# Differential System: A Self-Discovery Approach using Combinatoric and Modular Arithmetic Dynamics

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**Abstract:- This study provides a new dimension (shortest route) to handling and solving differential problems involving reiterative or successive differentiation using the combinatoric and modular arithmetic techniques. The application of the technique is identified to be very easy to apply in dealing with higher order differentiation of linear, trigonometric and exponential functions with attention drawn to the resulting identities and signs. The successive differentiation of a function with negative power is also explored and validated. The study is therefore targeted to aid learning, teaching and applications.**

***Keywords:-*** *Combinatoric, modular, differentiation, trigonometric, and successive.*

## I. INTRODUCTION

Problems on differential equation have posed diverse challenges to learners and teachers over the years. This could be due to the manner and methods available in solving the underlined differential equation. In an instance where there is need to investigate the rate of change of a variable with respect to another variable (independent) in a high-dimension successive times, a challenge is then posed and time is consumed meaninglessly. On this note, this study will help identify shortest possible ways of solving simple differential equations using the concept of combination and permutation. It is an additional knowledge to the view of Subramanian (2018).

## II. METHODS

The techniques adopted in this study are combinatoric and modular arithmeticmethods of solving simple differential equations and it is a self-discovery approach dependent on basic knowledge of the aforementioned concepts. The fundamental counting principle is centered on factorial denoted by $n!$, the possible number of ways $n$ object can be arranged on a straight path. Mathematically, it is expressed as:

$$n! = n(n-1)(n-2)\dots(2)(1)$$

If there exist an integer which is less than $n$ by $r$ i.e. $n-r$, then the possible number of ways $n-r$ object can be arranged on a straight path is $(n-r)!$. Thus, the quotient $\frac{n!}{(n-r)!}$ becomes the number of ways $r$ objects can be arranged on a straight path among $n$. This development is termed permutation and its is expressed as

$$p(n,r) = \frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)(n-r-1)\dots(2)(1)}{(n-r)!}$$

$$= \frac{n(n-1)(n-2)\dots(n-r)!}{(n-r)!}$$

$$= n(n-1)(n-2)\dots(n-r+1)$$

which is having $r$ number of terms.

## III. DISCUSSION AND FINDINGS

### A. *Permutation Representation of $\mathbf{r^{th}}$ Order Successive Differentiation*

- ***Case I****: Let* $y = x^n, \frac{dy}{dx} = nx^{n-1}, \frac{d^2y}{dx^2} = n(n-1)x^{n-2}, \frac{d^3y}{dx^3} = n(n-1)(n-2)x^{n-3}$;

  *(Australia Curriculum, 2004; Good and Melinda, 2006)*

  By implication/induction

$$\frac{d^ry}{dx^r} = n(n-1)(n-2)\dots(n-[r-1])x^{n-r}$$

$$= n(n-1)(n-2)\dots(n-r+1)x^{n-r}$$

$$= p(n,r)x^{n-r} = \frac{n!}{(n-r)!}x^{n-r}$$

The combinatoric form of $\frac{d^r y}{dx^r}$ becomes $r!\,c(n,r)x^{n-r}$

where $c(n,r) = \frac{n!}{r!(n-r)!} = \frac{1}{r!}p(n,r)$.

- ***Special Case I:****when $r = n$, we have the $n^{th}$ order successive differentiation. Then*

$$\frac{d^r y}{dx^r} = \frac{d^n y}{dx^n} = p(n,n)x^{n-n} = p(n,n) \times 1$$

$$= \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

Here, $n$ is the highest power of $x$ over which $y$ is defined.

