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Modified-Extended Rani Distribution: Baseline Distribution Generalization Approach

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Abstract:- In this study, an exponentiated form of the Rani distribution is proposed to capture the trend of increasing failure rate via reparameterization by adopting the generalization technique. The newly proposed distribution is termed Modified-Extended Rani Distribution adopting a baseline generalization approached. The distribution is evaluated to be legitimate and all necessary properties which include moments, Renyi entropy, reliability function, hazard function were established. The graphical representation of the distribution attributed to diverse shapes for comparison and asymptotic direction is explored to ascertained the stand of the distribution.

I. INTRODUCTION

Amon the main factors underlying the study of lifetime events entails the indulgence of reliability analysis and probability distribution modelling approach in order to be able to ascertain the prevailing dynamics in the datasets thereby gain deep knowledge about the intrinsic behavioural traits of the data among other features.

Over the years, researchers have proposed a lot of distributions to achieve this aim attributed to diverse field of study such as Applied Science, Engineering, Finance, Insurance and lot more because of the peculiarity of data set. These data sets often possess different shapes, mean residual life, hazard rate etc. and requires a unique distribution which gives the best fit. Some of the one-parameter distributions that has been proposed includes the Exponential distribution;[1], Lindley distribution [2], Akash distribution [3], Lomax distribution [4], Ishita distribution [5], Sujatha distribution [6], Devya distribution [7], Akshaya distribution [8] Amongst others.

These distributions often have advantages over one another for instance, the exponential and Lindley distribution are more popular due to the superiority of their survival function as they cannot be expressed in closed form [9]. In terms of modelling lifetime datasets, the Ishita distribution as proposed by Shanker and Shukla [5] is shown to provide better fits than the Akash, Lindley and Exponential distributions. The Rama distribution [10] also gave better fits than the Akash, Lindley, Exponential, Amarendra, and Shanker distributions. Rani distribution [10] have been found to fit better than the Akash, Devya, Akshaya, Ishita, Sujatha and Lindley distributions.

These one-parameter distributions are constrained in terms of flexibility hence the need to improve on the distributions. One of the methods of improving flexibility is by generalizing the baseline distribution. This is often done by adding one or more extra shape/parameter(s) to the baseline distribution.

The exponentiation method of Mudholkar and Srivastava [11] is popularly used when generalizing a baseline distribution due to its ability to be more flexible and give better fit. Some of the distributions that have been generalized using this method by researchers includes the exponentiated Lindley distribution [12], the exponentiated Frechet distribution [13], exponentiated inverted Weibull distribution [14], the Extended Pranav distribution [15], the Exponentiated Akash distribution [16] among others. These exponentiated distributions were shown to be superior and more flexible than their baseline distribution using real life datasets.

The aim of this study therefore is to generalize Rani Distribution (Shanker, 2017) using the exponentiation technique and show the flexibility of this new distribution as well as its ability to provide better fit to some real-life data sets. The rest of the paper is sectionalized to introduce the Modified-Extended Rani distribution, the reliability analysis of the distribution, the characteristics of the new distribution, the maximum likelihood estimation of the new distribution, and draws necessary conclusion.

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II. THE EXTENDED RANI DISTRIBUTION

- *A Random Variable x is said to have an Exponentiated Distribution if the Pdf and Cdf are given Respectively by*

$$g(x) = \theta[F(x)]^{\theta-1}f(x) \quad (1)$$

$$G(x) = [F(x)]^{\theta} \quad (2)$$

- *Given that the Pdf of a Rani Distribution is given by*

$$f(x) = \left(\frac{\alpha^5}{\alpha^5+24}\right)(\alpha + x^4)e^{-\alpha x} \quad (3)$$

- *And the Cdf of a Rani Distribution is given by*

$$F(x) = 1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x} \quad (4)$$

Hence, a random variable x is said to have an exponentiated Rani distribution if the pdf and cdf are given respectively by substituting equations (1) & (2) into equation (3) & (4) as shown below.

$$g(x) = \alpha\left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3 + 4\alpha^2x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right]e^{-\theta x}\right]^{\alpha-1}\left(\frac{\alpha^5}{\alpha^5 + 24}\right)(\alpha + x^4)e^{-\alpha x}$$

$$g(x) = \alpha\left(\frac{\alpha^5}{\alpha^5+24}\right)(\alpha + x^4)e^{-\alpha x}\left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\theta x}\right]^{\theta-1} \quad (5)$$

- *Equation 5 is the pdf of the exponentiated Rani distribution.*

$$G(x) = \left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\theta} \quad (6)$$

