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# Development of Morley's Theorem on Right Triangles for Inner, Outer or Supplementary Angle Trisectors

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**Abstract:- Basically, Morley's Theorem gives the trisector of the angles in all three angles at any $\Delta ABC$ so that from the points of intersection, three points have obtained that form an equilateral triangle. But, if the angle trisector is given only at two angles, then an equilateral triangle cannot be formed, either using an inner angle trisector, an outer angle trisector, or a supplementary angle trisector. Based on these problems, it will be shown that by providing an inner angle trisector or an outer angle trisector at both non-right angles of any right triangle *ABC* an equilateral triangle can be formed. But by providing the supplementary angle trisector at a non-right angle, it will form a rhombus.**

***Keywords:-*** *Inner Angle Trisector, Outer Angle Trisector, Supplementary Angle Trisector, Morley's Theorem.*

## I. INTRODUCTION

An angle trisector is two lines that divide an angle into three equal parts. One theorem that immediately comes to mind when discussing angle trisectors is Morley's Theorem [1]. Basically, Morley's Theorem gives the trisector of the interior angle of any triangle. From the points of intersection, three points are obtained that form an equilateral triangle. This equilateral triangle became known as the Morley Triangle.

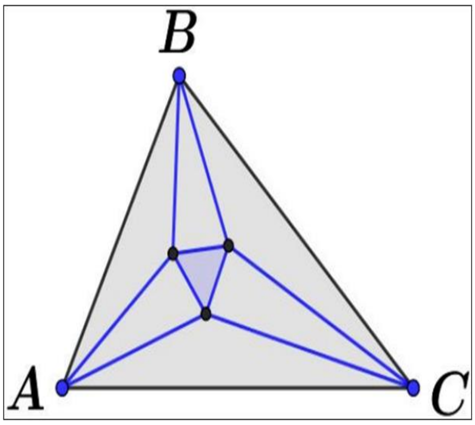

Fig 1 Equilateral Triangle with Inner Angle Trisector

Since its preface, Morley's Theorem has attracted the attention of researchers so many articles have been produced that discuss it, both for proof and development. Based on the type of angle trisector, there are three fairly easy ways to form an equilateral triangle. The first is an equilateral triangle formed using an inner angle trisector at all three angles at any $\Delta ABC$. This is the basis of Morley's Theorem. See Figure 1.

The second is an equilateral triangle formed by giving the outer angle trisector all three angles at any $\Delta ABC$ [2]. From the extension of the outer corner trisector lines, three points of intersection are obtained which will form an equilateral triangle. See Figure 2.

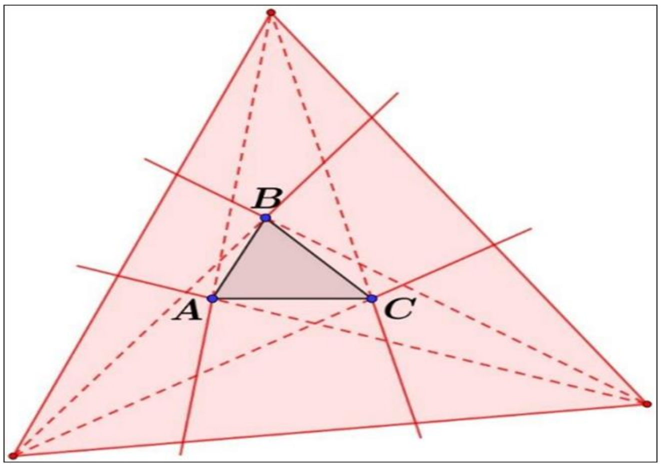

Fig 2 Equilateral Triangle with Outer Angle Trisector

The last one is an equilateral triangle which is formed by giving the supplementary angle trisector at all three angles of any triangle [3]. By giving the supplementary angle trisector at each angle, three intersection points have been obtained that form an equilateral triangle. See Figure 3.