➢ ***Case I: Successive Differential of a Function with Coefficient Term***

Let $y = ax^n$

$$\Rightarrow \frac{dy}{dx} = a \times nx^{n-1} \Rightarrow \frac{d^2 y}{dx^2} = a \times n(n-1)x^{n-2}$$

$$\Rightarrow \frac{d^3 y}{dx^3} = a \times n(n-1)(n-2)x^{n-3}$$

By implication/induction,

$$\frac{d^r y}{dx^r} = a \times n(n-1)(n-2)\,....(n-r+1)x^{n-r}$$

$$= a \times p(n,r)x^{n-r} = \frac{an!}{(n-r)!}x^{n-r}$$

Dividing both numerator and denominator by $r!$, the combinatorial form becomes

$$ar!\,c(n,r)x^{n-r}$$

- ***Special Case II:*** *For $r = n$,*

$$\frac{d^r y}{dx^r} = \frac{d^n y}{dx^n} = an!$$

Note: If $n < r$, then $\frac{d^r y}{dx^r} = 0$ or undefined.

*B. Successive Differentiation of an Exponential Function*

Let $y = e^x \;\; \Rightarrow \frac{dy}{dx} = e^x$*(TES, 2013; Kreyszig, G. E., 2011)*

Since, taking the natural log of the function arrives at

$$lny = x \;\Rightarrow \frac{1}{y}\frac{dy}{dx} = 1 \Rightarrow y = e^x$$

Let $y = e^{ax} \Rightarrow \frac{dy}{dx} = ae^{ax} \Rightarrow \frac{d^2 y}{dx^2} = a^2 e^{ax} \Rightarrow \frac{d^3 y}{dx^3} = a^3 e^{ax}$

By implication/induction,

$$\frac{d^r y}{dx^r} = a^r e^{ax}$$

*C. Direct Successive Differentiation of Trigonometric Functions via Modular Arithmetic*

➢ ***Axiom I:*** *Let $y = sinax$, then the $r^{th}$ order successive differential coefficient is given established to be*

$$\frac{d^r y}{dx^r} = \begin{cases} a^r sinax \quad or \; a^r cosax \\ if\; r = 1 \; or \; rMod(4) = 0,1 \; then \; (+)sign \\ \qquad\qquad otherwise \; (-)sign \\ \qquad if\; r = \begin{cases} odd\; number, & cos \\ even\; number, & sin \end{cases} \end{cases}$$

- <u>Validation:</u>

Sign notation for successive differential rth order for $y = sinax$

| $r =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ⋯ |
|---|---|---|---|---|---|---|---|---|---|---|
| $sign$ | + | - | - | + | + | - | - | + | + | ⋯ |

$\Rightarrow$ For $\frac{d^r y}{dx^r}$ to be positive, $r$ equals to

| 1 | 4 | 5 | 8 | 9 | 12 | 13 | ⋯ |
|---|---|---|---|---|---|---|---|
| | | | | | | | |
| | 4 | 4 | 4 | 4 | 4 | ⋯ | ⋯ |

Depicting $r = 1$ (first instance), considering modulo 4, it is observed that, for $\frac{d^r y}{dx^r}$ to be positive, $rMod(4) = 0,1$ (second instance), otherwise negative.

- <u>Revalidation:</u>

It is of interest to note that if $y = sinx$, then the following holds

$$\frac{d}{dx}(Sinx) = Cosx$$

$$\frac{d}{dx}(Cosx) = -Sinx$$

$$\frac{d}{dx}(-Sinx) = -Cosx$$

$$\frac{d}{dx}(-Cosx) = Sinx$$

Thus, the identity follows as:

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ⋯ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Identity$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | ⋯ |
| $Sign$ | + | - | - | + | + | - | - | + | + | ⋯ |

Hence if $r = odd;\; \frac{d^r y}{dx^r} \to Cos$ while $r = even; \frac{d^r y}{dx^r} \to Sin$

➢ ***Axiom II:*** *Let $y = Cosax$, then the $r^{th}$ order successive differential coefficient is established to be*

$$\frac{d^r y}{dx^r} = \begin{cases} a^r Sinax \quad or \; a^r Cosax \\ if\; r = 3 \; or \; rMod(4) = 0,3, then \; (+)sign \\ \qquad\qquad otherwise, (-) \\ \qquad if\; r = \begin{cases} odd\; number, & Sin \\ even\; number, & Cos \end{cases} \end{cases}$$

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- Validation:
  Given that $y$ is a $Cos$ function given as $y = Cosax$, then the following sign notations hold.