- *Equation (6) is the Cdf of the Exponentiated Rani Distribution.*

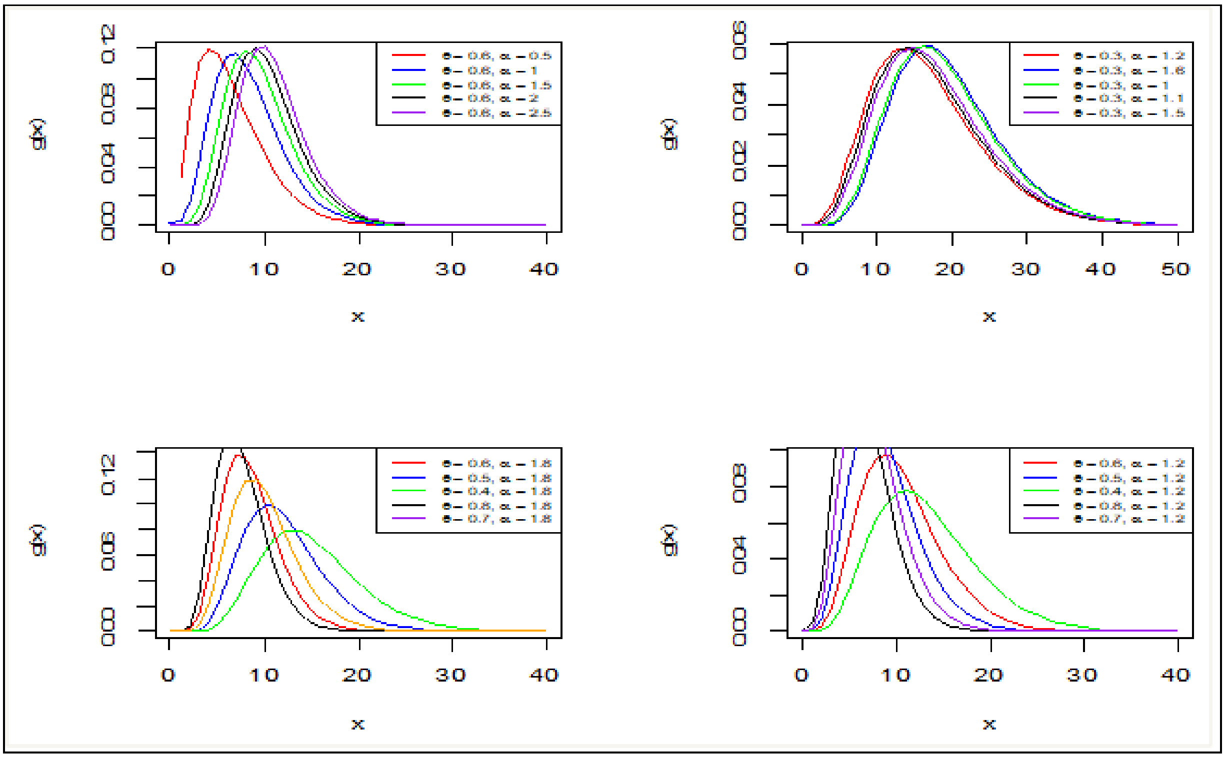

Fig 1 The Plots of the Pdf of the Exponentiated Rani Distribution

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III. RELIABILITY ANALYSIS

The survival function and hazard rate function which are important in the reliability analysis of any given probability distribution is presented in this section.

➢ *Survival Function*

The survival function is defined as the probability that an item does not fail prior to some time, t. The survival function of the exponentiated Rani distribution is given by

$$S(x) = 1 - G(x) \tag{7}$$

$$S(x) = 1 - \left[1 - \left(1 + \frac{\alpha x(\alpha^3 x^3 + 4\alpha^2 x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right) e^{-\alpha x}\right]^{\theta} \tag{8}$$

- *The Survival Function Plot Shown in Fig. 3 Indicates a Monotone Decreasing Function.*