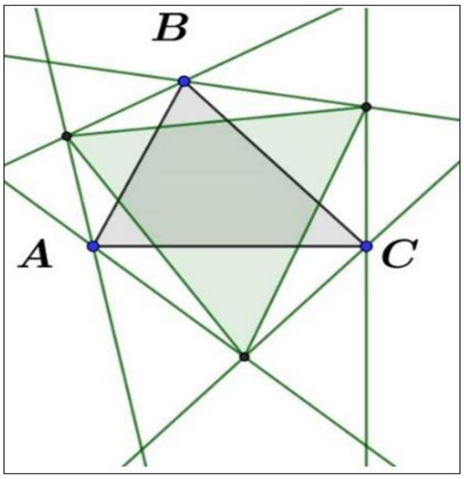

Fig 3 Equilateral Triangle with Supplementary Angle Trisector

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By three formations of equilateral triangles above, they have one thing in common, that is, the three types of angle trisectors are given at the three angles of any triangle. However, if the angle trisector is given only at two angles then an equilateral triangle cannot be formed. Based on this, this article discusses the inner angle trisector, the outer angle trisector, and the supplementary angle trisector in a right triangle with each angle trisector given only two non-right angles.

## II. LITERATURE REVIEW

Before entering the discussion, it is important to understand about the inner angle trisector, outer angle trisector, supplementary angle trisector, and several other basics first.

### *A. Inner Angle Trisector*

Given any $\Delta ABC$ with $BC=a$, $AC=b$, $\angle BAC = 3\alpha$, $\angle ABC = 3\beta$, and $\angle BCA = 3\gamma$. In $\Delta ABC$ there are $\angle BAC$ facing to $a$, $\angle ABC$ facing to $b$, and $\angle BCA$ facing to $c$. These three angles are called the interior angles of a triangle [4]. If the angle trisector is given to $\angle BAC$, it will divide $\angle BAC$ into three equal parts, that is $\alpha$. This angle trisector is then called the inner angle trisector. The inner angle trisector on $\angle BAC$ is denoted by $Ti_1\angle A$ and $Ti_2\angle A$ in clockwise notation order. See Figure 4 below.

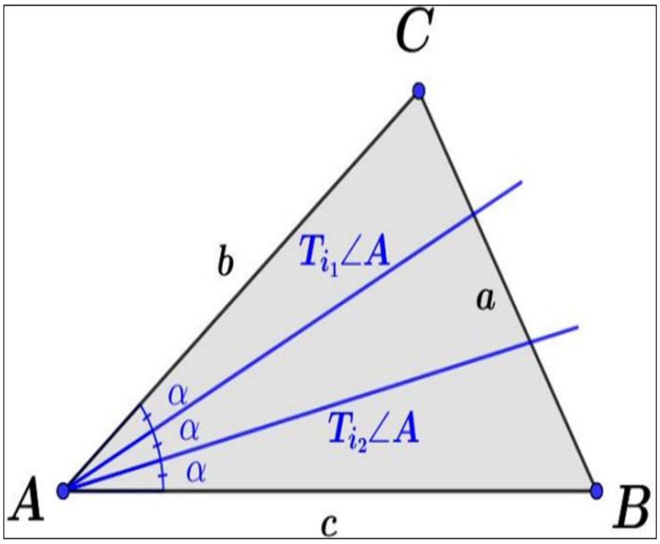


Fig 4 Inner Angle Trisector on $\angle BAC$

### *B. Outer Angle Trisector*

In addition to the interior angle, there is also an explementary angle [2]. An explementary angle is an angle that completes the inside angle to make one complete rotation. For example, on $\Delta ABC$ with one of the interior angles, that is $\angle BAC = 3\alpha$, then there is an explementary angle with an angle of $360° - 3\alpha$. If the explementary angle is given an angle trisector then this angle trisector will divide the explementary angle $BAC$ into three equal sizes, that is $120° - \alpha$. This angle trisector became known as the outer angle trisector. In this article, the outer angle trisector on $\angle BAC$ will be denoted by $To_1\angle A$ and $To_2\angle A$. See Figure 5.

