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Marshall-Olkin Chris-Jerry Distribution and its Applications

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Abstract:- In this paper, a new lifetime distribution known as the Marshall-Olkin Chris-Jerry (MOCJ) distribution is proposed. The proposition is motivated by Marshall-Olkin family of distributions and the one-parameter Chris-Jerry distribution. Some of its useful mathematical properties were derived and the derivation of the Pietra measure of inequality lends this distribution to wider application especially in income and population distributions. Two real data sets were used to illustrate the proposed model. From the results, the MOCJ distribution performed better than the other fitted distributions.

Keywords: *Chris-Jerry distribution, Marshall-Olkin distribution, Marshall-Olkin Chris-Jerry distribution, Holomorphic extension, Incomplete Moment, Normalized Incomplete Moment, Pietra Measure of Inequality*.

I. INTRODUCTION

The major aim of the modification of any probability model is the need for flexibility in applications. No particular distribution fits all situations while retaining its parsimony. It is on the above basis that researchers have continued to introduce new distributions as well as modify existing distributions. In the end, evolving life events are modeled easily.

Onyekwere and Obulezi (2022) proposed a one-parameter life distribution named Chris-Jerry distribution following the tradition involved in the derivation of the Lindley distribution by Lindley (1958). Chris-Jerry distribution is apt due to having one-parameter and yet fits data from taxes, health and engineering. The wider applicability of the Chris-Jerry distribution has motivated its extension in this paper. Other distributions derived in this fashion are Pranav distribution by KK (2018). Kamlesh Kumar Shukla and Rama Shanker (2019) proposed the 'Shukla distribution'. Sujatha Distribution with an increasing hazard rate for modelling lifetime data was suggested by Rama Shanker (2016). Shanker and K. Shukla (2017) introduced the Ishita distribution. Shanker (2015a) studied a one-parameter lifetime distribution named Akash distribution. Shanker (2017a) studied a one-parameter lifetime distribution named Rani distribution. Shanker (2017b) studied a one-parameter lifetime distribution named Rama distribution. Sen, Maiti, and Chandra (2016) studied a one-parameter lifetime distribution named XGamma distribution. Shanker (2016) studied a one-parameter lifetime distribution named Aradhana distribution. Shanker (2015b) studied a one-parameter lifetime distribution named Shanker. Each of the cited distributions above following the pattern of Lindley distribution has a unique mixing proportion.

Some useful distribution modifications in the literature are Modification of Shanker distribution using quadratic rank transmutation map by Onyekwere, Okoro, et al. (n.d.), Zubair-Exponential distribution by Anabike et al. (2023), Hassan et al. (2019) proposed the alpha power transformed power Lindley distribution, a generalization of the power Lindley distribution that provides a better fit. An extension of the Lindley distribution which offers a more flexible model for lifetime data was introduced by Nadarajah, Bakouch, and Tahmasbi (2011). Marshall and Olkin (1997) provides a unique way of enhancing a distribution model to yield wider applicability. Many authors have advanced this thought using various baseline distributions. Alsultan (2022) proposed the Marshall-Olkin Pranav distribution with a better fit to numerous data than the Pranav distribution. Ikechukwu and Eghwerido (2022) studied the Marshall-Olkin Sujatha distribution with similar advantages like the Marshall-Olkin Pranav. Importantly, all such modifications with using Marshall-Olkin family of distributions and any one-parameter lifetime distribution obtained from the mixture of Exponential and Gamma distributions have proven to be better with wider applications than their baseline distributions. On the above premise, we propose the Marshall-Olkin Chris-Jerry distribution in this paper.

The rest of this paper is organized as follows; in section 2, we derive the Marshall-Olkin Chris-Jerry distribution and present the pdf, cdf, survival and hazard rate functions and their associated plots. In section 3, we derive some useful mathematical properties. In section 4, we apply the proposed distribution to two real data sets and we conclude the paper in section 5.

➢ *The Marshall-Olkin Chris-Jerry (MOCJ) Distribution*

Marshall and Olkin (1997) developed a method for improving the flexibility of family of distributions.

Definition 1 Let $X \sim Q(x)$, a baseline c.d.f whose p.d.f is $q(x)$, then the c.d.f and p.d.f of the Marshall-Olkin family of distributions are respectively

$$F(x,\beta) = \frac{\beta Q(x)}{1-(1-\beta)\bar{Q}(x)} \quad (1)$$

And

$$f(x,\beta) = \frac{\beta q(x)}{\left[1-(1-\beta)\bar{Q}(x)\right]^2} \quad (2)$$

Where $-\infty < x < \infty$

Definition 2 Let $X \sim$ Chris-Jerry (θ) due to Onyekwere and Obulezi (2022) with p.d.f and c.d.f given as

$$q(x,\theta) = \frac{\theta^2}{\theta+2}(1+\theta x^2)e^{-\theta x}; \quad x, \quad \theta > 0 \quad (3)$$

And

$$Q(x,\theta) = 1 - \left[1 + \frac{\theta x\,(\theta x+2)}{\theta+2}\right]e^{-\theta x} \quad (4)$$

The c.d.f and p.d.f of Marshall-Olkin Chris-Jerry (MOCJ) distribution are therefore given respectively as

$$F(x,\theta,\beta) = \frac{\beta\left\{\theta+2-[\theta+2\theta x\,(\theta x+2)]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\,(\theta x+2)\right\}e^{-\theta x}} \quad (5)$$

and

$$f(x,\theta,\beta) = \frac{\beta\theta^2(\theta+2)(1+\theta x^2)e^{-\theta x}}{\{\theta+2-(1-\beta)\,[\theta+2+\theta x(\theta x+2)]e^{-\theta x}\}^2} \quad (6)$$

where $\beta > 0$ is the tilt parameter.

