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A Density Matrix Description based Approach to Calculation of Arithmetic Mean and Standard Deviation of a Set of Replicate Measurements

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Abstract:- The research article presents a mathematical framework for calculation of arithmetic mean and associated standard deviation of a set of replicate measurements, based on assignment of data point weightage as diagonal entries of the Density matrix descriptions generated from the set of data points under consideration. The framework presented provides flexibility in choice of the assigned weightage distributions by allowing for evolution of these weightage contributions under the effect of completely positive trace preserving transformations implemented through Unitary Quantum de-coherence channels [1, 8, 10, 11, 12, 14, 15, 17]. The presented formulation contains the conventional calculation procedure as a special case which involve the tuning parameter 'θ' being set equal to zero. Numerical case studies presented in the paper provide appropriate illustration of the mathematical constructs and terminology introduced.

Keywords:- *Arithmetic mean and Standard deviation of a set of measurements, Density Matrix description associated with mathematical constructs, Completely Positive Trace Preserving transformations, Kraus operators, Quantum Channels and Quantum de-coherence, Obliqueness factor associated with mean-variance partitioning of a set of measurements.*

I. INTRODUCTION

The computation of arithmetic mean and standard deviation of a set of replicate measurements is a basic calculation to estimate the central tendency and dispersion associated with the set of data points. In the conventional approach used for the computation, the mean is estimated from the projection of the data vector $X_{n\times 1}$ into one dimensional subspace spanned by the unit vector in $R^n(R)$ and this estimated arithmetic mean, denoted by $\bar{x}$, is a convex combination of the data points, where the weightage associated with each data point is equal to $1/n$. The deviation vector $d_{n\times 1}$ is the difference of the data vector and its projection into the unit vector subspace, it is confined in the 'n-1' dimensional orthogonal complement subspace; the magnitude of the deviation vector scaled appropriately by its available degrees of freedom gives an estimate of the standard deviation associated with the set of data points.

The research paper attempts to construct a mathematical formalism for assignment of the weightage to the data points of a set of replicate measurements based on the diagonal elements of Density matrix descriptions constructed from the data vector $X_{n\times 1}$ and of the Density matrix descriptions generated under the effect of Unitary Quantum de-coherence channels modeled through Completely Positive Trace Preserving transformations [1, 8, 10, 11, 12, 14, 15, 17]

In the present research initiative, an amalgamation of the approach and mathematical constructs developed in the previous studies [2, 3, 4, 5, 6, 7] is utilized to formulate the mathematical framework of the Standard form and the General form Density matrix descriptions associated with a given set of replicate measurements, thereby allowing the formulation of the Standard form and General form Weightage vectors and hence, the Standard form Arithmetic mean $\bar{x}(S\,|\,\theta)$ and the General form Arithmetic mean $\bar{x}(G\,|\,\theta)$.

The presented mathematical formalism therefore allows for a broader range of convex combinations of data points corresponding to the set of replicate measurements be realized as the arithmetic mean. The data vector and the estimated Standard form and General form arithmetic means allows for an estimate of deviation vectors not necessarily confined in the 'n-1' dimensional orthogonal complement subspace, The associated deviation vectors $|d(S\,|\,\theta)\rangle$ and $|d(G\,|\,\theta)\rangle$ allows quantification of Standard form standard deviation $S.D(S\,|\,\theta)$ and General form standard deviation $S.D(G\,|\,\theta)$ and the respective Obliqueness factors $\eta(S\,|\,\theta)$ and $\eta(G\,|\,\theta)$ which provide a quantitative estimate of the departure of the framework from Orthogonality.

The paper presents the mathematical formalism underlying the framework and provides an illustration of the introduced mathematical constructs through appropriately chosen numerical case studies. The paper concludes with a discussion of the numerical results and of the observations and obtained insights.

II. NOTATIONS

- N denotes the set of all Natural numbers
- R denotes the Real number field
- $M_{n\times n}(R)$ denotes the Real Matrix Space of order 'n'
- $R^n(R)$ denotes the Real co-ordinate space of dimension 'n'

