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Estimation of the Parameters of the Power Size Biased Chris-Jerry Distribution

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Abstract:- This paper extends the one-parameter Chris-Jerry distribution to the power size biased Chris-Jerry distribution, a lifetime distribution in the class of the Lindley distribution. We derive the r^{th} moment and particularly estimated the parameters using six classical methods and the Bayesian method. Results of the two real data analyses show that the proposed PSBCJ distribution is better than the likes of Marshall-Olkin Sujatha (MOS) distribution, Two-Parameter Lindley (TPL) distribution, Kumaraswamy-Weibull (KW) distribution, ZubairExponential (ZE) distribution, Lindley Distribution (LD), Exponential Distribution (ED), Pareto (P) distribution, Lindley-Lomax (LL) distribution and Lindley-Pareto (LP) distribution. The weighted least squares is the best method for estimating the parameters of the proposed distribution since the standard errors are the least among other methods.

Keywords: *Chris-Jerry Distribution, Power Size Biased Chris-Jerry Distribution, Classical Estimation, Bayesian Estimation.*

I. INTRODUCTION

In the last decades, many researchers have been involved in proposing a distribution that will have the best fit and flexibility in modelling real time data sets. There are lots of continuous distributions which have been used in modelling real time data sets such as Exponential, Ishita, Gamma, Shanker, Xgamma, Lognormal, Weibull, and Pranav distribution which are available in statistical literature.

Onyekwere and Obulezi [1] proposed Chris-Jerry distribution, a one-parameter distribution for modeling lifetime data. Anabike et al [2] used different classical methods and bayesian method under various loss functions to estimate the parameters of the Zubair-Exponential distribution. Onyekwere et al [3] modified Shanker distribution using quadratic rank transmutation map. Khaldoon Alhayasat, ibrahim kamarulzaman , Amer Ibrahim -Omari and mohd Aftar Abubakar in [4] introduced a power size two parameter Akash distribution (PSBTPAD).

Akash distribution has been extended, generalized and modified including its detailed application in reliability and other areas of knowledge by different scholars namely samuel aderoju and isaac adeniyi [5], Rama shanker, Kamlesh shukla and MK Tiwari [6], Al-khazaleh [7], Shanker [8], Rama Shanker and Berhance Adebe [9] shanker in[10] explained its statistical and mathematical characteristics the shape for altering values of parameter , moment based measures , hazard rate function, mean residual life function, stochastic ordering, mean deviation, order statistics renyi entropy measures, bonferroni, lorenz curves, stress strength reliability along the estimation of parameter and application for modelling waiting times. Moreso two parameter distribution by shanker has shown a real life dataset has been presented to test for the goodness of fit of QPAD over exponential and lognormal distribution . Onyekwere and Obulezi [1] in their paper titled Chris-Jerry distribution and its applications discussed the statistical properties namely the moments, shapes for different values of parameter, stochastic ordering, coefficient of skewness, coefficient of kurtosis, index dispersion, quantile function, order statistics, maximum likelihood, survival function, stress-strenghth reliability which showed superiority over the competitive distribution; the exponential, Ishita, Akash, Rama, Pranav, Rani, Sujatha, lindley, Ranaldhana, Shanker and Xgama distributions. Again, its real life application in health sciences , insurance and financial institution were explored. Shanker [11] discussed (Ishita distribution) and highlighted its mathematical properties, the hazard rate function, mean residual life function, stochastic ordering, mean residual deviation, order statistics, bonferroni and lorenz curves renyi enthropy, stress strength reliability, and goodness of fit which makes it better than the competitive distribution. Akash , Lindley, and exponential distribution and its real life application in aircraft industry, and education (Physics) and also dominant in the literature.

There are several conditions these distributions (Lindley, exponential and Akash for instance) may not be suitable either. Therefore, to obtain a new distribution which will fit better and flexible than the existing distributions for modelling lifetime data in reliability and in terms of its hazard rate shape, we introduced a new one parameter distribution called Power Size Biased Chris-Jerry distribution.