Definition 3 (Linear Representation)**.** To obtain a tractable function for the p.d.f of the proposed MOCJ distribution, consider the following known binomial expansions

$$(a-b)^{-n} = \sum_{\tau=0}^{\infty}(-1)^{\tau}\binom{n+\tau-1}{\tau}b^{\tau}a^{-(n+\tau)} \quad (7)$$

and

$$(a+b)^n = \sum_{\tau=0}^{n}\binom{n}{\tau}a^{n-\tau}b^{\tau}; \quad n > 0 \quad (8)$$

We can write

$$\left\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}^{-2}=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}x^{j+k}e^{-i\theta x}$$

Where

$$P_{ijk}=\binom{i+1}{i}\binom{i}{j}\binom{j}{k}(-1)^{i}2^{j-k}(1-\beta)^{i}(\theta+2)^{-(j+2)}\theta^{j+k} \quad (9)$$

Therefore, the p.d.f of MOCJ distribution in equation (6) can be linearly represented as

$$f(x,\theta,\beta)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}\beta\theta^{2}(\theta+2)(1+\theta x^{2})x^{j+k}e^{-\theta x(i+1)} \quad (10)$$

The Reliability rate, hazard rate, reversed hazard rate, cumulative hazard rate and the odd functions are respectively

$$S(x,\theta,\beta)=1-\frac{\beta\left\{\theta+2-\left[\theta+2\theta x\left(\theta x+2\right)\right]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\left(\theta x+2\right)\right\}e^{-\theta x}} \quad (11)$$

$$h(x,\theta,\beta)=\frac{\theta^{2}(1+\theta x^{2})}{\left[\theta+2+\theta x(\theta x+2)\right]\left[\theta+2-(1-\beta)\left\{\theta+2\theta x(\theta x+2)\right\}e^{-\theta x}\right]} \quad (12)$$

$$rhrf(x,\theta,\beta)=\frac{\beta\theta^{2}(1+\theta x^{2})e^{-\theta x}}{\left\{\theta+2-\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}\left\{\theta+2-(1-\beta)\left[\theta+2\theta x(\theta x+2)\right]e^{-\theta x}\right\}} \quad (13)$$

$$chrf(x,\theta,\beta)=-\ln\left(\frac{\beta\left\{\theta+2+\theta x(\theta x+2)\right\}e^{-\theta x}}{\theta+2-(1-\beta)\left\{\theta+2+\theta x(\theta x+2)\right\}e^{-\theta x}}\right) \quad (14)$$

and

$$O(x,\theta,\beta)=\frac{\beta\left\{\theta+2-\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}}{\beta(\theta+2)+(1-2\beta)\left\{\theta+2-\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}} \quad (15)$$

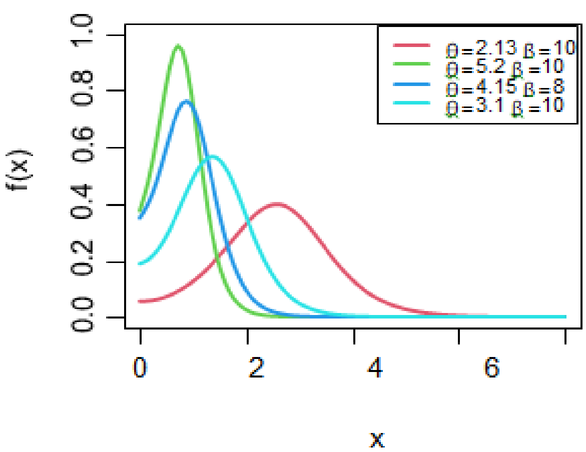

Fig 1 (a) PDF of MOCJ distribution

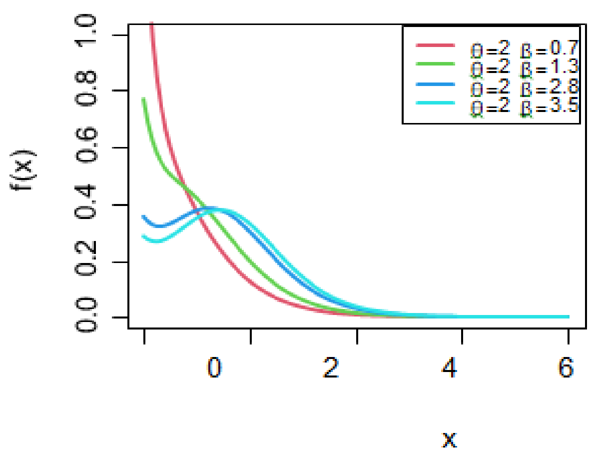

Fig 1 (b) PDF of MOCJ distribution

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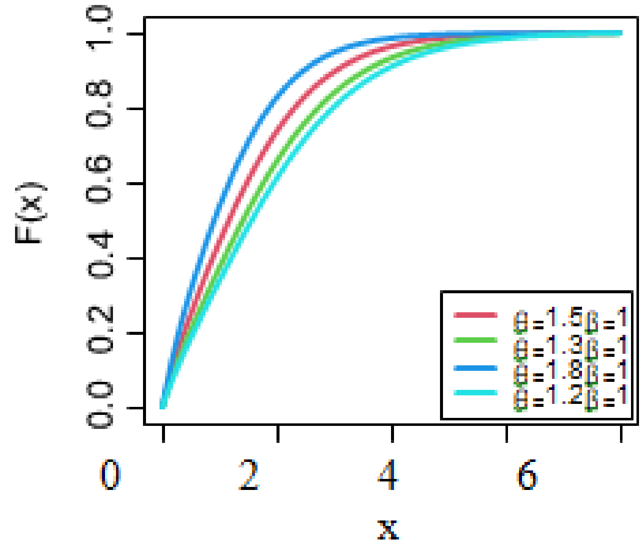