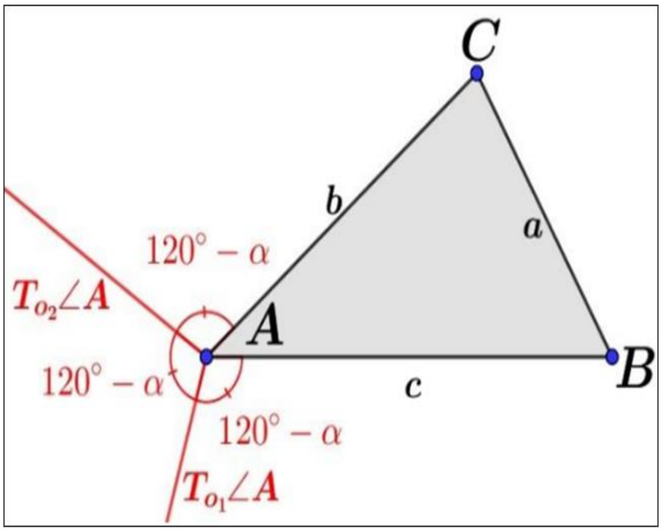


Fig 5 Outer Angle Trisector at $\angle BAC$

Furthermore, if the outer angle trisector at $\angle BAC$ is extended through point A then it will form an angle of $60° - 2\alpha$ concerning the nearest side $\Delta ABC$, as shown in Figure 6.

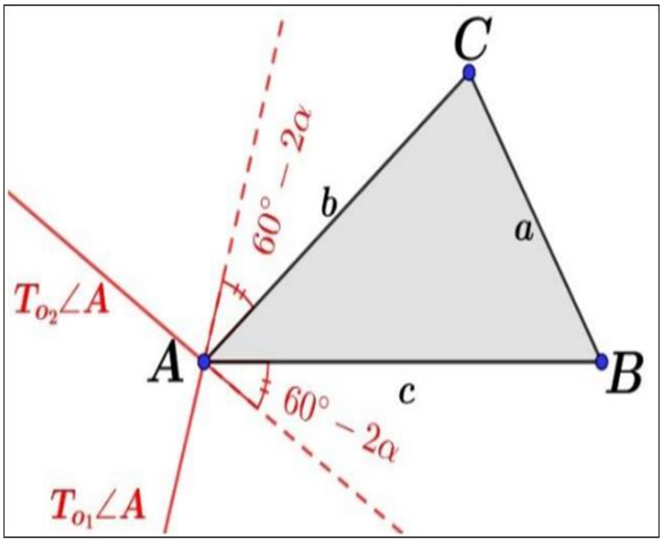


Fig 6 Extension of the Outer Angle Trisector at $\angle BAC$

### *C. Supplementary Angle Trisector*

In addition to the interior and explementary angles, there are also exterior angles or supplementary angles [3]. A supplementary angle is an angle that completes an angle to form a half-turn angle or 180°. At $\Delta ABC$ with $\angle BAC = 3\alpha$, side $BA$ is extended through point $A$ and stops at a point (namely point $P$). Then the $CA$ side is also extended through point $A$ and stops at a point (namely point $Q$). Because $BP$ and $CQ$ intersect at $A$, $\angle CAP$ and $\angle BAQ$ have the same angles (vertical angles). These two angles are called supplementary angles. Because $\angle CAP$ and $\angle BAQ$ are supplementary angles, they have an angle measure of $180° - 3\alpha$. If an angle trisector is given to these two angles, then this angle trisector will divide the angles into three equal parts, that is $60° - \alpha$. This angle trisector is known as the supplementary angle trisector. In this article, the supplementary angle trisector at $\angle BAC$ will be denoted by $Ts_1\angle A$ and $Ts_2\angle A$. This can be seen in the following figure.

