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Bounded Linear Operators on Function Spaces and Sequences Spaces

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Abstract. This paper will analyse the existence of sequence space and function space.It will be proved that these two spaces form a normed space in general and an inner product space in particular.Similarly, the space of linear operators will be introduced. Next, we will define the finite operator and prove its properties and give some examples related to the problem.

Keywords:- *Linear operators, bounded operators, sequence spaces, integrable function spaces.*

I. INTRODUCTION

Vector space is the most important part of Functional Analysis.In this paper the vector space is clearly defined [1, 6]. Furthermore, a vector space that has a norm function in it is called a norm space [2,5,7]. As for the clearly defined finite linear operator [3,4,5].

- **Definition 1.1 [**1**]**Suppose Vbe a vector space over the field F. Define a real function as follows :

$$\|\,.\,\| \,:\, V \times V \longrightarrow \mathbb{R}$$

Who fulfils :

$$\|x\| \geq 0$$

$$\|x\| = 0 \Leftrightarrow x = 0$$

$$\|\alpha\, x\| = |\alpha|\|x\|$$

$$\|x + y\| \leq \|x\| + \|y\|$$

Shared vector space with norms function$\|\,.\,\|$is called a normed space denoted by$(V, \|\,.\,\|)$.

- **Definition 1.2:** [5]Suppose V and Wbe are vector spaces a mapping T from the vector space V to the vector space Y satisfies$T\,(x + y) = T(x) + T(y)$and$T(\alpha x) = \alpha T(x)$for each$x, y \in Vand\alpha \in F. T$ is called a linear operator.

- **Definition 1.2 [**1**]**Suppose V and Wbe are vector spaces. A mapping T from the norm space V to the vector space Y is called a bounded linear mapping if there exists $c \in F$ such that it satisfies$\|T(x)\| \leq c\|x\|$for each$x \in V$

- **Definition 1.3 [5]**Suppose V and Wbe are vector spaces.

- DefineB$(U, V) = \{T \mid T : U \longrightarrow V,\ T\ linear\}$. Further it can be proved that $B(U, V)$ is a vector space. Further it can be proved that $B(U, V)$ is a vector space.

- **Definition 1.4 [2,6]**Suppose $(x_n\,)$be a sequence on the space of$\mathbb{R}$.

Define the sequence space:

$$\boldsymbol{\ell^p}(\mathbb{R}) = \left\{(x_n) \subseteq \mathbb{R} \middle| \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}$$

Furthermore, it can be proven that$\boldsymbol{\ell^p}(\mathbb{R})$is a vector space. Apart from that, the norm can also be defined:

$$\|x\|_{\ell^p} = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$$

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More specific to$\boldsymbol{p} = 2$, maka ruang $\boldsymbol{\ell}^{2}(\mathbb{R})$is an inner product space with inner product defined :

$$\langle x_n, y_n \rangle = \sum_{n=1}^{\infty} |x_n y_n|^2$$

Theorem 1.1[5]Any space$\boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R})$with$p \neq 2$, is not an inner product space.

Proof. Using the properties of parallelograms, select$x = (1,1,0,0,0\dots,) \in \boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R})andy = (1,-1,0,0,0,\dots,) \in \boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R})$then obtained:

$$\|x\| = \|y\| = 2^{\frac{1}{p}} \quad dan \ \ \|x+y\| = \|x-y\| = 2$$

It appears that the equation holds$p = 2$■

Theorem 1.2 [5]Any space$\boldsymbol{C}[\boldsymbol{a}, \boldsymbol{b}]$is not an inner product space.

Proof. Suppose$\|x\| = \max_{t \in [a,b]} |x(t)|$is the norm on the space$\boldsymbol{C}[\boldsymbol{a}, \boldsymbol{b}]$**.** This theorem can be proved using parallelogram theory. Suppose it is determined that$x(t) = 1,\ y(t) = \frac{t-a}{b-a}$then$\|x\| = 1and\|y\| = 1$so that it is obtained:

$$\boldsymbol{x}(t) + y(t) = 1 + \frac{t-a}{b-a}$$

$$\boldsymbol{x}(t) - y(t) = 1 - \frac{t-a}{b-a}$$

so that it is obtained:

$\|x+y\| = 2$, $\|x-y\| = 1$and$\|x+y\|^2 + \|x-y\|^2 = 5$even though$2(\|x\|^2 + \|y\|^2) = 4$.