Fig 1 (c) CDF of MOCJ distribution

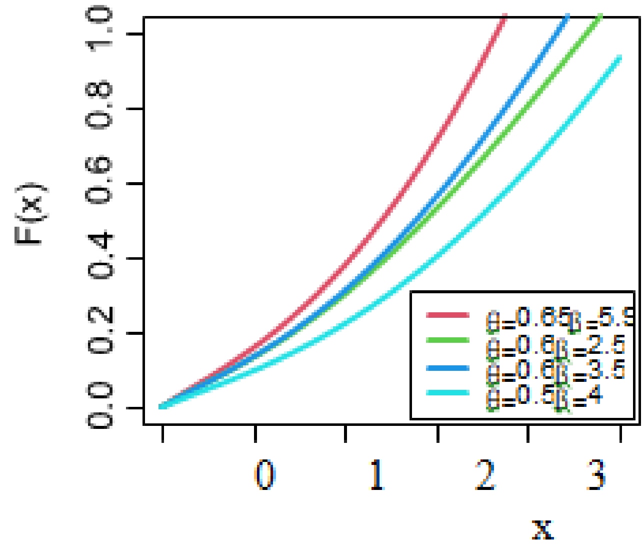

Fig 1 (d) CDF of MOCJ distribution

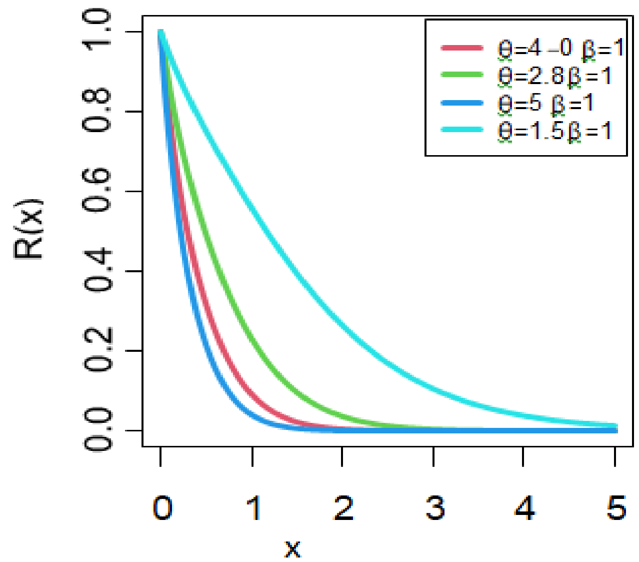

Fig 2 (a) Reliability function of MOCJ distribution

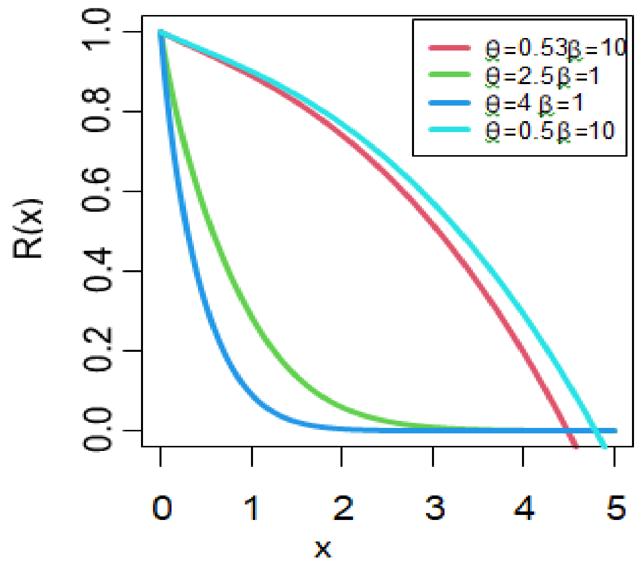

Fig 2 (b) Reliability function of MOCJ distribution

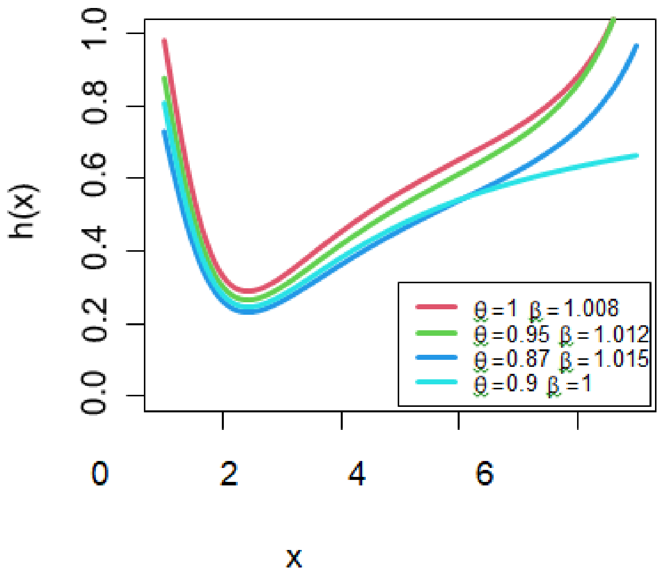

Fig 2 (c) Hazard function of MOCJ distribution

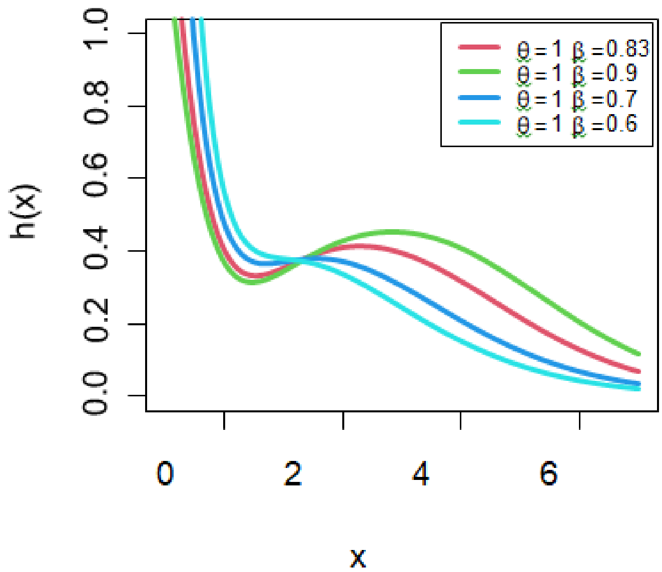

Fig 2 (d) Hazard function of MOCJ distribution

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➢ *Mathematical Properties of the MOCJ distribution*

In this section, we derive some useful mathematical properties of the proposed MOCJ distribution.