The remaining part of this paper is organized as follows; in section 2, we derive the power size biased Chris-Jerry distribution and in section 3, we derive the moment. In section 4, we estimate the parameters using some classical methods while the bayesian estimation is done in section 5. In section 6, we applied the model to two real life data sets and make appropriate conclusion in section 7.

- *Power Size Biased Chris-Jerry (PSBCJ) Distribution*

The probability density function (pdf) and the cumulative distribution function (cdf) of the Chris-Jerry distributed random variable *X* are respectively

$$f(x) = \frac{\theta^2}{\theta+2}(1+\theta x^2)\mathrm{e}^{-\theta x}; \quad x, \quad \theta > 0 \tag{1}$$

And

$$F_w(x) = 1 - \left[1 + \frac{\theta x(\theta x + 2)}{\theta + 2}\right]\mathrm{e}^{-\theta x} \tag{2}$$

For any non-negative continuous random variable *X* with a probability density function (pdf) *f*(*x*), the pdf of the weighted random variable X_w is defined as

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]} = \frac{w(x)f(x)}{\mu_x} \tag{3}$$

where *w*(*x*) is a non-negative weight function. A special case of equation (3) arises when the weight function $w(x) = x^\beta$. This is referred to as size biased distribution of order β with pdf given as

$$f_\beta(x) = \frac{x^\beta f(x)}{\int x^\beta f(x)dx} \tag{4}$$

when $\beta = 1$ *or* 2, the resulting distribution is known as length-biased and area-biased distribution respectively. Notice that the mean of the Chris-Jerry distribution is $\mu_x = \frac{\theta+6}{\theta(\theta+2)}$ hence the pdf of a weighted Chris-Jerry random variable $w(x) = x$ is given as

$$f_w(x) = \frac{\theta^3}{\theta+6}(1+\theta x^2)x\mathrm{e}^{-\theta x}; x > 0 \quad \theta > 0 \tag{5}$$

while the cumulative distribution function is given in the following equation

$$F_w(x) = 1 - \left[1 + \frac{\theta x(6+\theta+3\theta x+\theta^2 x^2)}{\theta+6}\right]\mathrm{e}^{-\theta x} \tag{6}$$

In this paper we modified the Chris-Jerry distribution to a power size biased Chris-Jerry (PSBCJ) distribution. By 1 making a change of variable $Y = X^{\beta}$ in equation (5) a pdf and cdf of a random variable *Y* can be respectively defined as

$$f_{PSBCJ}(y,\theta,\beta) = \frac{\beta\theta^2}{\theta+2}(1+\theta y^{2\beta})y^{\beta-1}\mathrm{e}^{-\theta y^\beta}; \quad y,\theta,\beta > 0 \tag{7}$$

And

$$F_{PSBCJ}(y,\theta,\beta) = \frac{1}{\theta+2}\left\{\theta\left(1-e^{-\theta y^\beta}\right) + \gamma\left(3,\theta y^\beta\right)\right\} \tag{8}$$

The survival function and hazard rate function of the PSBCJ distributed random variable *X* are respectively as

$$S_{PSBCJ}(y,\theta,\beta) = 1 - \frac{1}{\theta+2}\left\{\theta\left(1-e^{-\theta y^\beta}\right) + \gamma\left(3,\theta y^\beta\right)\right\} \tag{9}$$

and

$$H_{PSBCJ}(y,\theta,\beta)=\frac{\beta\theta^{2}(1+\theta y^{2\beta})y^{\beta-1}\mathrm{e}^{-\theta y^{\beta}}}{\theta+2-\theta\left(1-e^{-\theta y^{\beta}}\right)-\gamma(3,\theta y^{\beta})} \tag{10}$$