■ **Definition 1.5 [3,7]**Suppose $[a, b] \subseteq \mathbb{R}andf : [a, b] \longrightarrow \mathbb{R}$ is a real function on$\mathbb{R}$ .Defined

$$\boldsymbol{L}^{\boldsymbol{p}}([a,b]) = \left\{ f \ \ | \ \ \int_a^b |f(x)|^p \ dx \ < \ \infty \right\}$$

Furthermore, it can be proven that$\boldsymbol{L}^{\boldsymbol{p}}([a, b])$is a norm space, with norm:

$$\|f\|_{\boldsymbol{L}^{\boldsymbol{p}}} = \left(\int_a^b |f(x)|^p \ dx \right)^{\frac{1}{p}}$$

More specific to$\boldsymbol{p} = 2$, then the space$\boldsymbol{L}^{\boldsymbol{2}}(\mathbb{R})$is an inner product space with inner product defined :

$$\langle f(x), g(x) \rangle = \left(\int_a^b f(x) g(x)\, dx \right)^{\frac{1}{2}}$$

II. RESULTS

In this section, the results of this research will be described, namely by proving some properties of finite linear mappings on a normed space and on the inner product space..

Theorem 2.1. Suppuse$\boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R})$beis a normed space with norm defined as follows:

$$\|x\| = \sum_{n=1}^{\infty} |x_n|^p$$

Then there is a linear mapping shift left$f$which is bounded.

Proof. Suppose$f : \ \boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R}) \longrightarrow \boldsymbol{\ell}^{\boldsymbol{p}}(\mathbb{R})$, bewith the linkage defined:

$x = (x_1, x_2, x_3, x_4, \dots) \mapsto f(x) = (x_2, x_3, x_4, x_5, \dots)$.It will be shown that f is a linear mapping.Let's take an arbitrary$x, y \in \boldsymbol{\ell^p}(\mathbb{R})and\alpha, \beta \in \mathbb{R}$, write down$x = (x_1, x_2, x_3, x_4, \dots)andy = (y_1, y_2, y_3, y_4, \dots)$notice that:

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1),\ (\alpha x_2 + \beta y_2), \dots) \qquad = (0,\ \alpha x_2,\ \alpha x_3, \dots) + (0,\ \beta y_2,\ \beta y_3, \dots) \quad \alpha f(x) + \beta f(y)$$

Means f is linear.

For an arbitrary vector$x = (x_1, x_2, x_3, x_4, \dots) \in \boldsymbol{\ell^p}(\mathbb{R})$apply:

$$\|f(x)\|^p = \|(x_2, x_3, x_4, x_5, \dots)\|^p$$
$$= \sum_{n=1}^{\infty} |x_n|^p$$
$$\leq \sum_{n=1}^{\infty} |x_n|^p$$
$$= \|x\|^p$$

So obtained$\|f(x)\| \leq \|x\|$, $\forall\ x \in \boldsymbol{\ell^p}(\mathbb{R})$, means f is bounded. Furthermore, without prejudice to the generality of writing$\|f\| \leq 1$ ……………………………. (1)

Meanwhile, the vector$e = (0,1,00, \dots) \in \boldsymbol{\ell^p}(\mathbb{R})$.It is clear that$\|e\| = 1and\|f(e)\| = \|(1,0,0,0, \dots)\| = 1$thus obtained$\|f\| = \sup_{\|x\|=1} \|f(x)\| \geq 1$……………….……….(2)

From equations (1) and (2), it is concluded that$\|f\| = 1$■

Example 2.1. *Suppose$H = \boldsymbol{\ell^2}(\mathbb{R})$be. If $\boldsymbol{H}$ is a Hilbert space then the right shift mapping is a linear and bounded mapping.*

Proof. Suppose$f : \boldsymbol{\ell^2}(\mathbb{R}) \longrightarrow \boldsymbol{\ell^2}(\mathbb{R})$with the attribution defined :

$x = (x_1, x_2, x_3, x_4, \dots) \mapsto f(x) = (0, x_1, x_2, x_3, x_4, x_5, \dots)$. For an arbitrary$x = (x_1, x_2, x_3, x_4, \dots) \in \boldsymbol{\ell^2}(\mathbb{R})$apply:

$$\|f(x)\|^2 = \|(0, x_1, x_2, x_3, x_4, x_5, \dots)\|^2$$
$$= \sum_{n=1}^{\infty} |x_n|^2$$
$$\leq \sum_{n=1}^{\infty} |x_n|^2$$
$$= \|x\|^2$$

So it is obtained$\|f(x)\| \leq \|x\|$, $\forall\ x \in \boldsymbol{\ell^2}(\mathbb{R})$, means f is bounded.