Definition 1 (Moment) Let $X \sim$ MOCJ (θ, β) ,then its r^{th} crude incomplete moment is given as

$$I(x;r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2) \times \left(\{\theta(i+1)\}^{-(j+k+r+1)}\gamma(j+k+r+1,\theta x(i+1)) + \theta\{\theta(i+1)\}^{-(j+k+r+3)}\gamma(j+k+r+3,\theta x(i+1))\right) \quad (16)$$

Where γ is the lower incomplete gamma function and P_{ijk} is as defined in equation (9)

Proof. Let us define the r^{th} lower incomplete moment as

$$I(x;r) = \int_0^x y^r f(y)dy; \quad y < x \quad (17)$$

substituting accordingly into equation (17), we obtain

$$I(x;r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\int_0^x y^r(1+\theta y^2)y^{j+k}e^{-\theta y(i+1)}dy$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left(\int_0^x y^{j+k+r}e^{-\theta y(i+1)}dy + \theta\int_0^x y^{j+k+r+2}e^{-\theta y(i+1)}dy\right) \quad (18)$$

Since $\frac{1}{a^{s+1}}\gamma(s+1,x) = \int_0^x y^s e^{-ay}dy$, then equation (16) is valid deduction from equation (18). This completes the prove.

The holomorphic extension of the MOCJ incomplete moment in equation (16) is

$$I(x;r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+1)}\left(\frac{\{x\theta(i+1)\}^{\delta}\Gamma(j+k+r+1)}{\Gamma(j+k+r+\delta+2)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+r+3)}{\Gamma(j+k+r+\delta+4)}\right) \quad (19)$$

This is obtained by recursive relation.

Definition 2 (Mean of the MOCJ Distribution)**.** Let $X \sim$ MOCJ (θ, β) the arithmetic mean μ is obtained from equation (19) by replacing r by 1 which yields

$$\mu = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+1)}\left(\frac{\{x\theta(i+1)\}^{\delta}\Gamma(j+k+\delta+2)}{\Gamma(j+k+\delta+3)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+\delta+4)}{\Gamma(j+k+\delta+5)}\right) \quad (20)$$

Definition 3 Let $X \sim$ MOCJ (θ, β), the second, third and fourth moments μ_2,μ_3 and μ_4 are obtained from equation (19) by replacing r by 2,3 and 4 respectively which yield

$$\mu_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+1)}\left(\frac{\{x\theta(i+1)\}^{\delta}\Gamma(j+k+\delta+3)}{\Gamma(j+k+\delta+4)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+\delta+5)}{\Gamma(j+k+\delta+6)}\right)$$

$$\mu_3 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+1)}\left(\frac{\{x\theta(i+1)\}^{\delta}\Gamma(j+k+\delta+4)}{\Gamma(j+k+\delta+5)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+\delta+6)}{\Gamma(j+k+\delta+7)}\right) \quad (21)$$

$$\mu_4 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+1)}\left(\frac{\{x\theta(i+1)\}^{\delta}\Gamma(j+k+\delta+5)}{\Gamma(j+k+\delta+6)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+\delta+7)}{\Gamma(j+k+\delta+8)}\right)$$

Definition 4 (Normalized Incomplete Moment)**.** Butler and McDonald (1989) explained that normalized incomplete moment can be helpful in measuring inequalities in income distribution and hence the r^{th} incomplete moment defined as

$$\phi(x;r) = \frac{I(x;r)}{E(y^r)} \quad (22)$$

Where $E(y^r) = \lim_{x\to\infty} I(x;r)$

$$\Rightarrow E(y^r) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left(\int_0^{\infty} y^{j+k+r}e^{-\theta y(i+1)}dy + \theta\int_0^{\infty} y^{j+k+r+2}e^{-\theta y(i+1)}dy\right)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left(\{\theta(i+1)\}^{-(j+k+r+1)}\Gamma(j+k+r+1) + \theta\{\theta(i+1)\}^{-(j+k+r+3)}\Gamma(j+k+r+3)\right) \quad (23)$$

Therefore, the normalized incomplete moment of the MOCJ distribution is

$$\Phi(x;r) = \frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta x(i+2)}\left[\frac{\{\theta x(i+1)\}^{\delta}\Gamma(j+k+r+1)}{\Gamma(j+k+r+\delta+2)} + \frac{\theta^{\delta+1}\{x(i+1)\}^{\delta}\Gamma(j+k+r+3)}{\Gamma(j+k+r+\delta+4)}\right]}{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left[\{\theta(i+1)\}^{-(j+k+r+1)}\Gamma(j+k+r+1) + \theta^{\delta+1}\{(i+1)\}^{\delta}\Gamma(j+k+r+1)\right]} \quad (24)$$

As noted by Butler and McDonald (1989) an interesting property of the normalized incomplete moment is that $0 \leq \varphi(x;r) \leq 1$ and $\varphi'(x;r) \geq 0$

A useful measure of inequality mathematically tractable in MOCJ distribution is the Pietra index or measure P

Definition 5 (Pietra measure of Inequality)**.** Let $x \sim$ MOCJ (θ,β) , the Pietra measure of inequality is given as

$$P = \phi(\mu;0) - \phi(\mu,1) \quad (25)$$

Where

$$\Phi(\mu;0) = \frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta(i+2)}\left[\frac{\{\theta\mu(i+1)\}^{\delta}\Gamma(j+k+1)}{\Gamma(j+k+\delta+2)} + \frac{\theta^{\delta+1}\{\mu(i+1)\}^{\delta}\Gamma(j+k+3)}{\Gamma(j+k+r+\delta+4)}\right]}{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left[\{\theta(i+1)\}^{-(j+k+1)}\Gamma(j+k+1) + \theta\{\theta(i+1)\}^{-(j+k+3)}\Gamma(j+k+3)\right]} \quad (26)$$