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Sign$ | - | - | + | + | - | - | + | + | - | $\cdots$ |

This suffices that for $\frac{d^r y}{dx^r}$ to be positive, the following prevails

| 3 | 4 | 7 | 8 | 11 | 12 | 15 | $\cdots$ |
|---|---|---|---|---|---|---|---|
| | 4 | 4 | 4 | 4 | 4 | $\cdots$ | $\cdots$ |

The possible instances being $r = 3$ or $rMod(4) = 0, 3$. Otherwise, the sign will be negative.

- Revalidation:
  The trigonometric outcome of the function is derived as follows based on fundamentals:

$$y = Cosx$$

$$\frac{d}{dx}(Cosx) = -Sinx$$

$$\frac{d}{dx}(-Sinx) = -Cosx$$

$$\frac{d}{dx}(-Cosx) = Sinx$$

$$\frac{d}{dx}(Sinx) = Cosx$$

Thus, the notation identity is presented as:

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $Identity$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $Cosx$ | $Sinx$ | $\cdots$ |
| $Sign$ | - | - | + | + | - | - | + | + | - | $\cdots$ |

Hence, for $r = odd$; $\frac{d^r y}{dx^r} = Sin$ while $r = even$; $\frac{d^r y}{dx^r} = Cos$.

*D. Successive Differentiation with Negative Powers*

➢ *Axiom I: Let $y = x^n$ for $n < 0$, then $\frac{d^r y}{dx^r}$ which is the $r^{th}$ order differentiation is established to be:*

$$\frac{d^r y}{dx^r} = \frac{(-1)^r(|n| + r - 1)!\, x^{n-r}}{(|n| - 1)!}$$

- **Justification:**
  By induction approach, let $y = x^{-b}$ for $n = -b$; then,

$$\frac{dy}{dx} = -bx^{-b-1}$$

$$\frac{d^2 y}{dx^2} = (-b)(-b-1)x^{-b-2}$$

$$\frac{d^3 y}{dx^3} = (-b)(-b-1)(-b-2)x^{-b-3} \vdots$$

$$\frac{d^r y}{dx^r} = (-b)(-b-1)(-b-2)\cdots(-b-[r-1])x^{-b-r}$$

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$$= (-1)^r (b)(b+1)(b+2)\cdots(b+r-1)x^{-b-r}$$

$$= \frac{(-1)^r (1)(2)(3)\cdots(b-1)(b)(b+1)\cdots(b+r-1)x^{-b-r}}{(1)(2)(3)\cdots(b-1)}$$

$$= \frac{(-1)^r (b+r-1)!\, x^{-b-r}}{(b-1)!}$$

Recall that $n = -b \Rightarrow |n| = |-b| = b$

By substituting $|n|$ for $b$, we have

$$\frac{d^r y}{dx^r} = \frac{(-1)^r (|n|+r-1)!\, x^{n-r}}{(|n|-1)!}$$

## IV. CONCLUSION

Findings have shown with much justification from the study that some of the challenges posed while dealing with successive differential equations can be handled at ease by adopting the fundamental principles of combination, permutation and modular arithmetics. The study gives a directional approach with better stand point for its application. It is therefore recommended that this technique be adopted in diverse situations where the need is required since it is legitimate and saves time in solving related differential equation problems.

## REFERENCES

[1]. Australian Curriculum, (2014). Mathematics: Overview. https://www.australiacurriculum.edu.au/mathematics
[2]. Good and Melinda (2006). Differentiated Instruction: Principles and Techniques for the Elementary Grades. *San Rafael CA: Dominican University of California.*
[3]. Kreyszig G. E. (2011). Advanced Engineering Mathematics. *10th Edition, John Wiley & Sons Ltd., Amsterdam.*
[4]. TES. (2013). Using Differentiation in Mixed-ability Classes. https://newteacher.tes.co.uk.
[5]. Subramanian P. K. (2018). Successive Differentiation and Leibniz's Theorem.*The College of Mathematics Journal.*