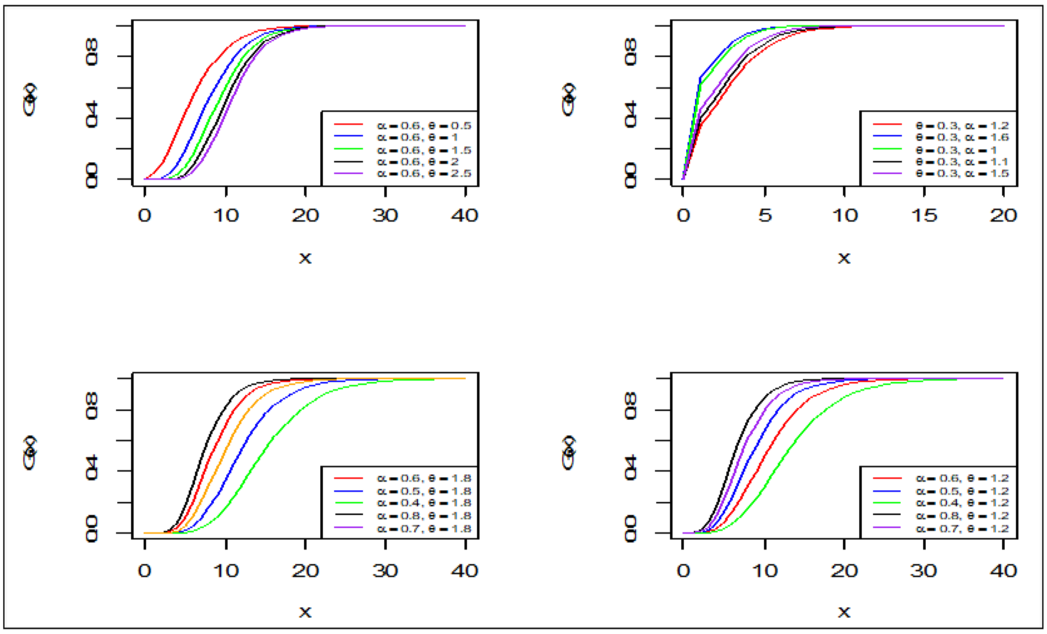

Fig 2 The Plots of the Cdf of the Exponentiated Rani Distribution

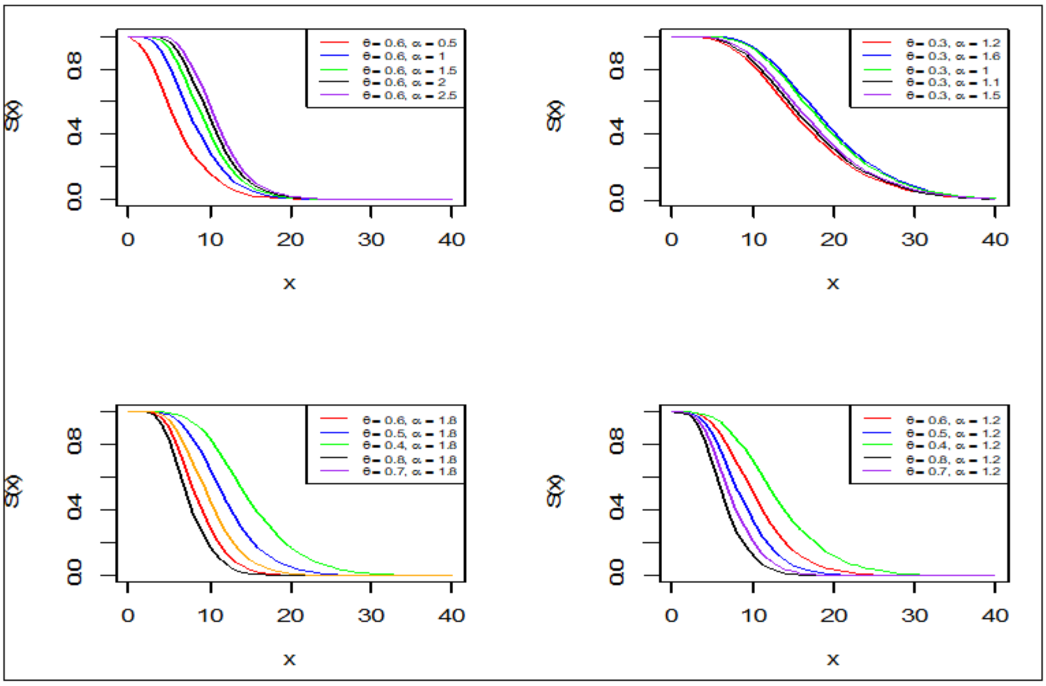

Fig 3 The Survival Plot of the Exponentiated Rani Distribution

➢ *Hazard Rate Function*

The hazard rate function on the other hand can be seen as the conditional probability of failure, given it has survived to the time, t. The hazard rate function of the exponentiated Rani distribution is given by

$$h(x) = \frac{g(x)}{1-G(x)} \tag{9}$$

$$h(x) = \frac{\frac{\theta\alpha^5}{\alpha^5+24}(\alpha+x^4)\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1}e^{-\alpha x}}{1-\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta}} \tag{10}$$