And

$$\Phi(\mu;1) = \frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}\sum_{\delta=0}^{\infty} P_{ijk}\beta\theta^2(\theta+2)e^{-\theta(i+2)}\left[\frac{\{\theta\mu(i+1)\}^{\delta}\Gamma(j+k+2)}{\Gamma(j+k+\delta+3)} + \frac{\theta^{\delta+1}\{\mu(i+1)\}^{\delta}\Gamma(j+k+4)}{\Gamma(j+k+r+\delta+5)}\right]}{\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j} P_{ijk}\beta\theta^2(\theta+2)\left[\{\theta(i+1)\}^{-(j+k+2)}\Gamma(j+k+2) + \theta\{\theta(i+1)\}^{-(j+k+4)}\Gamma(j+k+4)\right]} \quad (27)$$

μ is as defined in equation (20).

Definition 6 (Maximum Likelihood Estimation)**.** Let $X \sim$ MOCJ (θ,β) , the likelihood function of a random samples $x_1, x_2, \ldots, x_n$ of size n drawn from MOCJ $(x;\theta,\beta)$

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$$L(x_1,x_2,\ldots,x_n;\theta,\beta)=\prod_{i=1}^{n}f(x_i;\theta,\beta)=\beta^n\theta^{2n}(\theta+2)^n e^{-\theta\sum_{i=1}^{n}x}\prod_{i=1}^{n}\frac{(1+\theta x^2)}{\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\}^2} \quad (28)$$

Let

$$\ell=n\ln\beta+2n\ln\theta+n\ln(\theta+2)+\sum_{i=1}^{n}\ln\left(1+\theta x^2\right)-\theta x-2\sum_{i=1}^{n}\ln\psi \quad (29)$$

Take the log of likelihood function in equation (29)

$$\psi=\left\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}^2 \quad (30)$$

Differentiate equation (30) partially with respect to θ and β

$$\frac{\partial\ell}{\partial\theta}=\frac{2n}{\theta}+\frac{n}{\theta+2}+\sum_{i=1}^{n}\frac{x^2}{1+\theta x^2}-x-2\sum_{i=1}^{n}\left(\frac{1-\left(1-\theta x-\beta+\theta\beta x-\theta^2x^3+\theta^2\beta x^3\right)e^{-\theta x}}{\psi}\right) \quad (31)$$

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}-2\sum_{i=1}^{n}\frac{\left(\theta+2+\theta^2x^2+2\theta x\right)e^{-\theta x}}{\psi} \quad (32)$$

Equations (31) and (32) do not have closed-form solution hence not tractable. Its solutions will be obtained using iterative procedure in R.

Definition 7 (Moment Generating Function). Let $X\sim$ MOCJ (θ,β), the moment generating function is obtained as follows

$$M_X(t)=E(e^{tx})=\int_0^\infty e^{tx}f(x;\theta,\beta)dx=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}\beta\theta^2(\theta+2)\int_0^\infty(1+\theta x^2)x^{j+k}e^{-x\{\theta(i+1)-t\}}dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}\beta\theta^2(\theta+2)\left(\{\theta(i+1)-t\}^{-(j+k+1)}\Gamma(j+k+1)+\theta\{\theta(i+1)-t\}^{-(j+k+3)}\Gamma(j+k+3)\right) \quad (33)$$

Definition 8 (Characteristic Function). Let $X\sim$ MOCJ (θ,β), the characteristic function is obtained as follows

$$\phi_X(it)=E(e^{itx})=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}\beta\theta^2(\theta+2)\left(\{\theta(i+1)-it\}^{-(j+k+1)}\Gamma(j+k+1)+\theta\{\theta(i+1)-it\}^{-(j+k+3)}\Gamma(j+k+3)\right) \quad (34)$$

Definition 9 (Stochastic Ordering of MOCJ Distribution). The stochastic ordering is used in comparing the behaviour of system components. A random variable X is said to be smaller than another random variable Y in the

- Stochastic order $(X\leq_{st}Y)$ if $F_X(x)\geq F_Y(x)\forall x$
- Hazard rate order $(X\leq_{hr}Y)$ if $h_X(x)\geq h_Y(x)\forall x$
- Mean residual life order $(X\leq_{mrl}Y)$ if $m_X(x)\geq m_Y(x)\forall x$
- Likelihood ratio order $(X\leq_{lr}Y)$ if $\frac{f_X(x)}{F_Y(x)}$ decreases in x

This implies that $X\leq_{lr}Y\Rightarrow X\leq_{hr}Y\Rightarrow X\leq_{st}Y\Rightarrow X\leq_{mrl}$

Here, we prove that MOCJ distribution is ordered with respect to the strongest "likelihood ratio" as shown in theorem below.

Theorem 1 Let $X \sim$ MOCJ (θ_1, β_1) and $Y \sim$ MOCJ (θ_2, β_2). If $\theta_1 > \theta_2$ then $X \leq_{lr} Y$ hence $X \leq_{mrl} Y$, $X \leq_{hr} Y$, and $X \leq_{st} Y$

Proof

$$\frac{f_X(x)}{f_Y(x)} = \frac{\beta_1\theta_1^2(\theta_1+2)(1+\theta_1 x^2)e^{-(\theta_1-\theta_2)x}}{\beta_2\theta_2^2(\theta_2+2)(1+\theta_2 x^2)}\,\frac{\theta_2+2-(1-\beta_2)\left\{\theta_2+2+\theta_2 x(\theta_2 x+2)\right\}e^{-\theta_2 x}}{\theta_1+2-(1-\beta_1)\left\{\theta_1+2+\theta_1 x(\theta_1 x+2)\right\}e^{-\theta_1 x}}$$

