$\hat{A}_{2\times3} = \begin{bmatrix}1.57867 & -0.402266 & 0.588202\\-0.402266 & 1.57867 & 0.588202\end{bmatrix} \text{, rank}(\hat{A}_{2\times3}) = 2 \text{, } (\|\hat{A}_{2\times3}\|_F)^2 = 6$$

$\sigma(\hat{A}_{2\times3}):\hat{\sigma}_1=1.980936,\hat{\sigma}_2=1.440796$

Example 3:

$$A_{2\times3}=\begin{bmatrix}1&1&-1\\1&-1&1\end{bmatrix},\ \text{rank}(A_{2\times3})=2,\ (\|A_{2\times3}\|_F)^2=6$$

$\sigma(A_{2\times3}):\sigma_1=2,\sigma_2=\sqrt{2}=1.414214$

We have the following set of numerical results:

$$\rho_{4\times4}=\begin{bmatrix}2.162678&-0.792664&0.207336&0.525783\\-0.792664&2.251994&-0.748006&0.237108\\0.207336&-0.748006&1.251994&0.237108\\0.525783&0.237108&0.237108&0.333333\end{bmatrix},\ \text{rank}(\rho_{4\times4})=3,\ \text{trace}(\rho_{4\times4})=6$$

$$\hat{\rho}_{4\times4}=\begin{bmatrix}2.01546&-0.863371&0.136629&-0.334836\\-0.863371&2.257798&-0.742202&-0.050894\\0.136629&-0.742202&1.257798&-0.050894\\-0.334836&-0.050894&-0.050894&0.468943\end{bmatrix},\ \text{rank}(\hat{\rho}_{4\times4})=4,\ \text{trace}(\hat{\rho}_{4\times4})=6$$

$$\bar{A}_{2\times3}=\begin{bmatrix}1.142414&-0.618094&0.405227\\-0.495917&1.175274&0.1408\end{bmatrix},\ \text{rank}(\bar{A}_{2\times3})=2,\ (\|\bar{A}_{2\times3}\|_F)^2=3.498388$$

$\sigma(\bar{A}_{2\times3}):\bar{\sigma}_1=1.728967,\bar{\sigma}_2=0.713485$, $\eta=73.66\ \%$

$$\hat{A}_{2\times3}=\begin{bmatrix}1.545743&-0.53426&0.653768\\-0.389534&1.618976&0.353574\end{bmatrix},\ \text{rank}(\hat{A}_{2\times3})=2,\ (\|\hat{A}_{2\times3}\|_F)^2=6$$

$\sigma(\hat{A}_{2\times3}):\hat{\sigma}_1=2.059158,\hat{\sigma}_2=1.3266$

❖ $m=3,n=4$, therefore $s=5$

$$P_{4\times5}=\begin{bmatrix}1&0&0&0&\frac{1}{4}\\0&1&0&0&\frac{1}{4}\\0&0&1&0&\frac{1}{4}\\0&0&0&1&\frac{1}{4}\end{bmatrix},\ Q_{5\times3}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix},\ \Gamma_{5\times5}=\begin{bmatrix}1&0&0&0&1\\0&1&0&0&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}$$

$$W_{5\times 4}=(\frac{1}{4\sqrt{5}})\begin{bmatrix} 2+3\sqrt{5} & 2-\sqrt{5} & 2-\sqrt{5} & 2-\sqrt{5} \\ 2-\sqrt{5} & 2+3\sqrt{5} & 2-\sqrt{5} & 2-\sqrt{5} \\ 2-\sqrt{5} & 2-\sqrt{5} & 2+3\sqrt{5} & 2-\sqrt{5} \\ 2-\sqrt{5} & 2-\sqrt{5} & 2-\sqrt{5} & 2+3\sqrt{5} \\ 2 & 2 & 2 & 2 \end{bmatrix},$$