- *The Plots of the Hazard Rate Function Shown in Fig. 4 Indicates a Monotone Increasing Function*

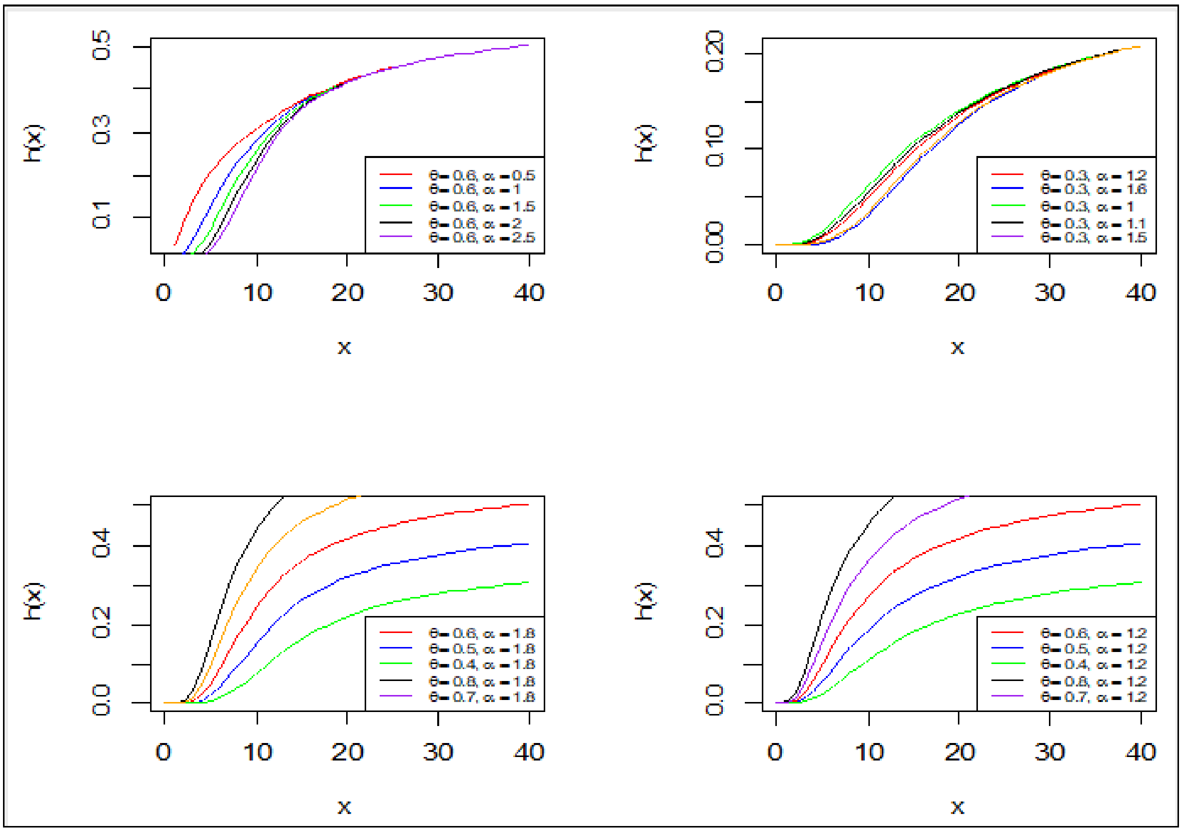

Fig 4 The Hazard Rate Function Plot of the Exponentiated Rani Distribution

IV. MATHEMATICAL CHARACTERISTICS

We present in this section, the mathematical characteristics of the exponentiated Rani distribution

A. Moments

➢ *Theorem 1*

Given a random variable X, following an exponentiated Rani distribution, the rth order moment about origin, $E(X^r)$ of the exponentiated Rani distribution is given by

$$E(X^r) = A_{ijk}\frac{(r+4j-k-l-m)!}{[i+1]^{r+4j-k-l-m+1}} + B_{ijk}\frac{(r+4j-k-l-m+4)!}{[i+1]^{r+4j-k-l-m+5}}, \tag{11}$$

Where

$$A_{i,j,k,l} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^{-r+5}\alpha 4^k 3^l 2^m}{(\theta^5+24)^{j+1}}$$