- $|v\rangle \in R^n(R)$, $|v\rangle = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{bmatrix}_{n\times 1}$, $\langle v| = [v_1 \quad v_2 \quad . \quad . \quad v_n]_{1\times n}$, $v_1, v_2, ..., v_n \in R$
- $|a\rangle \in R^n(R)$, $|b\rangle \in R^n(R)$, $H_{n\times n} \in M_{n\times n}(R)$ where $H_{n\times n} = [h_{ij}]_{n\times n}$, therefore $\langle a| H_{n\times n} |b\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij} a_i b_j$
- $|e_1\rangle = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}_{n\times 1}$, $|e_2\rangle = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{bmatrix}_{n\times 1}$,..., $|e_n\rangle = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$, $|n\rangle = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$, $|n_1\rangle = \begin{bmatrix} 1 \\ 2 \\ . \\ . \\ n \end{bmatrix}_{n\times 1}$, $|n_2\rangle = \begin{bmatrix} n \\ n-1 \\ . \\ . \\ 1 \end{bmatrix}_{n\times 1}$
- $\langle n|n\rangle = n$,
- $I_{n\times n}$ denotes the Identity matrix of order 'n'
- A^T denotes the Transpose of the matrix A
- $(X_{n\times n})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{n\times n}$, i.e. $(X_{n\times n})^{-1} X_{n\times n} = X_{n\times n}(X_{n\times n})^{-1} = I_{n\times n}$
- $X_{n\times n} \in M_{n\times n}(R)$ such that $X_{n\times n}$ is symmetric and positive definite , then $(X_{n\times n})^{-1/2} \in M_{n\times n}(R)$, $(X_{n\times n})^{-1/2}$ is symmetric and positive definite such that: $(X_{n\times n})^{-1/2}(X_{n\times n})^{-1/2} = (X_{n\times n})^{-1}$, we also have $(X_{n\times n})^{1/2} \in M_{n\times n}(R)$, $(X_{n\times n})^{1/2}$ is symmetric and positive definite, such that: $(X_{n\times n})^{1/2}(X_{n\times n})^{1/2} = X_{n\times n}$
- The symbol '×' denotes scalar multiplication
- $[revdiag(1...1)]_{n\times n}$ denotes a square matrix of order 'n', which has 1's along the reverse diagonal (the diagonal opposite to the main diagonal of the matrix) and 0's everywhere else.

$$|X\rangle = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} , \quad |\hat{X}\rangle = \begin{bmatrix} x_n \\ x_{n-1} \\ . \\ . \\ x_1 \end{bmatrix}_{n\times 1}$$

- 'θ' denotes the Tuning parameter, where $\theta \in [0,1]$
- '$\overline{x}$' denotes the simple Arithmetic mean of a set of replicate measurements
- '$S.D(\overline{x})$' denotes the simple Standard deviation of a set of replicate measurements
- '$S.D(S|\theta)$' denotes the Standard form Standard deviation of a set of replicate measurements
- '$S.D(G|\theta)$' denotes the General form Standard deviation of a set of replicate measurements

III. MATHEMATICAL FRAMEWORK

Let $x_1, x_2 ...., x_n$ denote a set of replicate measurements from an experimental system, we have

$$x_j \in R \quad \forall j \in \{1,2,....,n\} \ , \ x_1 \le x_2 \le .... \le x_n$$

where $n \in N$ and $n \ge 2$

$\Sigma_{n\times n} = [revdiag(1...1)]_{n\times n}$, therefore $(\Sigma_{n\times n})^T = (\Sigma_{n\times n})^{-1} = \Sigma_{n\times n}$

We define the following associated vectors:

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therefore we have $\left|\hat{X}\right\rangle = \Sigma_{n\times n}\left|X\right\rangle$, $\left|X\right\rangle = \Sigma_{n\times n}\left|\hat{X}\right\rangle$ and

We have $\left\langle X\middle|X\right\rangle = \left\langle \hat{X}\middle|\hat{X}\right\rangle = m_X = \sum_{j=1}^{n} x_j^{\ 2}$

$\left\langle n\middle|n\right\rangle = n$, $\left\langle n_1\middle|n_1\right\rangle = \left\langle n_2\middle|n_2\right\rangle = (\frac{1}{6})n(n+1)(2n+1)$

We define the following associated matrices:

$U(1)_{n\times n} = I_{n\times n} - (\frac{2}{n})\left|n\right\rangle\left\langle n\right|$, $U(2)_{n\times n} = I_{n\times n} - (\frac{12}{n(n+1)(2n+1)})\left|n_1\right\rangle\left\langle n_1\right|$ and $U(3)_{n\times n} = I_{n\times n} - (\frac{12}{n(n+1)(2n+1)})\left|n_2\right\rangle\left\langle n_2\right|$

Therefore we have the result: $U(j)_{n\times n} = (U(j)_{n\times n})^T = (U(j)_{n\times n})^{-1}$ for $j = 1,2,3$

We define the matrices $\Omega(X,\hat{X})_{n\times n}$ and $\Omega_S(X,\hat{X})_{n\times n}$ as following:

$\Omega(X,\hat{X})_{n\times n} = (\frac{1}{2})(\left|X\right\rangle\left\langle X\right| + \left|\hat{X}\right\rangle\left\langle\hat{X}\right|)$, $\Omega_S(X,\hat{X})_{n\times n} = (\frac{1}{4})(\Omega(X,\hat{X})_{n\times n} + \sum_{j=1}^{3} U(j)_{n\times n}\Omega(X,\hat{X})_{n\times n}(U(j)_{n\times n})^T)$

$trace(\Omega(X,\hat{X})_{n\times n}) = trace(\Omega_S(X,\hat{X})_{n\times n}) = m_X$, both the matrices $\Omega(X,\hat{X})_{n\times n}$ and $\Omega_S(X,\hat{X})_{n\times n}$ are symmetric, Positive semi-definite or Positive definite

- *The framework of the associated Completely Positive Trace Preserving transformation*

$V(j)_{n\times n} \in M_{n\times n}(R)$ Where $j = 1,2,....,n$, $(V(j)_{n\times n})^T = (V(j)_{n\times n})^{-1}$ $\forall j = 1,2,....,n$

$p_j \in [0,1]$ Such that $\sum_{j=1}^{n} p_j = 1$

We define the matrix $\Omega_G(X,\hat{X})_{n\times n}$ as following:

$\Omega_G(X,\hat{X})_{n\times n} = \sum_{j=1}^{n} p_j V(j)_{n\times n}\Omega_S(X,\hat{X})_{n\times n}(V(j)_{n\times n})^T$, therefore the matrix $\Omega_G(X,\hat{X})_{n\times n}$ is symmetric, Positive semi-definite or Positive definite

We have: $trace(\Omega_G(X,\hat{X})_{n\times n}) = trace(\Omega_S(X,\hat{X})_{n\times n}) = trace(\Omega(X,\hat{X})_{n\times n}) = m_X$

- *Mathematical formulation of the Standard form and the General form Density Matrix descriptions*

$$\rho_S(X,\hat{X}\,|\,\theta)_{n\times n} = (\frac{1}{n+\theta m_X})(I_{n\times n} + \theta\Omega_S(X,\hat{X})_{n\times n})$$

- The matrix $\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}$ is termed as the "Standard form Density Matrix description" associated with the set of replicate measurements

- The matrix $\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, Positive definite $\forall\theta \in [0,1]$

$$\rho_G(X,\hat{X}\,|\,\theta)_{n\times n} = \sum_{j=1}^{n} p_j V(j)_{n\times n}\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}(V(j)_{n\times n})^T = (\frac{1}{n+\theta m_X})(I_{n\times n} + \theta\Omega_G(X,\hat{X})_{n\times n})$$

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- The matrix $\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}$ is termed as the "General form Density Matrix description" associated with the set of replicate measurements
- The matrix $\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}$ is symmetric, Positive definite $\forall\theta\in[0,1]$
- $trace(\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}) = trace(\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}) = 1 \;,\; \forall\theta\in[0,1]$

➢ *Mathematical formulation of the Standard form and the General form Weightage vectors*

$$\left|w(S\,|\,\theta)\right\rangle = \begin{bmatrix} w_1(S\,|\,\theta) \\ w_2(S\,|\,\theta) \\ . \\ . \\ w_n(S\,|\,\theta) \end{bmatrix}_{n\times 1} = \begin{bmatrix} \left\langle e_1\right|\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_1\right\rangle \\ \left\langle e_2\right|\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_2\right\rangle \\ . \\ . \\ \left\langle e_n\right|\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_n\right\rangle \end{bmatrix}_{n\times 1}$$

$$\left|w(G\,|\,\theta)\right\rangle = \begin{bmatrix} w_1(G\,|\,\theta) \\ w_2(G\,|\,\theta) \\ . \\ . \\ w_n(G\,|\,\theta) \end{bmatrix}_{n\times 1} = \begin{bmatrix} \left\langle e_1\right|\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_1\right\rangle \\ \left\langle e_2\right|\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_2\right\rangle \\ . \\ . \\ \left\langle e_n\right|\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_n\right\rangle \end{bmatrix}_{n\times 1}$$

- The vector $\left|w(S\,|\,\theta)\right\rangle$ is termed as the "Standard form Weightage vector" associated with the set of replicate measurements
- The vector $\left|w(G\,|\,\theta)\right\rangle$ is termed as the "General form Weightage vector" associated with the set of replicate measurements
- $w_j(S\,|\,\theta) > 0 \;,\; w_j(G\,|\,\theta) > 0 \quad \forall j = 1,2,....,n$ and $\theta\in[0,1]$
- $\left\langle n\right|w(S\,|\,\theta)\rangle = \left\langle n\right|w(G\,|\,\theta)\rangle = 1 \;,\; \forall\theta\in[0,1]$