$$G_{5\times 3}=(\frac{1}{3\sqrt{5}})\begin{bmatrix} \sqrt{3}+2\sqrt{5} & \sqrt{3}-\sqrt{5} & \sqrt{3}-\sqrt{5} \\ \sqrt{3}-\sqrt{5} & \sqrt{3}+2\sqrt{5} & \sqrt{3}-\sqrt{5} \\ \sqrt{3}-\sqrt{5} & \sqrt{3}-\sqrt{5} & \sqrt{3}+2\sqrt{5} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix},$$

$$Z_{5\times 5}=\begin{bmatrix} 0.926777 & -0.073223 & -0.073223 & -0.073223 & 0.353553 \\ -0.073223 & 0.926777 & -0.073223 & -0.073223 & 0.353553 \\ -0.073223 & -0.073223 & 0.926777 & -0.073223 & 0.353553 \\ -0.073223 & -0.073223 & -0.073223 & 0.926777 & 0.353553 \\ -0.353553 & -0.353553 & -0.353553 & -0.353553 & 0.707107 \end{bmatrix},$$

$$p_1=\frac{16}{31}, p_2=\frac{8}{31}, p_3=\frac{4}{31}, p_4=\frac{2}{31}, p_5=\frac{1}{31}$$

Example 1:

$$A_{3\times 4}=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ \text{rank}(A_{3\times 4})=1, \ (\|A_{3\times 4}\|_F)^2=12$$

$$\sigma(A_{3\times 4}):\sigma=\sqrt{12}=3.464102$$

We have the following set of numerical results:

$$\rho_{5\times 5}=(\frac{12}{5})\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{5\times 1}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{1\times 5}, \ \text{rank}(\rho_{5\times 5})=1, \ \text{trace}(\rho_{5\times 5})=12$$

$$\hat{\rho}_{5\times 5}=\begin{bmatrix} 1.829032 & 1.829032 & 1.829032 & 1.829032 & -1.219355 \\ 1.829032 & 1.829032 & 1.829032 & 1.829032 & -1.219355 \\ 1.829032 & 1.829032 & 1.829032 & 1.829032 & -1.219355 \\ 1.829032 & 1.829032 & 1.829032 & 1.829032 & -1.219355 \\ -1.219355 & -1.219355 & -1.219355 & -1.219355 & 4.683871 \end{bmatrix},$$

rank($\hat{\rho}_{5\times5}$) = 2 , trace($\hat{\rho}_{5\times5}$) =12

$$\bar{A}_{3\times4} = \begin{bmatrix} 0.616382 & 0.616382 & 0.616382 & 0.616382 \\ 0.616382 & 0.616382 & 0.616382 & 0.616382 \\ 0.616382 & 0.616382 & 0.616382 & 0.616382 \end{bmatrix}, \text{ rank}(\bar{A}_{3\times4}) = 1, \ (\|\bar{A}_{3\times4}\|_F)^2 = 4.559118 ,$$

$\eta = 55.06\,\%$, $\sigma(\bar{A}_{3\times4}) : \bar{\sigma} = 2.135209$

$$\hat{A}_{3\times4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \text{ rank}(\hat{A}_{3\times4}) = 1, \ (\|\hat{A}_{3\times4}\|_F)^2 = 12 , \ \sigma(\hat{A}_{3\times4}) : \hat{\sigma} = 3.464102$$

<u>Example 2:</u>

$$A_{3\times4} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \text{ rank}(A_{3\times4}) = 2, \ (\|A_{3\times4}\|_F)^2 = 12$$