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and

$$B_{ijk} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i \sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^{-r}\alpha 4^k 3^l 2^m}{(\theta^5+24)^{j+1}}.$$

➢ *Proof*

The rth moment of a random variable X is given by;

$$E(X^r) = \int x^r\, g(x)dx$$

$$= \int x^r\left(\frac{\theta\alpha^5}{\alpha^5+24}\right)(\alpha + x^4)e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1} dx$$

$$= \int \frac{x^r\theta\alpha^6}{\alpha^5+24}e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1} dx + \int \frac{x^{r+4}\theta\alpha^5}{\alpha^5+24}e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1} dx$$

- Using the binomial series expansion method and substituting, we obtain

$$= \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha\alpha^{4j-k-l-m+6}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\int x^{4j-k-l-m+r}\, e^{-\alpha x(i+1)}dx + \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{4j-k-l-m+5}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\int x^{4j-k-l-m+4+r}\, e^{-\alpha x(i+1)}dx$$

- Also, applying the Gamma properties, $\int_0^\infty x^n e^{-\alpha x}dx = \frac{\Gamma(n+1)}{a^{n+1}}$ *and* $\Gamma(\alpha) = (\alpha-1)!$

$$E(X^r) = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^{-r+5}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\frac{(r+4j-k-l-m)!}{[i+1]^{r+4j-k-l-m+1}} +$$

$$\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{-r}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\frac{(r+4j-k-l-m+4)!}{[i+1]^{r+4j-k-l-m+5}}.$$

Let $A_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{-r+5}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}$

And

$$B_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{-r}\alpha 4^k3^l2^m}{(\alpha^5+24)^{j+1}}$$

Therefore;

$$A_{ijk}\frac{(r+4j-k-l-m)!}{[i+1]^{r+4j-k-l-m+1}} + B_{ijk}\frac{(r+4j-k-l-m+4)!}{[i+1]^{r+4j-k-l-m+5}}$$

B. Moment Generating Function

Here, we propose the moment generating function for the exponentiated Rani distribution.

➢ *Theorem 2*

Let X have an exponentiated Rani distribution. Then the moment generating function of X,$M_X(t)$ is given by

$$M_X(t) = \sum_{n=0}^{\infty}\left(\frac{t}{\alpha}\right)^n\left[A_{ijk}\frac{(4j-k-l-m+n)!}{n![i+1]^{4j-k-l-m+n+1}} + B_{ijk}\frac{(4j-k-l-m+n+4)!}{n![i+1]^{4j-k-l-m+n+5}}\right], \quad (12)$$

Where

$$A_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^5\theta 4^k 3^l 2^m}{(\alpha^5+24)^{j+1}}$$

And

$$B_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta 4^k 3^l 2^m}{(\alpha^5+24)^{j+1}}$$

- *Proof*

The moment generating function of a random variable X is given by

$$M_X(t) = E(e^{tx}) = \int e^{tx} g(x)dx$$

$$= \int e^{tx}\left(\frac{\theta\alpha^5}{\alpha^5+24}\right)(\alpha+x^4)e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1}dx$$

$$= \int \frac{e^{tx}\theta\alpha^6}{\alpha^5+24}e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1}dx$$
$$+\int \frac{x^4\theta\alpha^5 e^{tx}}{\alpha^5+24}e^{-\alpha x}\left[1-\left(1+\frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1}dx$$

Using the binomial series expansion method and the property $e^{tx} = \sum_{n=0}^{\infty}\frac{(tx)^n}{n!}$, we obtain;

$$M_X(t)$$
$$= \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\alpha^{4j-k-l-m+6}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\int x^{4j-k-l-m+n}e^{-\alpha x(i+1)}dx$$
$$+ \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{\alpha^{4j-k-l-m+5}\theta 4^k3^l2^m}{(\alpha^5+24)^{j+1}}\int x^{4j-k-l-m+4+n}e^{-\alpha x(i+1)}dx$$

Also, applying the Gamma properties, $\int_0^{\infty}x^n e^{-\alpha x}dx = \frac{\Gamma(n+1)}{\alpha^{n+1}}$ *and* $\Gamma(\alpha) = (\alpha-1)!$

$$M_X(t) = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{\infty}\left(\frac{t}{\alpha}\right)^n\frac{\alpha^5\theta4^k3^l2^m}{(\alpha^5+24)^{j+1}}\frac{(4j-k-l-m+n)!}{[i+1]^{r+4j-k-l-m+1}} +$$
$$\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{\infty}\left(\frac{t}{\alpha}\right)^n\frac{\theta4^k3^l2^m}{(\alpha^5+24)^{j+1}}\frac{(4j-k-l-m+n+4)!}{[i+1]^{r+4j-k-l-m+5}}.$$