➢ *Mathematical formulation of the Standard form and the General form Arithmetic means*

$$\bar{x}(S\,|\,\theta) = \sum_{j=1}^{n} x_j w_j(S\,|\,\theta) = \sum_{j=1}^{n} x_j \left\langle e_j\right|\rho_S(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_j\right\rangle$$

$$\bar{x}(G\,|\,\theta) = \sum_{j=1}^{n} x_j w_j(G\,|\,\theta) = \sum_{j=1}^{n} x_j \left\langle e_j\right|\rho_G(X,\hat{X}\,|\,\theta)_{n\times n}\left|e_j\right\rangle$$

- $\bar{x}(S\,|\,\theta)$ is termed as the "Standard form Arithmetic mean" associated with the set of replicate measurements
- $\bar{x}(G\,|\,\theta)$ is termed as the "General form Arithmetic mean" associated with the set of replicate measurements

- $\theta = 0 \Rightarrow \rho_S(X, \hat{X} \mid \theta = 0)_{n\times n} = \rho_G(X, \hat{X} \mid \theta = 0)_{n\times n} = (\frac{1}{n})I_{n\times n}$ therefore we have $\bar{x}(S \mid \theta = 0) = \bar{x}(G \mid \theta = 0) = \bar{x} = (\frac{1}{n})\sum_{j=1}^{n} x_j$

- $\bar{x}(S \mid \theta) \in [x_1, x_n]$ and $\bar{x}(G \mid \theta) \in [x_1, x_n] \quad \forall \theta \in [0,1]$

➢ *Mathematical formulation of the Standard form and the General form Standard deviation and associated analytical results*

$$\left|d(S \mid \theta)\right\rangle = \left|X\right\rangle - \bar{x}(S \mid \theta)\left|n\right\rangle \ , \ \left|d(G \mid \theta)\right\rangle = \left|X\right\rangle - \bar{x}(G \mid \theta)\left|n\right\rangle$$

$$\left|d_C(S \mid \theta)\right\rangle = (I_{n\times n} - (\frac{1}{n})\left|n\right\rangle\left\langle n\right|)\left|d(S \mid \theta)\right\rangle \ , \ \left|d_C(G \mid \theta)\right\rangle = (I_{n\times n} - (\frac{1}{n})\left|n\right\rangle\left\langle n\right|)\left|d(G \mid \theta)\right\rangle$$

$$\left|d_m(S \mid \theta)\right\rangle = (\frac{1}{n})\left\langle n \middle| d(S \mid \theta)\right\rangle\left|n\right\rangle \ , \ \left|d_m(G \mid \theta)\right\rangle = (\frac{1}{n})\left\langle n \middle| d(G \mid \theta)\right\rangle\left|n\right\rangle$$

$$S.D(S \mid \theta) = ((\frac{1}{n})\left\langle d(S \mid \theta) \middle| d(S \mid \theta)\right\rangle)^{1/2} \ , \ S.D(G \mid \theta) = ((\frac{1}{n})\left\langle d(G \mid \theta) \middle| d(G \mid \theta)\right\rangle)^{1/2} \quad \text{where } \theta \in (0,1]$$

- The vector $\left|d(S \mid \theta)\right\rangle$ is termed as the “Standard form Deviation vector” associated with the set of replicate measurements

- The vector $\left|d(G \mid \theta)\right\rangle$ is termed as the “General form Deviation vector” associated with the set of replicate measurements

Therefore, we have the following set of results:

- $\left|X\right\rangle = \bar{x}(S \mid \theta)\left|n\right\rangle + \left|d_m(S \mid \theta)\right\rangle + \left|d_C(S \mid \theta)\right\rangle = (\bar{x}(S \mid \theta) + (\frac{1}{n})\left\langle n \middle| d(S \mid \theta)\right\rangle)\left|n\right\rangle + \left|d_C(S \mid \theta)\right\rangle$
- $\left|X\right\rangle = \bar{x}(G \mid \theta)\left|n\right\rangle + \left|d_m(G \mid \theta)\right\rangle + \left|d_C(G \mid \theta)\right\rangle = (\bar{x}(G \mid \theta) + (\frac{1}{n})\left\langle n \middle| d(G \mid \theta)\right\rangle)\left|n\right\rangle + \left|d_C(G \mid \theta)\right\rangle$