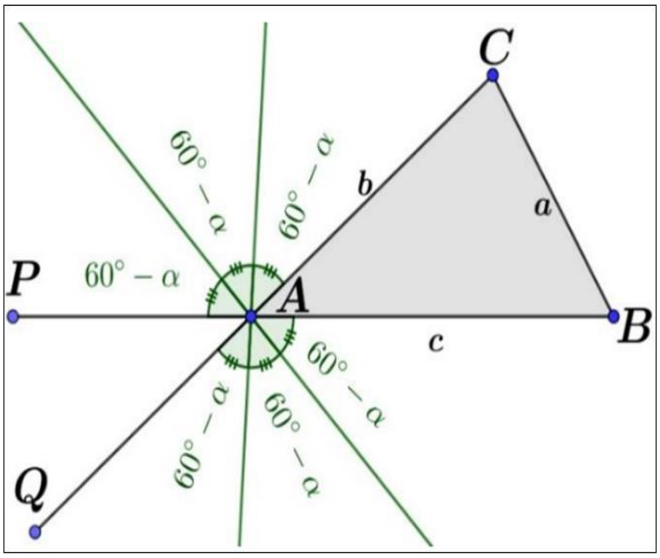


Fig 7 Supplementary Angle Trisector at ∠*BAC*

*D. Sine Rule*

In any Δ*ABC* with *BC* = *a*, *AC* = *b*, *AB* = *c*, ∠*BAC* = *α*, ∠*ABC* = *β*, and ∠*BCA* = *γ* we can form the circumcircle. Each side and angle leading to it has the same ratio, which is twice the radius of the circumcircle Δ*ABC* (2*R*). This has been stated in the following theorem [5-7].

- *Ttheorem 2.1. (Sine Rule) Suppose a, b, and c are side lengths on ΔABC, then*

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

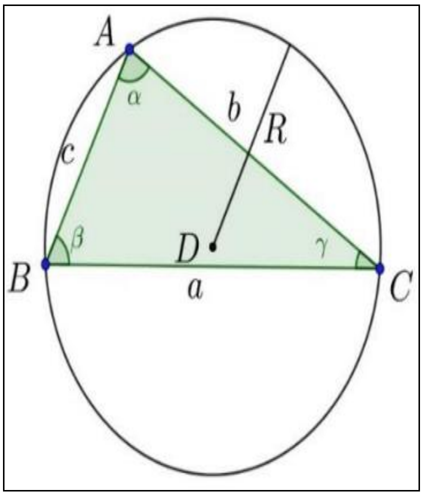


Fig 8 Sine Rule

Proof. See [5-7]

*E. Morley's Theorem*

In the previous discussion, it can be seen that the inner angle trisector is given to ∠*BAC*. Furthermore, by giving the inner angle trisector on ∠*BAC*, ∠*ABC*, and ∠*BCA* three intersection points are obtained which produce an equilateral triangle. This condition is mentioned in a theorem, namely *Morley's Theorem* [3]. Here's the theorem.

- *Theorem 2.1. (Morley's Theorem) In any triangle the trisectors of its angles, proximal to the three sides respectively, meet at the vertices of an equilateral.*