Let $A_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^5\theta4^k3^l2^m}{(\alpha^5+24)^{j+1}}$

And

$$B_{ijk} = \sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta 4^k 3^l 2^m}{(\alpha^5+24)^{j+1}}$$

- *Therefore;*

$$M_X(t) = \sum_{n=0}^{\infty}\left(\frac{t}{\alpha}\right)^n\left[A_{ijk}\frac{(4j-k-l-m+n)!}{n!\,[i+1]^{4j-k-l-m+n+1}} + B_{ijk}\frac{(4j-k-l-m+n+4)!}{n!\,[i+1]^{4j-k-l-m+n+5}}\right]$$

C. Order Statistics

The order statistics of the exponentiated Rani distribution is presented below.

- *Theorem 3*

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponentiated Rani distribution. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ Denote the corresponding order statistics. Then, the probability density function, pdf of the pth order statistics, say$X = X_{(p)}$, is given by

$$g_X(x) = \frac{n!\theta\alpha^5(\alpha+x^4)}{(p-1)!(n-p)!}\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\theta(p+i)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\times \frac{\alpha^{4k-l-m-n}x^{4k-l-m-n}4^l3^m2^n}{(\alpha^5+24)^{k+1}}e^{-\alpha x(j+1)} \quad (13)$$

While the Cdf is given by;

$$G_X(x) = \sum_{i-p}^{n}\binom{n}{i}\sum_{i=0}^{\infty}\binom{\theta(j+1)-1}{k}(-1)^i\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\sum_{q=0}^{n}\binom{n}{q}\frac{\alpha^{4l-m-n-q}x^{4l-m-n-q}4^m3^n2^q}{(\alpha^5+24)^l}e^{-k\alpha x} \quad (14)$$

- *Proof.*

The pdf of the p^th^ order statistics is given by

$$g_X(x) = \frac{n!}{(p-1)!\,(n-p)!}G^{p-1}(x)[1-G(x)]^{n-p}g(x)$$

$$g_X(x) = \frac{n!}{(p-1)!(n-p)!}G^{p-1+i}(x)g(x)\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i \quad (15)$$

Substituting for $G(x)$ and $g(x)$ in the equations above we obtain the pdf and cdf of the order statistics respectively as;

$$g_X(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\theta(p-1+i)}\left(\frac{\theta\alpha^5}{\alpha^5+24}\right)(\alpha + x^4)e^{-\alpha x}\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\theta-1}$$

$$g_X(x) = \frac{n!}{(p-1)!\,(n-p)!}\sum_{i=0}^{\infty}\binom{\theta-1}{i}(-1)^i\alpha\left(\frac{\theta\alpha^5}{\alpha^5+24}\right)(\alpha + x^4)e^{-\alpha x}\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\theta(p+i)-1}$$

Applying some binomial series expansion

$$\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\theta(p+i)-1}$$

$$= \sum_{j=0}^{\infty}\binom{\theta(p+i)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\frac{(\alpha x)^k}{(\alpha^5+24)^k}(\alpha^3x^3)^{k-l}\,4^l(\alpha^2x^2)^{l-m}3^m(\alpha x)^{m-n}2^n\,e^{-j\alpha x}.$$

- *Therefore,*

$$g_X(x) = \frac{n!\theta\alpha^5(\alpha+x^4)}{(p-1)!(n-p)!}\sum_{i=0}^{\infty}\binom{n-p}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\theta(p+i)-1}{j}(-1)^j\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\sum_{n=0}^{m}\binom{m}{n}\times \frac{\alpha^{4k-l-m-n}x^{4k-l-m-n}4^l3^m2^n}{(\alpha^5+24)^{k+1}}e^{-\alpha x(j+1)}.$$

The cdf of the p[th] order statistics is given by

$$G_X(x) = \sum_{i=p}^{n} \binom{n}{i} G^i(x)[1 - G(x)]^{n-i}$$

$$G_X(x) = \sum_{i=p}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j G^{i+j}(x)$$

$$G_X(x) = \sum_{i=p}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3 + 4\alpha^2x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right] e^{-\alpha x}\right]^{\theta(i+j)}$$