- If $\left|d(S \mid \theta)\right\rangle \neq 0_{n\times 1}$, we define the Standard form Obliqueness factor $\eta(S \mid \theta)$ and Standard form percentage Obliqueness factor % $\eta(S \mid \theta)$, as follows:

$$\eta(S \mid \theta) = \frac{\left\langle d_m(S \mid \theta) \middle| d_m(S \mid \theta)\right\rangle}{\left\langle d(S \mid \theta) \middle| d(S \mid \theta)\right\rangle} = 1 - (\frac{\left\langle d_C(S \mid \theta) \middle| d_C(S \mid \theta)\right\rangle}{\left\langle d(S \mid \theta) \middle| d(S \mid \theta)\right\rangle})$$

% $\eta(S \mid \theta) = \eta(S \mid \theta) \times 100$ where $\theta \in [0,1]$

- If $\left|d(G \mid \theta)\right\rangle \neq 0_{n\times 1}$, we define the General form Obliqueness factor $\eta(G \mid \theta)$ and General form percentage Obliqueness factor % $\eta(G \mid \theta)$, as follows:

$$\eta(G \mid \theta) = \frac{\left\langle d_m(G \mid \theta) \middle| d_m(G \mid \theta)\right\rangle}{\left\langle d(G \mid \theta) \middle| d(G \mid \theta)\right\rangle} = 1 - (\frac{\left\langle d_C(G \mid \theta) \middle| d_C(G \mid \theta)\right\rangle}{\left\langle d(G \mid \theta) \middle| d(G \mid \theta)\right\rangle})$$

% $\eta(G \mid \theta) = \eta(G \mid \theta) \times 100$ where $\theta \in [0,1]$

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IV. NUMERICAL CASE STUDIES

- The numerical computations are performed using the Scilab 5.4.1 computational platform
- In the numerical studies discussed in this section, the parameter 'θ' is set as: $\theta = \frac{1}{2}$

Case 1: n = 10

The CPTP transformation framework utilized in the discussion of the following numerical examples is given as following:

$$p_j = \frac{1}{10} \quad , \quad \forall j = 1,2,....,10$$

$$\Gamma_{10\times10} = \begin{bmatrix} I_{9\times9} & |9\rangle \\ 0_{1\times9} & 1 \end{bmatrix}, \quad Q_{10\times10} = \Gamma_{10\times10}[(\Gamma^T\Gamma)^{-\frac{1}{2}}]_{10\times10} \quad \text{therefore we have } (Q_{10\times10})^T = (Q_{10\times10})^{-1}$$

$$V(1)_{10\times10} = Q_{10\times10}, \quad V(2)_{10\times10} = Q_{10\times10}Q_{10\times10} = Q^2{}_{10\times10} \quad \text{and so on upto } V(10)_{10\times10} = Q^{10}{}_{10\times10}$$

<u>Example 1</u>

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1.15	1.25	1.28	1.35	1.36	1.39	1.45	1.65	1.71	1.80

We have the following associated results:

$\overline{x}(S\,|\,\theta=0) = \overline{x}(G\,|\,\theta=0) = \overline{x} = 1.44$ (Rounded upto 2 decimal places)

$S.D(\overline{x}) = 0.21$ (Rounded up to 2 decimal places)

Table 1: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S\,|\,\theta=\frac{1}{2})$	$w_2(S\,|\,\theta=\frac{1}{2})$	$w_3(S\,|\,\theta=\frac{1}{2})$	$w_4(S\,|\,\theta=\frac{1}{2})$	$w_5(S\,|\,\theta=\frac{1}{2})$
0.125307	0.107603	0.095272	0.087832	0.084256

$w_6(S\,|\,\theta=\frac{1}{2})$	$w_7(S\,|\,\theta=\frac{1}{2})$	$w_8(S\,|\,\theta=\frac{1}{2})$	$w_9(S\,|\,\theta=\frac{1}{2})$	$w_{10}(S\,|\,\theta=\frac{1}{2})$
0.084256	0.087832	0.095272	0.107603	0.125037

Table 2: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G\,|\,\theta=\frac{1}{2})$	$w_2(G\,|\,\theta=\frac{1}{2})$	$w_3(G\,|\,\theta=\frac{1}{2})$	$w_4(G\,|\,\theta=\frac{1}{2})$	$w_5(G\,|\,\theta=\frac{1}{2})$
0.099736	0.085916	0.076646	0.071377	0.070620

$w_6(G\,|\,\theta=\frac{1}{2})$	$w_7(G\,|\,\theta=\frac{1}{2})$	$w_8(G\,|\,\theta=\frac{1}{2})$	$w_9(G\,|\,\theta=\frac{1}{2})$	$w_{10}(G\,|\,\theta=\frac{1}{2})$
0.073977	0.081469	0.093926	0.109976	0.236358