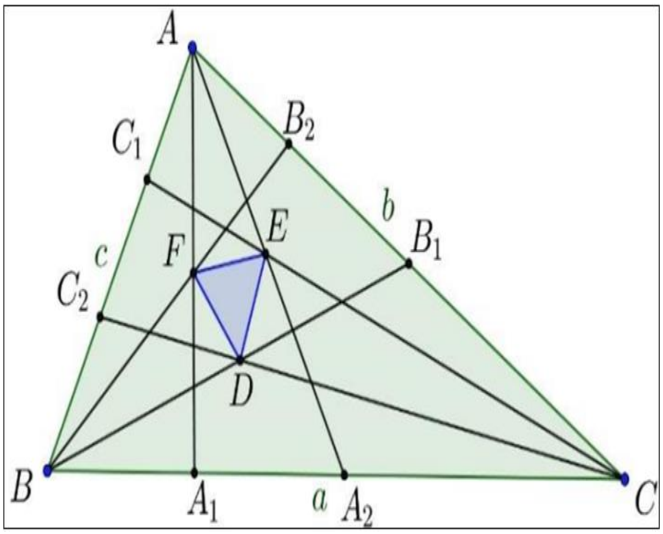


Fig 9 Morley's Theorem

- Proof. Various proof methods can be seen in [18–31], so that the side length ΔDEF = 8R sin (α) sin (β) sin (γ) is obtained.

## III. EQUILATERAL TRIANGLE WITH THE INNER AND OUTER TRISECTORS IN A RIGHT TRIANGLE

Suppose a Δ*ABC* with *BC* = *a*, *AC* = *b*, *AB* = *c*, ∠*BAC* = 3*α*, ∠*ABC* = 90°, and ∠*BCA* = 3*γ*. Because ∠*BAC* + ∠*ABC* + ∠*BCA* = 180° we get ∠*BCA* = 3(30° – *α*) or *γ* = 30° – *α*. By giving the inner angle trisector and outer angle trisector on ∠*BCA*, the inner angle trisector is 30° – *α*, the outer angle trisector is 90° + α, and the angle formed by the extension of the outer angle trisector with side *AC* or *BC* that is 2α. This can be seen in the Figure 10.

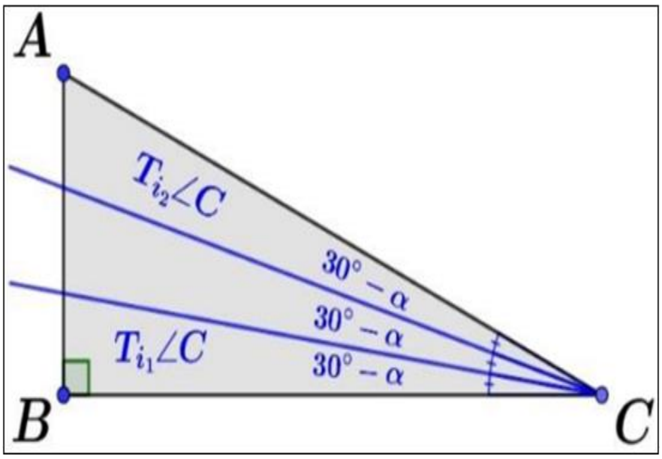


Fig 10 Inner Angle Trisector at ∠*BCA*

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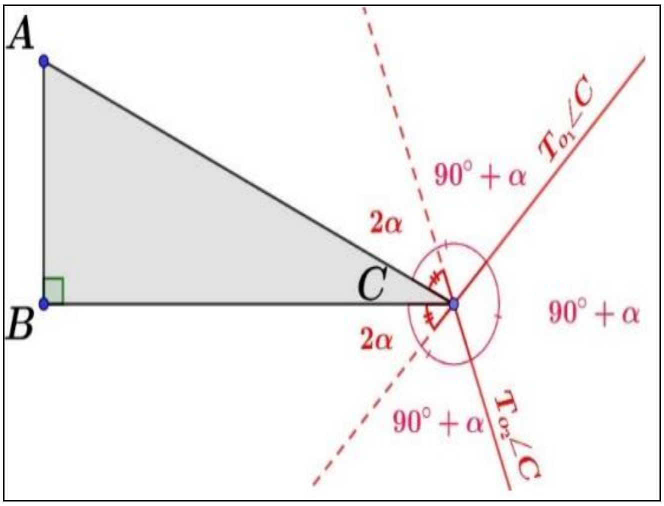

Fig 11 Outer Angle Trisector at $\angle BCA$

Furthermore, by applying the inner angle trisector to $\angle BAC$ and $\angle BCA$, the first equilateral triangle is obtained. Here's the theorem.