$\sigma(A_{3\times4}) : \sigma_1 = 2\sqrt{2} = 2.828427, \sigma_2 = 2$

$$\rho_{5\times5} = \begin{bmatrix} 3.110753 & -0.738979 & 1.261021 & -0.122401 & 0.877599 \\ -0.738979 & 3.41129 & -2.58871 & 1.361201 & 0.361201 \\ 1.261021 & -2.58871 & 3.41129 & -0.638799 & 0.361201 \\ -0.122401 & 1.361201 & -0.638799 & 1.533333 & 0.533333 \\ 0.877599 & 0.361201 & 0.361201 & 0.533333 & 0.533333 \end{bmatrix},$$

rank($\rho_{5\times5}$) = 4 , trace($\rho_{5\times5}$) = 12

$$\hat{\rho}_{5\times5} = \begin{bmatrix} 2.661721 & -0.946398 & 1.053602 & -0.410358 & -0.682885 \\ -0.946398 & 3.445484 & -2.554516 & 1.314857 & -0.065009 \\ 1.053602 & -2.554516 & 3.445484 & -0.685143 & -0.065009 \\ -0.410358 & 1.314857 & -0.685143 & 1.406452 & -0.270968 \\ -0.682885 & -0.065009 & -0.065009 & -0.270968 & 1.04086 \end{bmatrix}‘$$

rank($\hat{\rho}_{5\times5}$) = 5 , trace($\hat{\rho}_{5\times5}$) = 12

$$\bar{A}_{3\times4} = \begin{bmatrix} 1.293937 & -0.209364 & 0.098893 & 0.057884 \\ -0.424403 & 1.705637 & -0.843554 & 0.687426 \\ 0.069025 & -0.658384 & 1.576763 & 0.131020 \end{bmatrix}, \text{ rank}(\bar{A}_{3\times4}) = 3, \ (\|\bar{A}_{3\times4}\|_F)^2 = 8.946271$$

$\eta = 85.42\,\%$, $\sigma(\bar{A}_{3\times4}) : \bar{\sigma}_1 = 2.511327, \bar{\sigma}_2 = 1.281953, \bar{\sigma}_3 = 0.998051$

$$\hat{A}_{3\times4}=\begin{bmatrix}1.642592 & -0.159876 & 0.096802 & 0.126909\\ -0.409000 & 1.94716 & -0.776487 & 0.854234\\ 0.048967 & -0.586666 & 1.881434 & 0.270410\end{bmatrix}, \text{rank}(\hat{A}_{3\times4})=3,\ (\|\hat{A}_{3\times4}\|_F)^2=12$$

$$\sigma(\hat{A}_{3\times4}):\hat{\sigma}_1=2.706413,\hat{\sigma}_2=1.631353,\hat{\sigma}_3=1.41916$$

Example 3:

$$A_{3\times4}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{bmatrix},\ \text{rank}(A_{3\times4})=3,\ (\|A_{3\times4}\|_F)^2=12,\ \sigma(A_{3\times4}):\sigma_1=\sigma_2=\sigma_3=2$$

$$\rho_{5\times5}=\begin{bmatrix}3.133333 & 0.133333 & 0.133333 & -0.2 & 0.8\\ 0.133333 & 3.133333 & -0.866667 & 0.8 & 0.8\\ 0.133333 & -0.866667 & 3.133333 & 0.8 & 0.8\\ -0.2 & 0.8 & 0.8 & 1.8 & 0.8\\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8\end{bmatrix},$$

rank($\rho_{5\times5}$) = 4 , trace($\rho_{5\times5}$) = 12

$$\hat{\rho}_{5\times5}=\begin{bmatrix}2.943011 & -0.056989 & -0.056989 & -0.390323 & -0.406452\\ -0.056989 & 2.943011 & -1.056989 & 0.609677 & -0.406452\\ -0.056989 & -1.056989 & 2.943011 & 0.609677 & -0.406452\\ -0.390323 & 0.609677 & 0.609677 & 1.609677 & -0.406452\\ -0.406452 & -0.406452 & -0.406452 & -0.406452 & 1.56129\end{bmatrix},$$