Table 3: The table of the Standard form and General form mean and standard deviation and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\overline{x}(S\,|\,\theta=\frac{1}{2})$	$S.D(S\,|\,\theta=\frac{1}{2})$	$\%\eta(S\,|\,\theta=\frac{1}{2})$	$\overline{x}(G\,|\,\theta=\frac{1}{2})$	$S.D(G\,|\,\theta=\frac{1}{2})$	$\%\eta(G\,|\,\theta=\frac{1}{2})$
1.44	0.20	0.06	1.50	0.21	8.85

- *Example 2*

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
30.2	30.8	31.4	32.5	32.6	32.9	33.1	33.7	35.0	35.2

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We have the following associated results:

$$\bar{x}(S \mid \theta = 0) = \bar{x}(G \mid \theta = 0) = \bar{x} = 32.7$$ (Rounded upto 1 decimal place)

$$S.D(\bar{x}) = 1.6$$ (Rounded upto 1 decimal place)

Table 4: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S \mid \theta = ½)$	$w_2(S \mid \theta = ½)$	$w_3(S \mid \theta = ½)$	$w_4(S \mid \theta = ½)$	$w_5(S \mid \theta = ½)$
0.149086	0.116114	0.092128	0.075359	0.067313

$w_6(S \mid \theta = ½)$	$w_7(S \mid \theta = ½)$	$w_8(S \mid \theta = ½)$	$w_9(S \mid \theta = ½)$	$w_{10}(S \mid \theta = ½)$
0.067313	0.075359	0.092128	0.116114	0.149086

Table 5: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G \mid \theta = ½)$	$w_2(G \mid \theta = ½)$	$w_3(G \mid \theta = ½)$	$w_4(G \mid \theta = ½)$	$w_5(G \mid \theta = ½)$
0.098472	0.072448	0.054415	0.044717	0.043133

$w_6(G \mid \theta = ½)$	$w_7(G \mid \theta = ½)$	$w_8(G \mid \theta = ½)$	$w_9(G \mid \theta = ½)$	$w_{10}(G \mid \theta = ½)$
0.049699	0.064395	0.087299	0.118550	0.366872

Table 6: The table of the Standard form and General form mean and standard deviation (Rounded upto 1 decimal place) and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\bar{x}(S \mid \theta = ½)$	$S.D(S \mid \theta = ½)$	$\%\eta(S \mid \theta = ½)$	$\bar{x}(G \mid \theta = ½)$	$S.D(G \mid \theta = ½)$	$\%\eta(G \mid \theta = ½)$
32.7	1.6	0.00	33.5	1.8	21.04

Example 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
103	106	110	112	112	115	115	117	118	119

We have the following associated results:

$$\bar{x}(S \mid \theta = 0) = \bar{x}(G \mid \theta = 0) = \bar{x} = 113$$ (Rounded to nearest integer value)

$$S.D(\bar{x}) = 5$$ (Rounded to nearest integer value)

Table 7: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S \mid \theta = ½)$	$w_2(S \mid \theta = ½)$	$w_3(S \mid \theta = ½)$	$w_4(S \mid \theta = ½)$	$w_5(S \mid \theta = ½)$
0.149971	0.116709	0.091382	0.075040	0.066898

$w_6(S \mid \theta = ½)$	$w_7(S \mid \theta = ½)$	$w_8(S \mid \theta = ½)$	$w_9(S \mid \theta = ½)$	$w_{10}(S \mid \theta = ½)$
0.066898	0.075040	0.091382	0.116709	0.149971

Table 8: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G \mid \theta = ½)$	$w_2(G \mid \theta = ½)$	$w_3(G \mid \theta = ½)$	$w_4(G \mid \theta = ½)$	$w_5(G \mid \theta = ½)$
0.098427	0.072303	0.054394	0.044647	0.043066

$w_6(G \mid \theta = ½)$	$w_7(G \mid \theta = ½)$	$w_8(G \mid \theta = ½)$	$w_9(G \mid \theta = ½)$	$w_{10}(G \mid \theta = ½)$
0.049681	0.064383	0.087323	0.118443	0.367334

Table 9: The table of the Standard form and General form mean and standard deviation (Rounded upto nearest integer value) and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\bar{x}(S \mid \theta = ½)$	$S.D(S \mid \theta = ½)$	$\%\ \eta(S \mid \theta = ½)$	$\bar{x}(G \mid \theta = ½)$	$S.D(G \mid \theta = ½)$	$\%\ \eta(G \mid \theta = ½)$
112	5	0.37	115	5	13.27