➢ *Theorem 3.1. On $\Delta ABC$ with right angles at B, given the inner angle trisectors at A ($Ti_1\angle A$ and $Ti_2\angle A$) and C ($Ti_1\angle C$ and $Ti_2\angle C$). $Ti_1\angle A$ and $Ti_2\angle C$ intersect at D, $Ti_2\angle A$ and BC intersect at E, $Ti_1\angle C$ and BA intersect at F. Points D, E, and F form an equilateral.*

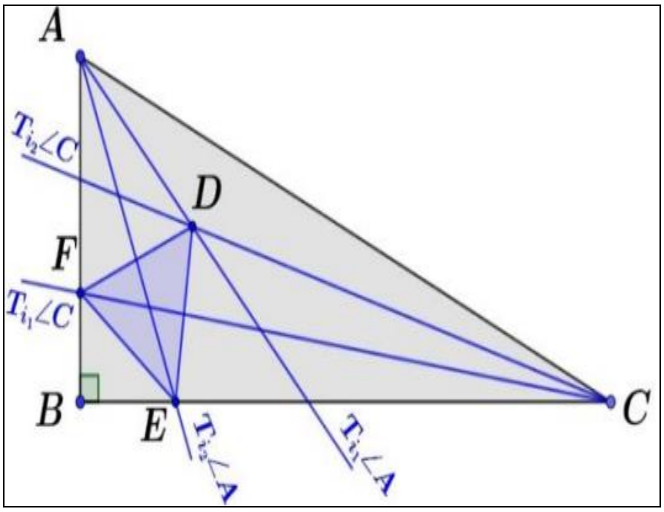

Fig 12 Equilateral Triangle with Inner Angle Trisectors at *A* and *C*

- *Proof*. The proof is done by showing two things, there are $ED = EF$ and $FD = FE$. Based on the inner angle trisector discussed earlier, we have $\angle DAC = \alpha$ and $\angle DCA = 30° - \alpha$, so we get $\angle ADC = 150°$. By using the sine rule on $\Delta ADC$ is obtained

$$AD = 2b \sin(30° - \alpha) \quad (1)$$

Furthermore, by paying attention to $\Delta AFC$, because $\angle FAC = 3\alpha$ and $\angle FCA = 2(30° - \alpha)$ it is easy to obtain $\angle AFC = 120° - \alpha$. By using the sine rule is obtained

$$AF = 2b \sin(30° - \alpha) \quad (2)$$

From equations (1) and (2) it can be seen that $AD = AF$. Next, by paying attention to $\Delta ADE$ and $\Delta AFE$ we have $AD = AF$, $\angle DAE = \angle DAF$, $AE = AE$ so that based on the congruence of the side-angle-side, we get that $\Delta ADE \cong \Delta AFE$. Since the two triangles are congruent, $ED = EF$.

By doing the same for $\Delta ADC$ and $\Delta AEC$ we get

$$CD = 2b \sin \alpha \quad (3)$$

$$CE = 2b \sin \alpha \quad (4)$$

From equations (3) and (4) it can be seen that $CD = CE$. Next, by paying attention to $\Delta CEF$ and $\Delta CDF$, we have $CE = CD$, $\angle ECF = \angle DCF$, $CF = CF$ so that based on the congruence of the side-angle-side, it is obtained that $\Delta CEF \cong \Delta CDF$. Since the two triangles are congruent, then $FD = FE$. With two conditions fulfilled ($ED = EF$ and $FD = FE$), then $\Delta DEF$ is an equilateral triangle. Thus, Theorem 3.1 is proven. ■

Furthermore, the second equilateral triangle is obtained by giving the outer angle trisectors to $\angle BAC$ and $\angle BCA$. This can be seen in the following theorem.