rank($\hat{\rho}_{5\times5}$) = 5 , trace($\hat{\rho}_{5\times5}$) = 12

$$\bar{A}_{3\times4}=\begin{bmatrix}1.501706 & -0.093390 & -0.093390 & 0.108547\\ -0.217051 & 1.580497 & -0.419503 & 0.479529\\ -0.217051 & -0.419503 & 1.580497 & 0.479529\end{bmatrix}, \text{rank}(\bar{A}_{3\times4})=3,(\|\bar{A}_{3\times4}\|_F)^2=8.186373$$

$$\eta=81.11\ \%\ ,\ \sigma(\bar{A}_{3\times4}):\bar{\sigma}_1=2,\bar{\sigma}_2=1.632993,\bar{\sigma}_3=1.232764$$

$$\hat{A}_{3\times4}=\begin{bmatrix}1.87468 & -0.063676 & -0.063676 & 0.181718\\ -0.213948 & 1.903186 & -0.392725 & 0.632534\\ -0.213948 & -0.392725 & 1.903186 & 0.632534\end{bmatrix},\ \text{rank}(\hat{A}_{3\times4})=3,\ (\|\hat{A}_{3\times4}\|_F)^2=12$$

$$\sigma(\hat{A}_{3\times4}):\hat{\sigma}_1=2.295911,\hat{\sigma}_2=1.984408,\hat{\sigma}_3=1.670603$$

III. DISCUSSION AND CONCLUSION

The numerical case studies involving numerical matrices belonging to (m=2, n=3) and (m=3, n=4) complex matrix spaces demonstrate the framework and presents some insights about the examples themselves. We observe that the matrix of all ones belonging to the (m=2, n=3) matrix space and the (m=3, n=4) matrix space retains its numerical rank under the mathematical scheme and the final matrix is equal to the input matrix within the tolerances of numerical precision available in the computational platform used in the study. The second example of (m=2, n=3) case and that of the (m=3, n=4) case undergo rank enhancement upon application of the mathematical scheme and the efficiency of the extraction is observed to be around 82 % and 85% respectively. The third examples for both (m=2, n=3) and (m=3, n=4) cases correspond to the situation of the maximal possible rank; it can be observed that for both these examples there is preservation of the numerical rank under application of the mathematical scheme. The extraction efficiency is around 74% for the (m=2, n=3) example and it is around 81% for the (m=3, n=4) example.

An observable drawback of the methodology is that it fails to provide an output when the matrix $\bar{A}_{m\times n}$ is obtained as zero, i.e. $\bar{A}_{m\times n} = 0_{m\times n}$. In this situation there are no specific set of Principal axes available from $\bar{A}_{m\times n}$ to assign the unaccounted variance contribution.

In the case of Overflow, i.e. the situation: $(\|\bar{A}_{m\times n}\|_F)^2 > (\|A_{m\times n}\|_F)^2$ the total variance is mapped onto a single degree of freedom, which is associated with the Principal axes pair corresponding to the largest singular value of $\bar{A}_{m\times n}$. Therefore in the case of Overflow, the matrix $\hat{A}_{m\times n}$ is a rank=1 matrix while the rank of the matrix $\bar{A}_{m\times n}$ is possibly greater or equal to that of $\hat{A}_{m\times n}$. In the case of Underflow which corresponds to the situation: $(\|A_{m\times n}\|_F)^2 > (\|\bar{A}_{m\times n}\|_F)^2$, the unaccounted variance is split equally among the available degrees of freedom from the matrix $\bar{A}_{m\times n}$ and thus all the available 'b' degrees of freedom from the matrix $\bar{A}_{m\times n}$ is used up in this variance readjustment process.

The article presents a specific CPTP transformation generated from the information on the embedding dimension. The mathematical elements associated with the Spacer component matrices can also be coupled with different sets of Kraus operators [8, 11, 12, 13, 15, 16, 18] and the resulting mathematical schematic can be used to study and understand the functioning and limitations of the presented mathematical framework at a much deeper level. Follow up studies focused on this aspect of the research problem is expected to shed more light and provide a more complete understanding of the mathematical framework.

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ISSN No:-2456-2165

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