Case 2: n = 6

The CPTP transformation framework utilized in the discussion of the following numerical examples is given as following:

$$\Gamma_{6\times6} = \begin{bmatrix} I_{5\times5} & |5\rangle \\ 0_{1\times5} & 1 \end{bmatrix} \quad , \quad Q_{6\times6} = \Gamma_{6\times6}[(\Gamma^T\Gamma)^{-1/2}]_{6\times6} \quad \text{therefore we have} \quad (Q_{6\times6})^T = (Q_{6\times6})^{-1}$$

$$V(1)_{6\times6} = Q_{6\times6} \quad , \quad V(2)_{6\times6} = Q_{6\times6}Q_{6\times6} = Q^2{}_{6\times6} \quad \text{and so on upto} \quad V(6)_{6\times6} = Q^6{}_{6\times6}$$

$$p_1 = \frac{1}{1000} \quad , \quad p_4 = \frac{999}{1000} \quad , \quad p_2 = p_3 = p_5 = p_6 = 0$$

Example 1

x_1	x_2	x_3	x_4	x_5	x_6
1.84	1.85	1.86	1.86	1.88	2.55

We have the following associated results:

$\bar{x}(S\,|\,\theta=0) = \bar{x}(G\,|\,\theta=0) = \bar{x} = 1.97$ (Rounded upto 2 decimal places)

$S.D(\bar{x}) = 0.28$ (Rounded upto 2 decimal places)

Table 10: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S\,|\,\theta=1/2)$	$w_2(S\,|\,\theta=1/2)$	$w_3(S\,|\,\theta=1/2)$	$w_4(S\,|\,\theta=1/2)$	$w_5(S\,|\,\theta=1/2)$	$w_6(S\,|\,\theta=1/2)$
0.200251	0.161384	0.138365	0.138365	0.161384	0.200251

Table 11: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G\,|\,\theta=1/2)$	$w_2(G\,|\,\theta=1/2)$	$w_3(G\,|\,\theta=1/2)$	$w_4(G\,|\,\theta=1/2)$	$w_5(G\,|\,\theta=1/2)$	$w_6(G\,|\,\theta=1/2)$
0.192787	0.148311	0.142100	0.159202	0.199768	0.157832

Table 12: The table of the Standard form and General form mean and standard deviation and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\bar{x}(S\,|\,\theta=1/2)$	$S.D(S\,|\,\theta=1/2)$	$\%\eta(S\,|\,\theta=1/2)$	$\bar{x}(G\,|\,\theta=1/2)$	$S.D(G\,|\,\theta=1/2)$	$\%\eta(G\,|\,\theta=1/2)$
1.99	0.26	0.75	1.97	0.26	0.05

Example 2

x_1	x_2	x_3	x_4	x_5	x_6
12.24	12.24	12.26	12.28	12.29	13.98

We have the following associated results:

$\bar{x}(S\,|\,\theta=0) = \bar{x}(G\,|\,\theta=0) = \bar{x} = 12.55$ (Rounded upto 2 decimal places)

$S.D(\bar{x}) = 0.70$ (Rounded upto 2 decimal places)

Table 13: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S\,|\,\theta=1/2)$	$w_2(S\,|\,\theta=1/2)$	$w_3(S\,|\,\theta=1/2)$	$w_4(S\,|\,\theta=1/2)$	$w_5(S\,|\,\theta=1/2)$	$w_6(S\,|\,\theta=1/2)$
0.220985	0.157016	0.121998	0.121998	0.157016	0.220985

Table 14: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G\,|\,\theta=1/2)$	$w_2(G\,|\,\theta=1/2)$	$w_3(G\,|\,\theta=1/2)$	$w_4(G\,|\,\theta=1/2)$	$w_5(G\,|\,\theta=1/2)$	$w_6(G\,|\,\theta=1/2)$
0.193912	0.141838	0.132804	0.158705	0.219537	0.153204

Table 15: The table of the Standard form and General form mean and standard deviation and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\bar{x}(S \mid \theta = \frac{1}{2})$	$S.D(S \mid \theta = \frac{1}{2})$	$\%\eta(S \mid \theta = \frac{1}{2})$	$\bar{x}(G \mid \theta = \frac{1}{2})$	$S.D(G \mid \theta = \frac{1}{2})$	$\%\eta(G \mid \theta = \frac{1}{2})$
12.64	0.65	1.99	12.53	0.64	0.12

Example 3

x_1	x_2	x_3	x_4	x_5	x_6
115	115	116	117	117	126

We have the following associated results:

$\bar{x}(S \mid \theta = 0) = \bar{x}(G \mid \theta = 0) = \bar{x} = 118$ (Rounded to nearest integer value)

$S.D(\bar{x}) = 4$ (Rounded to nearest integer value)