➢ *Theorem 3.2. On $\Delta ABC$ with the right angle at B, given the outer angle trisectors at A ($To_1\angle A$ and $To_2\angle A$) and C ($To_1\angle C$ and $To_2\angle C$). $To_2\angle A$ and $To_1\angle C$ intersect at D, $To_1\angle A$ and BC side extensions intersect at E, $To_2\angle C$ and BA side extensions intersect at F. Points D, E, and F form an equilateral.*

- *Proof*. The proof is done by showing two things, there are $ED = EF$ and $FD = FE$. Based on the outer angle trisector discussed earlier, we have $\angle DAC = 60° + \alpha$ and $\angle DCA = 90° - \alpha$, so we can easily obtain $\angle ADC = 30°$. By using the sine rule on $\Delta ADC$ is obtained

$$AD = 2b \cos \alpha \quad (5)$$

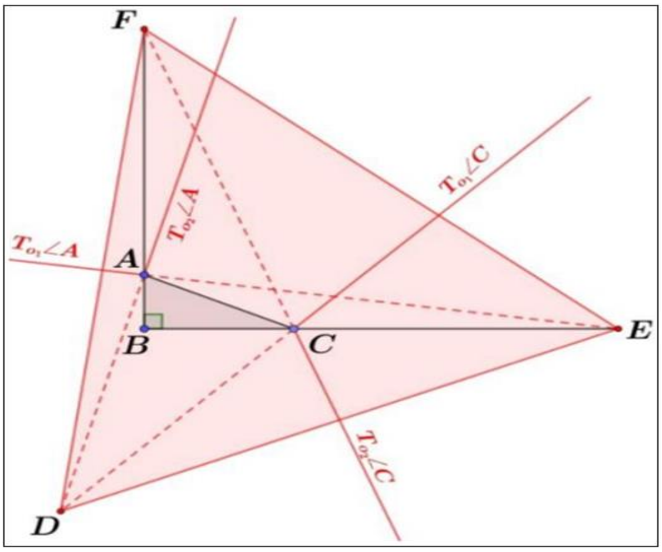

Fig 13 Equilateral Triangle with Outer Angle Trisectors at *A* and *C*

Furthermore, by paying attention to $\Delta AFC$, because $\angle FAC = 180° - 3\alpha$ and $\angle FCA = 2\alpha$, it is easy to obtain $\angle AFC = \alpha$. By using the sine rule is obtained

$$AF = 2b \cos \alpha \quad (6)$$

From equations (5) and (6) it can be seen that $AD = AF$. Next, by paying attention to $\Delta ADE$ and $\Delta AFE$ we get $AD = AF$, $\angle DAE = \angle FAE$, $AE = AE$ so that based on the congruence of the side-angle-side, we get that $\Delta ADE \cong \Delta AFE$. Since the two triangles are congruent, so $ED = EF$.

By doing the same for $\Delta ADC$ and $\Delta AEC$ we get

$$CD = 2b \cos (30° - \alpha) \quad (7)$$

$$CE = 2b \cos(30° - \alpha) \quad (8)$$

From equations (7) and (8) it can be seen that $CD = CE$. Next, by paying attention to $\Delta CEF$ and $\Delta CDF$, $CE = CD$, $\angle ECF = \angle DCF$, $CF = CF$ so that based on the congruence of the side-angle-side, it is obtained that $\Delta CEF \cong \Delta CDF$. Since the two triangles are congruent, then $FD = FE$. With two conditions fulfilled ($ED = EF$ and $FD = FE$), then $\Delta DEF$ is an equilateral triangle. Thus, Theorem 3.2 is proven. ■

## IV. ROMBUS WITH THE SUPPLEMENTARY TRISECTOR IN A RIGHT TRIANGLE

In the previous discussion, we discussed the angle trisector at $\angle BAC$ in any triangle. Furthermore, on $\Delta ABC$ with a right angle at $B$, a supplementary angle trisector is formed on $\angle BCA$ which divides the supplementary angle into three equal parts, that is $30° + \alpha$. This supplementary angle trisector is denoted by $Ts_1\angle C$ and $Ts_2\angle C$. By giving the supplementary angle trisector at $A$ and $C$, an equilateral triangle cannot be formed. However, from this supplementary angle trisector produces points that form a rhombus. This can be seen in the following theorem.