Using the binomial series expansion;

$$\left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right] e^{-\alpha x}\right]^{\theta(i+j)} = \sum_{i=0}^{\infty} \binom{\theta(j+1)-1}{k} (-1)^i \sum_{l=0}^{k} \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m} \sum_{n=0}^{m} \binom{m}{n} \sum_{q=0}^{n} \binom{n}{q} \frac{\alpha^{4l-m-n-q} x^{4l-m-n-q} 4^m 3^n 2^q}{(\alpha^5+24)^l}.$$

- *Therefore,*

$$G_X(x) = \sum_{i-p}^{n} \binom{n}{i} \sum_{i=0}^{\infty} \binom{\theta(j+1)-1}{k} (-1)^i \sum_{l=0}^{k} \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m} \sum_{n=0}^{m} \binom{m}{n} \sum_{q=0}^{n} \binom{n}{q} \frac{\alpha^{4l-m-n-q} x^{4l-m-n-q} 4^m 3^n 2^q}{(\alpha^5 + 24)^l} e^{-k\alpha x}$$

D. Entropy

Entropy measures the uncertainties associated with a random variable of a probability distributions. One of the type of entropy widely used is the Rényi's entropy [17].

- *Theorem IV*

Given a random variable X, which follows an exponentiated Rani distribution. The Rényi entropy is given by

$$T_R(x) = \frac{1}{1+\beta} \log\left[A_{ijk} \frac{(4j-k-l-m)!}{[\alpha(\beta+1)]^{4j-k-l-m+1}} + B_{ijk} \frac{(4\beta+4k-l-m)!}{[\alpha(\beta+1)]^{4\beta+4j-k-l-m+1}}\right] \tag{16}$$

Where

$$A_{ijk} = \sum_{i=0}^{\infty} \binom{\beta(\theta-1)}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m} \frac{\theta^\beta \alpha^{6\beta-4j-k-l-m} 4^k 3^l 2^m}{(\alpha^5+24)^{\beta+J}} \text{ and } B_{ijk} = \sum_{i=0}^{\infty} \binom{\beta(\theta-1)}{i} (-1)^i \sum_{j=0}^{i} \binom{i}{j} \sum_{k=0}^{j} \binom{j}{k} \sum_{l=0}^{k} \binom{k}{l} \sum_{m=0}^{l} \binom{l}{m} \frac{\theta^\beta \alpha^{5\beta-4j-k-l-m} 4^k 3^l 2^m}{(\alpha^5+24)^{\beta+J}}.$$

- *Proof.*

The Rényi entropy is given by;

$$T_R(x) = \frac{1}{1+\beta} \log\left[\int g_\theta^\beta(x) dx\right] \qquad ,\beta > 0, \beta \neq 1$$

$$T_R(x) = \frac{1}{1+\beta} \log\left[\int \left(\frac{\theta\alpha^5}{\alpha^5 + 24}\right)(\alpha + x^4)e^{-\alpha x} \left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3 + 4\alpha^2x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right] e^{-\alpha x}\right]^{\theta-1}\right]^{\beta} dx$$

$$T_R(x) = \frac{1}{1+\beta} \log\left[\int \left(\frac{\theta^\beta \alpha^{5\beta}}{(\alpha^5 + 24)^\beta}\right)(\alpha + x^4)^\beta e^{-\beta\alpha x} \left[1 - \left[1 + \frac{\alpha x\,(\alpha^3x^3 + 4\alpha^2x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right] e^{-\alpha x}\right]^{\beta(\theta-1)}\right] dx$$

$$T_R(x) = \frac{1}{1+\beta}\log\left[\int\left(\frac{\theta^\beta\alpha^{6\beta}}{(\alpha^5+24)^\beta}\right)e^{-\beta\theta x}\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\beta(\alpha-1)}\right]dx$$
$$+\frac{1}{1+\beta}\log\left[\int\left(\frac{\theta^\beta\alpha^{5\beta}x^{4\beta}}{(\alpha^5+24)^\beta}\right)e^{-\beta\alpha x}\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\beta(\theta-1)}\right]dx$$

Using binomial series expansion;

$$\left[1-\left[1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right]e^{-\alpha x}\right]^{\beta(\theta-1)}$$
$$=\sum_{i=0}^{\infty}\binom{\beta(\theta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{4j-k-l-m}x^{4j-k-l-m}4^k3^l2^m}{(\alpha^5+24)^{\beta+J}}e^{-\alpha x(\beta+I)}$$

$$T_R(x)$$
$$=\frac{1}{1+\beta}\log\left[\sum_{i=0}^{\infty}\binom{\beta(\theta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\alpha^{6\beta-4j-k-l-m}4^k3^l2^m\alpha^\beta}{(\alpha^5+24)^{\beta+j}}\int x^{4\beta-4j-k-l-m}e^{-\alpha x(\beta+i)}\,dx\right.$$
$$\left.+\sum_{i=0}^{\infty}\binom{\beta(\theta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^\beta\alpha^{5\beta-4j-k-l-m}4^k3^l2^m}{(\alpha^5+24)^{\beta+j}}\int x^{4\beta-4j-k-l-m}e^{-\alpha x(\beta+i)}\,dx\right]$$