Table 16: The table of Standard form weightage coefficients (Rounded upto 6 decimal places)

$w_1(S \mid \theta = \frac{1}{2})$	$w_2(S \mid \theta = \frac{1}{2})$	$w_3(S \mid \theta = \frac{1}{2})$	$w_4(S \mid \theta = \frac{1}{2})$	$w_5(S \mid \theta = \frac{1}{2})$	$w_6(S \mid \theta = \frac{1}{2})$
0.223525	0.156196	0.120279	0.120279	0.156196	0.223525

Table 17: The table of General form weightage coefficients (Rounded upto 6 decimal places)

$w_1(G \mid \theta = \frac{1}{2})$	$w_2(G \mid \theta = \frac{1}{2})$	$w_3(G \mid \theta = \frac{1}{2})$	$w_4(G \mid \theta = \frac{1}{2})$	$w_5(G \mid \theta = \frac{1}{2})$	$w_6(G \mid \theta = \frac{1}{2})$
0.191013	0.141892	0.133170	0.159471	0.220759	0.153694

Table 18: The table of the Standard form and General form mean and standard deviation (Rounded upto nearest integer value) and the associated percentage Obliqueness factors (Rounded upto 2 decimal places)

$\bar{x}(S \mid \theta = \frac{1}{2})$	$S.D(S \mid \theta = \frac{1}{2})$	$\%\eta(S \mid \theta = \frac{1}{2})$	$\bar{x}(G \mid \theta = \frac{1}{2})$	$S.D(G \mid \theta = \frac{1}{2})$	$\%\eta(G \mid \theta = \frac{1}{2})$
118	4	1.47	118	4	0.05

V. DISCUSSION AND CONCLUSION

The conventional approach to calculation of the arithmetic mean and the associated standard deviation of a set of replicate measurements suffers from some serious limitations; in particular when there are intrinsic systematic effects taking placing within the experimental system generating the data. There exists extensive literature on estimation of central tendency and dispersion of data generated from a system affected by non-random effects.

The paper attempts to draw parallels between the data generated from an experimental system affected by unknown and uncontrollable systematic effects with the evolution of the state of a quantum system in a dissipative environment; it is well known that the phenomenon of Quantum de-coherence can be modeled through Trace preserving and the Completely Positive Trace preserving (abbreviated as CPTP) transformations [1, 8, 10, 11, 12, 14, 15, 17], in this paper the problem of determination of the accurate estimate of the arithmetic mean is posed as accurate determination of the assigned weightage of the individual data points, which are assumed to be related to the diagonal matrix elements of the Density Matrix descriptions generated from the data. The Standard form Density matrix description $\rho_S(X, \hat{X} \mid \theta)_{n\times n}$ is generated from contributions coming from a direct channel linked to the data vector $X_{n\times 1}$ and three indirect channels representing three types of plausible systematic effects (uniform, strictly increasing and strictly decreasing across the magnitude ordered sequence of the data points). The density matrix description $\rho_S(X, \hat{X} \mid \theta)_{n\times n}$ acts as the reference point which under the effect of Completely Positive Trace preserving transformations representing other plausible intricate systematic effects occurring in the experimental system, leads to the General form Density Matrix description $\rho_G(X, \hat{X} \mid \theta)_{n\times n}$.

The tuning parameter ‘θ’ allows for an external control on the proportion of contribution of the intrinsic data point effects in the formulated framework; at θ = 0, both the General form and the Standard form Density matrix descriptions reduce to $(\frac{1}{n})I_{n\times n}$ and the estimated arithmetic means reduces the conventional arithmetic mean

$$\bar{x} = (\frac{1}{n})\sum_{j=1}^{n} x_j$$

:

The numerical studies illustrated in the paper consider sample data of size n=10 and n=6; in these studies empirical CPTP schematic frameworks are utilized and results obtained are observed to be in close proximity with those obtained by considering the conventional method. The percentage Obliqueness factors associated with the numerical case studies range from near zero to about 21%, indicating that considerable departure from Orthogonality is exhibited among some of the numerical samples forming part of the case study.

In conclusion, it is emphasized that a good estimate of the arithmetic mean and associated standard deviation for a given set of replicate measurements, in light of the formalism considered in the paper, effectively boils down to a proper choice of the CPTP framework that accurately models the effects of systematic and possibly unknown and uncontrolled processes occurring in the experimental system. Follow up studies dedicated upon unraveling the link between the mathematical frameworks of data generation by the experimental system and its modeling in terms of Density Matrix descriptions and their evolution under CPTP transformations, would lead to a better understanding and further development of the mathematical framework formulated in the present research endeavor.

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