Example 2.2. *Suppose$H = \boldsymbol{\ell^2}(\mathbb{R})$.If H is a Hilbert space then the left shift mapping is a linear and bounded mapping*.

Proof. It will be shown f is a linear mapping. Take any$x, y \in \boldsymbol{\ell^2}(\mathbb{R})and\alpha, \beta \in \mathbb{R}$, write $x = (x_1, x_2, x_3, x_4, \dots)andy = (y_1, y_2, y_3, y_4, \dots)$and realise that:

$$f(\alpha x + \beta y) = f(\alpha x_1 + \beta y_1), \qquad (\alpha x_2 + \beta y_2), \dots)$$
$$= (0,\ \alpha x_2,\ \alpha x_3, \dots) + (0,\ \beta y_2,\ \beta y_3, \dots)$$
$$= \alpha f(x) + \beta f(y)$$

This proves that f is linear. Next suppose$f : \boldsymbol{\ell^2}(\mathbb{R}) \longrightarrow \boldsymbol{\ell^2}(\mathbb{R})$with the attribution defined :

$x = (x_1, x_2, x_3, x_4, \dots) \mapsto f(x) = (x_2, x_3, x_4, x_5, \dots)$. For an arbitrary vector$x = (x_1, x_2, x_3, x_4, \dots) \in \boldsymbol{\ell^2}(\mathbb{R})$apply:

$$\|f(x)\|^2 = \|(x_2, x_3, x_4, x_5, \dots)\|^2$$
$$= \sum_{n=1}^{\infty} |x_n|^2$$
$$\leq$$
$$\sum_{n=1}^{\infty} |x_n|^2 = \|x\|^2$$

So it is obtained$\|f(x)\| \leq \|x\|$, $\forall\ x \in \boldsymbol{\ell}^2(\mathbb{R})$, means f is bounded. Furthermore, without prejudice to the generality of writing$\|f\| \leq 1$ … … … … … … … … … .. (1)

Meanwhile, the vector$e = (0,1,00, \dots) \in \boldsymbol{\ell}^2(\mathbb{R})$.It is clear that$\|e\| = 1and\|f(e)\| = \|(1,0,0,0, \dots)\| = 1$thus obtained$\|f\| = \sup_{\|x\|=1} \|f(x)\| \geq 1$ ……………...………..(2)

From equations (1) and (2) it can be deduced$\|f\| = 1$■

Example 2.3. *Suppose$H = \boldsymbol{\ell}^2(\mathbb{R})$be,If H is a Hilbert space then the right shift mapping is a linear and bounded mapping.*
Proof. Suppose$f : \boldsymbol{\ell}^2(\mathbb{R}) \longrightarrow \boldsymbol{\ell}^2(\mathbb{R})$with the attribution defined:

$x = (x_1, x_2, x_3, x_4, \dots) \mapsto f(x) = (0, x_1, x_2, x_3, x_4, x_5, \dots)$. For an arbitrary$x = (x_1, x_2, x_3, x_4, \dots) \in \boldsymbol{\ell}^2(\mathbb{R})$apply :

$$\|f(x)\|^2 = \|(0, x_1, x_2, x_3, x_4, x_5, \dots)\|^2$$
$$= \sum_{n=1}^{\infty} |x_n|^2$$
$$\leq \sum_{n=1}^{\infty} |x_n|^2$$
$$= \|x\|^2$$

So obtained$\|f(x)\| \leq \|x\|$, $\forall$ □ $\in \boldsymbol{\ell}^2(\mathbb{R})$, means f is bounded. Furthermore, without prejudice to the generality of writing$\|f\| \leq 1$ … … … … … … … … … … … … … …. (1)

Meanwhile, the vector$e = (1,0,00, \dots) \in \boldsymbol{\ell}^2(\mathbb{R})$.It is clear that$\|e\| = 1and\|f(e)\| = \|(0,1,0,0,0, \dots)\| = 1$so that it is obtained$\|f\| = \sup_{\|x\|=1} \|f(x)\| \geq 1$ ……..………..(2)