Taking natural log of the ratio will yield

$$\ln\frac{f_X(x)}{f_Y(x)} = \ln\beta_1 + 2\ln\theta_1 + \ln(\theta_1+2) + \ln(1+\theta_1 x^2) - (\theta_1-\theta_2)x - \ln\beta_2 - 2\ln\theta_2 - \ln(\theta_2+2) - \ln(1+\theta_2 x^2)$$

$$+\ln\left(\theta_2+2-(1-\beta_2)\left\{\theta_2+2+\theta_2 x(\theta_2 x+2)\right\}e^{-\theta_2 x} + \ln\left(\theta_1+2-(1-\beta_1)\left\{\theta_1+2+\theta_1 x(\theta_1 x+2)\right\}e^{-\theta_1 x}\right.\right.$$

Differentiating the natural log of the ratio with respect to x will yield

$$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{2\theta_1 x}{1+\theta_1 x^2} - (\theta_1-\theta_2) - \frac{2\theta_2 x}{1+\theta_2 x^2} + \frac{\left(\theta_2(1-\beta_2)(\theta_2+2) - \theta_2^2(1-\beta_2)\left[2x-\theta_2 x^2\right] - 2(1-\beta_2)\theta_2\left[1-\theta_2 x\right]\right)e^{-\theta_2 x}}{\theta_2+2-(1-\beta_2)\left\{\theta_2+2+\theta_2 x(\theta_2 x+2)\right\}e^{-\theta_2 x}}$$

$$+\frac{\left(\theta_1(1-\beta_1)(\theta_1+2) - \theta_1^2(1-\beta_1)\left[2x-\theta_1 x^2\right] - 2(1-\beta_1)\theta_1\left[1-\theta_1 x\right]\right)e^{-\theta_1 x}}{\theta_1+2-(1-\beta_1)\left\{\theta_1+2+\theta_1 x(\theta_1 x+2)\right\}e^{-\theta_1 x}}$$

$\theta_2 > \theta_1$, and $\beta_1 = \beta_2 = 1$, $\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} < 0$, and $\frac{f_X(x;\theta_1,\beta_1)}{f_Y(x;\theta_2,\beta_2)}$ is decreasing in x. That is, $X \leq_{lr} Y$ and hence, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$

Definition 10 (Distribution of the Order Statistics). Suppose $X_1, X_1, \ldots, X_n$ is a random sample of $X_{(r)}; (r = 1, 2, \ldots, n)$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude $\ni X_1 \leq X_2 \leq \ldots \leq X_r$ and $X_1 = min(X_1, X_2, \ldots, X_r)$, $X_r = max(X_1, X_2, \ldots, X_r)$ then the probability density function of the order statistics of the MOCJ distribution is given by

$$f_{r:n}(x;\theta,\beta) = \frac{n!}{(r-1)!(n-r)!} f_{MOCJ}(x;\theta,\beta)\left[F_{MOCJ}(x;\theta,\beta)\right]^{r-1}\left[1-F_{MOCJ}(x;\theta,\beta)\right]^{n-r} \quad (35)$$

where $f(\cdot)$ and $F(\cdot)$ are the p.d.f and c.d.f of MOCJ distribution respectively. Hence, we have

$$f_{r:n}(x;\theta,\beta) = \frac{n!}{(r-1)!(n-r)!}\,\frac{\beta\theta^2(\theta+2)(1+\theta x^2)e^{-\theta x}}{\left\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}^2}\left\{\frac{\beta\left\{\theta+2-\left[\theta+2\theta x\,(\theta x+2)\right]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\,(\theta x+2)\right\}e^{-\theta x}}\right\}^{r-1} \quad (36)$$

$$\times\left\{1-\frac{\beta\left\{\theta+2-\left[\theta+2\theta x\,(\theta x+2)\right]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\,(\theta x+2)\right\}e^{-\theta x}}\right\}^{n-r}$$

The p.d.f of the largest order statistics is obtained by setting $r = n$

$$f_{n:n}(x;\theta,\beta) = \frac{n\beta\theta^2(\theta+2)(1+\theta x^2)e^{-\theta x}}{\left\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\right\}^2}\left\{\frac{\beta\left\{\theta+2-\left[\theta+2\theta x\,(\theta x+2)\right]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\,(\theta x+2)\right\}e^{-\theta x}}\right\}^{n-1} \quad (37)$$

The pdf of the smallest order statistics is obtained by setting $r = 1$

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$$f_{1:n}(x;\theta,\beta)=\frac{n\beta\theta^2(\theta+2)(1+\theta x^2)e^{-\theta x}}{\{\theta+2-(1-\beta)\left[\theta+2+\theta x(\theta x+2)\right]e^{-\theta x}\}^2}\left\{1-\frac{\beta\left\{\theta+2-\left[\theta+2\theta x\left(\theta x+2\right)\right]e^{-\theta x}\right\}}{\theta+2-(1-\beta)\left\{\theta+2\theta x\left(\theta x+2\right)\right\}e^{-\theta x}}\right\}^{n-1} \tag{38}$$

Definition 11 (Reny Entrop). Let $X \sim$ MOCJ (θ, β), the Reny Entropy is given as

$$R_E=\frac{1}{1-a}\left\{\log\int_0^{\infty}\left[f(x;\theta,\beta\right]^a dx\right\};\quad a>0;\quad a\neq 0$$

$$=\frac{1}{1-a}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}P_{ijk}\beta\theta^2(\theta+2)\right]^a\left\{\{\theta(i+1)\}^{-(j+k+1)}\Gamma(j+k+1)+\theta\{\theta(i+1)\}^{-(j+k+3)}\Gamma(j+k+3)\right\}^a \tag{39}$$

- *Numerical Analysis*

In this section, two real data sets are introduced to check the performance of the MOCJ distribution.

The first data set is the survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal et al. (1960) shown in table 1.