- *Theorem 4.1. On $\Delta ABC$ with right angles at B, given trisectors with right angles at A ($Ts_1\angle A$ dan $Ts_2\angle A$) and at C ($Ts_1\angle C$ dan $Ts_2\angle C$). $Ts_1\angle A$ and $Ts_1\angle C$ intersect at D, $Ts_2\angle A$ and $Ts_2\angle C$ intersect at E. Points A, C, E, and D form a rhombus.*

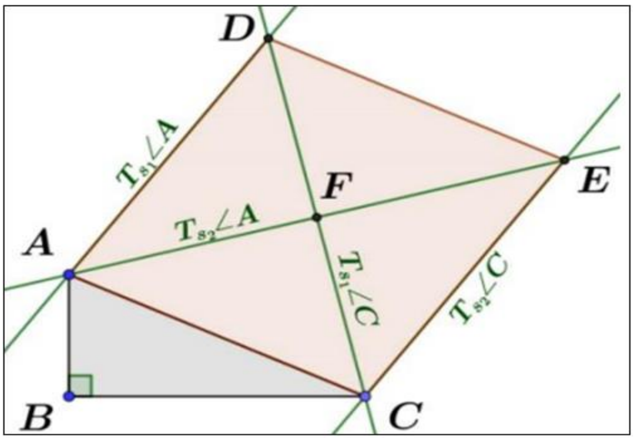

Fig 14 Rhombus with Angle Trisector at *A* and *C*

*Proof*. The proof is done by showing two things, there are $AC = CE = ED = DA$ and the opposite angles are equal. At $\Delta ACF$, based on the supplementary angle trisector discussed earlier, we have obtained $\angle CAF = 60° - \alpha$ and $\angle ACF = 30° + \alpha$ then obtained $\angle AFC = 90°$. Because $AE$ is a straight line, $\angle EFC = \angle AFC = 90°$. By paying attention to $\Delta AFC$ and $\Delta EFC$ we get $\angle AFC = \angle EFC$, $CF = CF$, and $\angle ACF = \angle ECF$ so that based on the congruence of the angle-side-angle we get $\Delta AFC \cong \Delta EFC$. Since the two triangles are congruent then $AC = CE$ and $AF = EF$. In the same way on $\Delta CFA$ and $\Delta DFA$ also obtained $AC = DA$ and $CF = DF$.

Furthermore, by considering $\Delta AFC$ and $\Delta EFD$, $\angle AFC$ and $\angle EFD$ are vertical angles so that $\angle AFC = \angle EFD$. Because $AF = EF$, $\angle AFC = \angle EFD$, and $CF = DF$, based on the congruence of the side-angle-side, we get $\Delta AFC \cong \Delta EFD$. Since the two triangles are congruent then $ED = AC$. Since $AC = CE = ED = DA$, the first condition is satisfied.

Previously we obtained $\Delta AFC \cong \Delta EFC$ so that $\angle CAF = \angle CEF$. For the same reasons, $\angle DAF = \angle DEF$, so $\angle CAF = \angle DAF = \angle CEF + \angle DEF$, or in other words $\angle CAD = \angle CED$. In the same way, $\angle ACE = \angle ADE$ is also obtained, so that the second condition is fulfilled. With both conditions fulfilled, the quadrilateral $ACED$ is a rhombus, and Theorem 4.1 is proven. ■

## V. CONCLUTION

On Morley's Theorem, an equilateral triangle can be formed by giving each angle trisector at all three angles of any triangle. If an equilateral triangle cannot be formed by giving each angle trisector at only two angles of any triangle, then by giving an inner angle trisector and an outer angle trisector to any right triangle, an equilateral triangle still can be formed. Slightly different for the supplementary angle trisector, from the points of intersection a rhombus can be formed.

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