Let $A_{ijk}=\sum_{i=0}^{\infty}\binom{\beta(\theta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^\beta\alpha^{6\beta-4j-k-l-m}4^k3^l2^m}{(\alpha^5+24)^{\beta+J}}$

and $B_{ijk}=\sum_{i=0}^{\infty}\binom{\beta(\theta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\theta^\beta\alpha^{5\beta-4j-k-l-m}4^k3^l2^m}{(\alpha^5+24)^{\beta+J}}$

Then, applying the Gamma properties, $\int_0^\infty x^n e^{-\alpha x}dx=\frac{\Gamma(n+1)}{\alpha^{n+1}}$ *and* $\Gamma(\alpha)=(\alpha-1)!$

$$T_R(x)=\frac{1}{1+\beta}\log\left[A_{ijk}\frac{(4j-k-l-m)!}{[\alpha(\beta+1)]^{4j-k-l-m+1}}+B_{ijk}\frac{(4\beta+4k-l-m)!}{[\alpha(\beta+1)]^{4\beta+4j-k-l-m+1}}\right]$$

V. MAXIMUM LIKELIHOOD ESTIMATION

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the exponentiated Pranav distribution. The log-likelihood function of parameters can be written as

$$LL(x;\theta,\alpha)=\prod_{i=1}^{n} In\,g(x) \tag{17}$$

$$=\prod_{i=1}^{n} In\frac{\theta\alpha^5}{\alpha^5+24}(\alpha+x^4)\left[1-\left(1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]^{\theta-1}e^{-\alpha x}$$

$$LL(x;\theta,\alpha)=n[In(\theta)+5In(\alpha)-In(\alpha^5+24)]+\sum In(\alpha+x^4)-\alpha\sum x+\theta-1\sum In\,P_i(x;\alpha) \tag{18}$$

Where,

$$P_i(x;\alpha)=\left[1-\left(1+\frac{\alpha x\,(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-\alpha x}\right]$$

In order to maximize the log likelihood, we solve the nonlinear likelihood equations obtained from the differentiation of (20) w.r.t αas shown below;

$$\frac{\partial LL}{\partial\alpha}=\partial\left\{n[5In(\alpha)-In(\alpha^5+24)]+\sum In(\alpha)-\alpha\sum x+\theta-1\sum In\,P_i(x;\alpha)\right\}$$

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$$\frac{\partial LL}{\partial \alpha} = \frac{120n}{\alpha(\alpha^5+24)} + \frac{1}{\alpha} - \sum x + \theta - 1 \sum \frac{\left[\frac{\alpha^8x^4 - 40\alpha^7x^3 - 60\alpha^6x^2 - 96\alpha^3x^4 - 288\alpha^2x^3 - 576\alpha x^2 - 576x}{\alpha^5+24^2}\right]}{\left[1 - \left(1 + \frac{\alpha x(\alpha^3x^3+4\alpha^2x^2+12\alpha x+24)}{\alpha^5+24}\right)e^{-x}\right]}$$

$$\frac{\partial LL}{\partial \theta} = \frac{n}{\theta} + \sum InP_i(x;\alpha)$$

$$\frac{\partial LL}{\partial \theta} = \frac{n}{\theta} + \left[1 - \left(1 + \frac{\alpha x(\alpha^3x^3 + 4\alpha^2x^2 + 12\alpha x + 24)}{\alpha^5 + 24}\right)e^{-\alpha x}\right]$$

The estimates of the parameters are obtained using the nonlinear equations above.

VI. CONCLUSION

The study proposed a new distribution known as the exponentiated Rani distribution using the widely used exponentiation technique. The mathematical properties of the newly developed distribution including the Order statistics, Entropy, Moments and Moment generating function and reliability analysis was also proposed and derived. The plots of the hazard rate function indicate that the Extended Rani distribution has an increasing failure rate while the survival rate function is a decreasing function. Furthermore, the maximum likelihood estimation was discussed to validate the distribution.

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