Table 1 Survival Times of Guinea Pigs Injected with different Amount of Tubercle Bacilli

10	33	44	56	59	72	74	77	92	93	96	100	100	102	105	107	107	108
108	108	109	112	113	115	116	120	121	122	122	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	183	195	196	197	202	213	215	216	222
230	231	240	245	251	253	254	255	278	293	327	342	347	361	402	432	458	555

Next, we illustrate the proposed MOCJ distribution by comparing its model performance with those of the Marshall-Olkin Sujatha (MOS) distribution, Kumaraswamy-Weibull (KW) distribution, Exponentiated-Weibull (EW) distribution and Exponential Distribution (ED) using the survival times of Guinea pigs injected with different amounts of tubercle bacilli, as shown in Table 2. The analytical measures of fitness, which include log-likelihood (LL), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and Kolmogorov–Smirnov (K-S) statistics, are such that the model with the smaller values of these analytical measures is best among others. See Uzoma, Jeremiah, et al. (2016) for relevant modification on model performance criteria namely Bayesian Information Criterion (BIC).

Table 2 The Analytical Measures of Fitness and MLE Estimates for the Fitted distribution using Guinea Pigs data

Distribution	Parameters Estimates		Std. Error	LL	AIC	BIC	K-S
MOCJ	θ β	0.01603 0.85541	0.00353 0.53806	**-425.61**	**855.22**	**859.7733**	**0.17502**
MOS	θ β	0.01607 0.85574	0.00355 0.54223	-425.7702	855.5404	860.0937	0.50639
KW	a b c	0.36088 0.05534 0.62883	0.01638 0.00654 0.00194	-474.6325	957.265	966.3717	0.38542
	l	0.51970	0.00248				
EW	a k	2.65415 1.16037	1.53624 0.30810	-425.6656	857.3311	864.1611	0.08912
	l	112.8844	46.29453				
ED	θ	0.00568	0.00065	-444.6156	8912312	893.5079	0.29585

From table 2, we see that the proposed Marshall-Olkin Chris-Jerry (MOCJ) distribution is better than the competing distributions fitted based on the criteria of model performance used.

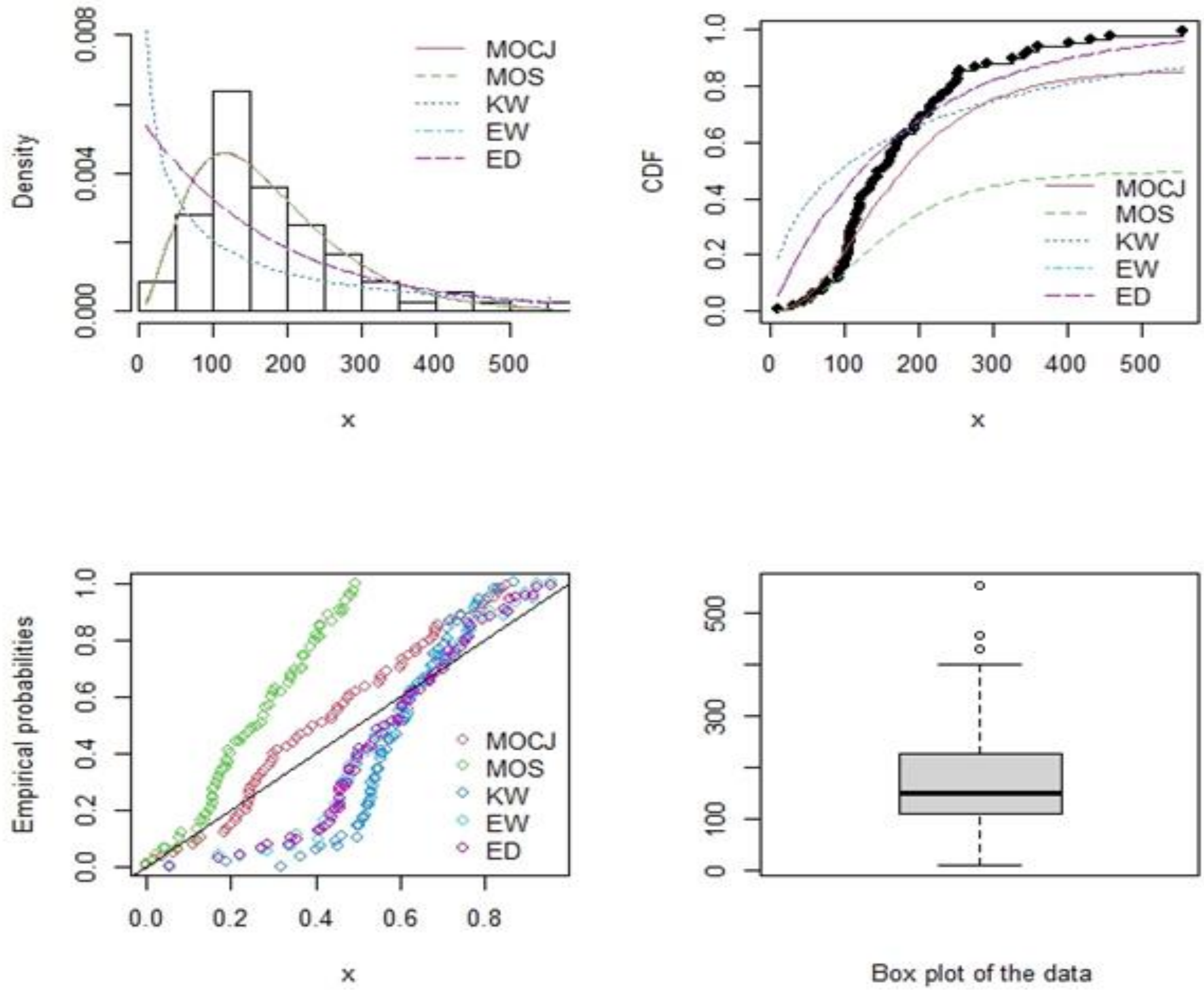

Fig 3 The Estimated PDF, CDF, Kaplan-Meier and Box Plots of the MOCJ other Fitted distributions using the Guinea Pigs Data

The plots in figure 3 also reveal that MOCJ is a better fit for the data than the other fitted distributions.