From equations (1) and (2) it can be deduced$\|f\| = 1$■

Theorem 2.2. Given two norm spaces$(X, \|\cdot\|_1)$and$(Y, \|\cdot\|_2)$.Suppose $T \in B(X, Y)$ *be*, defined

$$\|T\| = \inf_{x \in X} \{ M : \|T(x)\|_2 \leq M \|x\|_1 \} \text{then}$$

$$\|T\| = \sup_{x \in x,\ x \neq 0} \frac{\|T(x)\|_2}{\|x\|_1}$$

And

$$\|T\| = \sup_{\|x\|_1 < 1} \|T(x)\|_2$$

Proof. Note that the :

$$\frac{\|T(x)\|_2}{\|x\|_1} \leq M, \quad \forall\ x \in X, \text{with } x \neq 0$$

From the definition obtained :

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$$\|T\| = \inf_{x\in X}\{M : \|T(x)\|_2 \le M\,\|x\|_1\}$$
$$= \sup_{x\in x,\ x\neq 0} \frac{\|T(x)\|_2}{\|x\|_1} \qquad (1)$$

Now suppose that$y = \frac{x}{\|x\|_1}$for$\forall\ x \in X,\ x \neq 0$its accuracy is obtained$y \in Xand\|y\| = 1$. From equation (1) is obtained:

$$\|T\| = \sup_{x\in x,\ x\neq 0} \frac{\|T(x)\|_2}{\|x\|_1}$$

$$= \sup_{\|y\|=1} \left\| T\left(\frac{\|x\|_1 y}{\|x\|_1}\right)\right\|_2$$

$$= \sup_{\|y\|=1} \|T(y)\|_2$$

$$= \sup_{\|x\|=1} \|T(x)\|_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots.\ldots.(2)$$

From equation (2) is obtained :

$$\|T\| = \sup_{\|x\|_1} \|T(x)\|_2$$

$$\le \sup_{\|x\|_1} \|T(x)\|_2$$

$$\le \sup_{x\in X,\ \|x\|_1\le 1} \frac{\|T(x)\|_2}{\|x\|_1}$$

$$\le \sup_{x\in X,\ x\neq 0} \frac{\|T(x)\|_2}{\|x\|_1}$$

$$= \|T\|$$

$$\text{then}\|T\| = \sup_{\|x\|_1\le 1} \|T(x)\|_2$$

Furthermore, suppose that, $A = \{x \in X : \|x\|_1 \le 1\}andA^0 = \{x \in X : \|x\|_1 < 1\}$, because it is recognised that$\|T\| = \sup_{x\in A} \|T(x)\|_2$then there is a sequence$(x_n) \in A$so that:

$$\|T\| = \lim_{n\to\infty} \|T(x_n)\|_2$$

Note that the sequence$(y_n)$with$y_n = \left(1 - \frac{1}{2^n}\right)x_n$.It is clear that$y_n \in A^0$, such that for all$n \in \mathbb{N}$apply

$$\lim_{n\to\infty} \|T(x_n)\|_2 = \lim_{n\to\infty} \left\| T\left(1 - \frac{1}{2^n}\right)x_n \right\|_2$$

$$= \lim_{n\to\infty} T\left(1 - \frac{1}{2^n}\right)x_n \|T(x_n)\|_2$$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) \lim_{n\to\infty} \|T(x_n)\|_2$$

$$= \|T\|$$

Therefore:

$$\sup_{x\in A^0} \|T(x)\|_2 \ge \|T\|$$

on the other hand $\sup_{x\in A^0} \|T(x)\|_2 \le \sup_{x\in A} \|T(x)\|_2$

$$= \|T\| \blacksquare$$

- **Exsamle 2.3.** Suppose it is known that the space$X = C[0,1]$,with the maximum norm. . The integral operator is defined as follows : $\varphi : C[0,1] \longrightarrow C[0,1]$with :

$$\varphi f(x) = \int_0^1 f(y)\ dy$$

Then φ is bounded.