The second application is on the emission times (in months) of 128 bladder cancer used by Okasha, Mohammed, and Lio (2021).We demonstrate the proposed MOCJ distribution by comparing its model performance with those of the Marshall-Olkin Sujatha (MOS) distribution, Exponential Distribution (ED), Lindley Distribution (LD) and Transmuted Power Lomax (TPL) distribution using the survival times of Guinea pigs injected with different amounts of tubercle bacilli, as shown in Table 3. The analytical measures of fitness, which include log-likelihood (LL), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and Kolmogorov–Smirnov (K-S) statistics, are such that the model with the smaller values of these analytical measures is best among others.

Table 3 Analytical Measures of Fitness and MLE Estimates for the Fitted distributions using 128 Bladder Cancer Data

Distr	Parameter	Estimates	Std. Error	LL	AIC	BIC	K-S
MOCJ	β	0.093	0.0315	-410.8146	825.6291	831.3332	0.955
	θ	0.0451	0.0341				
MOS	θ	0.1172	0.0278	-415.2429	834.4858	840.1898	1.9846
	β	0.0498	0.0315				
ED	θ	0.1068	0.0094	-414.3419	830.6838	833.5358	0.08464
LD	θ	0.19610	0.0123	-419.5299	841.0598	843.9118	0.1164
TPL	θ	0.1068	0.0209	-414.3419	832.6838	838.3879	3.4982
	α	0.00001	0.0186				

From table 3, we see that the proposed Marshall-Olkin Chris-Jerry (MOCJ) distribution is better than the competing distributions fitted based on the criteria of model performance used based on the data on emission times (in months) of 128 bladder cancer patients.

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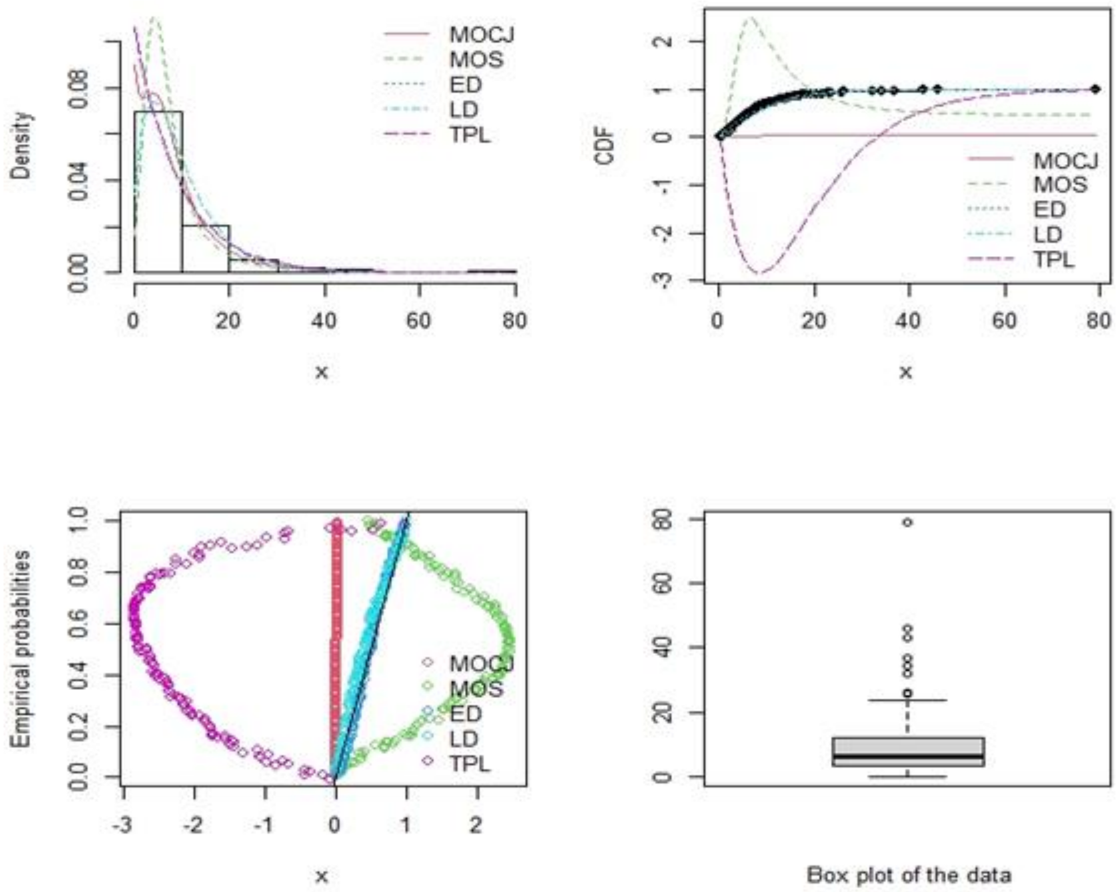

Fig 4 The Estimated PDF, CDF, Kaplan-Meier and Box plots of the MOCJ and other Fitted distributions using Emission Times (in months) of 128 Bladder Cancer Patients

Figure 4 is a weird visualization but a closer study reveals that MOCJ performed better than the other competing distributions.

II. CONCLUSION

In this paper, an extension of the Chris-Jerry distribution has been investigated with the Pietra Measure of inequality derived and the holomorphic extension of the MOCJ incomplete moment. The normalized incomplete moment was also derived which enabled the derivation of the Pietra Index. A meaningful application of the Pietra Measure of inequality and possibly the Gini index can be studied using income or population data. Practical applications of the proposed distribution were carried out using two data sets and the proposed MOCJ distribution performed better than the other fitted distributions in both scenarios.

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