Proof. Note that the:

$$\|\varphi f\| \le \max_{x \in [0,1]} \int_0^x |f(y)|\ dy$$

$$\le \int_0^1 |f(y)|\ dy$$

$$\le \|f\|$$

Because$\|\varphi\| \le 1$, and$1 \in X$, $\varphi\ 1 = x$then$\|\varphi\ 1\| = 1$. ■

Exsamle 2.4. Suppose$\boldsymbol{L}^2[0,1]$, bewith norms$\|x\|_2$. Define the integral operator:

$$\varphi : \boldsymbol{L}^2[0,1] \longrightarrow \boldsymbol{L}^2[0,1]\ with\ \varphi f(x) = \int_0^x f(y)\ dy$$

Then φ is bounded.

Proof.

$$\|\varphi f\|_2^2 = \int_0^1 \left| \int_0^t f(s) ds \right|^2 dt$$

$$= \int_0^1 \left| \int_0^t \sqrt{\cos\frac{\pi s}{2}} \frac{f(s)}{\sqrt{\cos\frac{\pi s}{2}}}\ ds \right|^2 dt$$

$$\le \int_0^1 \left(\int_0^t \cos\frac{\pi s}{2}\ ds \int_0^t \frac{|f(s)|^2}{\cos\frac{\pi s}{2}}\ ds \right) dt$$

$$= \frac{2}{\pi} \int_0^1 \left(\int_0^t \sin\frac{\pi t}{2} \cdot \frac{|f(s)|^2}{\cos\frac{\pi s}{2}}\ ds \right) dt$$

$$= \frac{2}{\pi} \int_0^1 \left(\int_0^t \sin\frac{\pi t}{2} \cdot \frac{|f(s)|^2}{\cos\frac{\pi s}{2}}\ dt \right) ds$$

$$= \frac{2}{\pi} \int_0^t \left(\int_0^1 \sin\frac{\pi t}{2}\ dt \right) \frac{|f(s)|^2}{\cos\frac{\pi s}{2}}\ ds$$

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$$= \left(\frac{2}{\pi}\right)^2 \int_0^t \frac{|f(s)|^2}{\cos\frac{\pi s}{2}} \; ds$$

This is the case when$f(s) = \cos\ \frac{\pi s}{2}$

Theorem 2.5 An inner productive space satisfies the Schwarz inequality and the triangle inequality, namely:

$|\langle x, y\rangle| \leq \|x\|\|y\|$ (Schwarz's inequality)

$\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Proof. In part a) it is easy to prove the bilan vector$\{x, y\}$linearly dependent i.e. suppose$y = tx$means it can be written as$|\langle x, y\rangle| \leq$

$$\|x\|\|y\| \Leftrightarrow |\langle x, tx\rangle| \leq \|x\|\|tx\|$$

$$\Leftrightarrow |t\langle x, x\rangle| \leq t\|x\|\|x\|$$

$$\Leftrightarrow |t|\|x\|^2 \leq t\|x\|^2$$

Furthermore, if$\{x, y\}$is linearly independent then : if$y = 0$then$0 = |\langle x, 0\rangle| \leq \|x\|\|0\| = 0$then it is proven. Now if$y \neq 0$. Suppose for any scalar$\boldsymbol{t}$ **,** note that : $0 \leq \|x - ty\|^2 = \langle x - ty, x - ty\rangle$

$$= \langle x, x\rangle - \bar{t}\langle x, y\rangle - t[\langle y, x\rangle - \bar{t}\langle y, y\rangle]$$

$$\text{by choosing}\,\overline{t} = \frac{\langle y, x\rangle}{\langle y, y\rangle}$$

then obtained :

$$0 \leq \langle x, x\rangle - \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle$$

$$= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

Because of this$\langle x, y\rangle = \overline{\langle y, x\rangle}$then by multiplying it with$\|y\|^2$then the above equation is proven. ■

For part b). Note that:

$$\|x + y\|^2 = \langle x + y, x + y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$

From equation a) is obtained :

$$|\langle x, y\rangle| = |\langle y, x\rangle| \leq \|x\|\|y\|$$

Furthermore, from

$$\|x + y\|^2 = \langle x + y, x + y\rangle$$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2|x, y| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

So it is proven■

III. CONCLUSION

From the description it can be concluded that in the space of sequence$\boldsymbol{\ell^p}(\mathbb{R})$can be constructed a bounded linear operator as well as in the space of functions$\boldsymbol{L^p}(\mathbb{R})$**.** In particular for $p = 2$ it can be shown that both spaces are inner product spaces and satisfy the Schwarz inequality and triangle inequality.

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