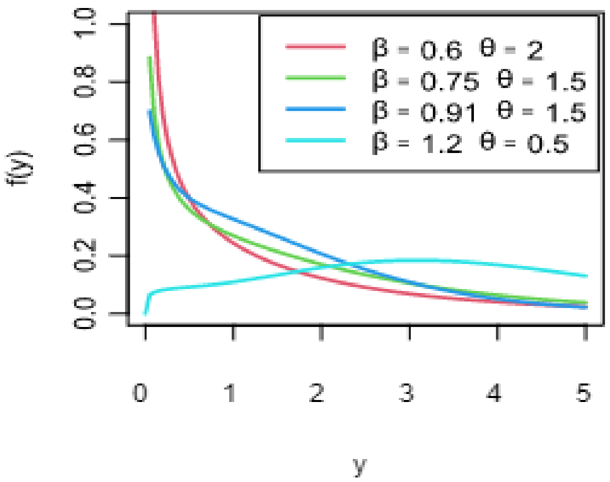

Fig 1 (a) PDF of PSBCJ distribution

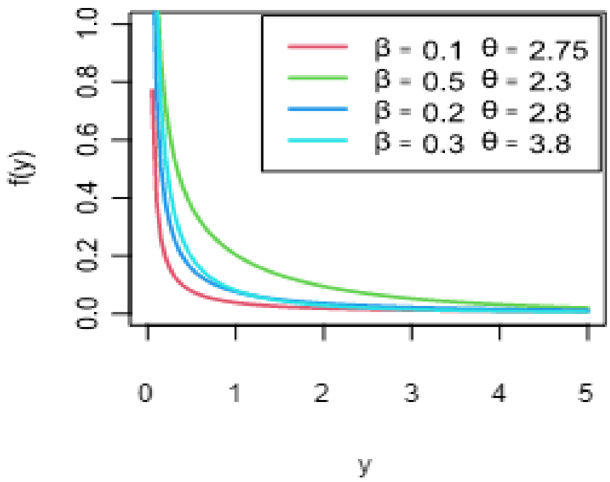

Fig 1 (b) PDF of PSBCJ distribution

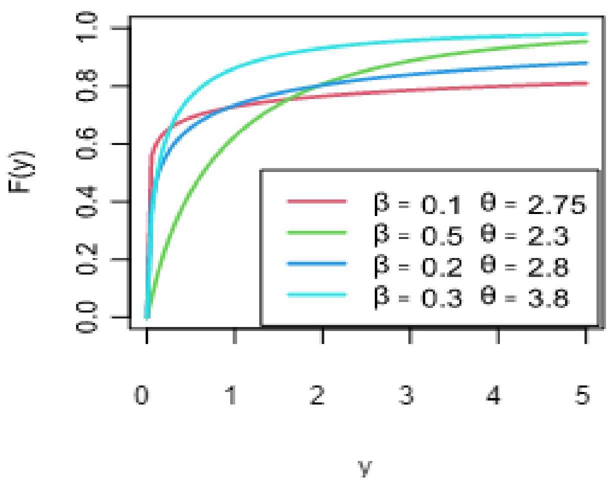

Fig 1(c) CDF of PSBCJ distribution

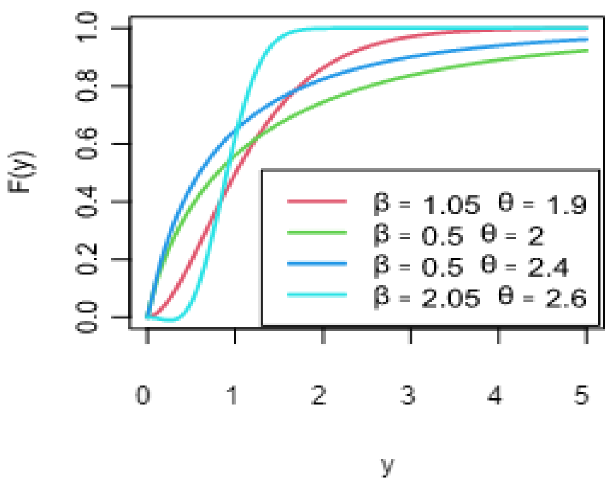

Fig 1 (d) CDF of PSBCJ distribution

Fig 1 PDF and CDF of the PSBCJ distribution

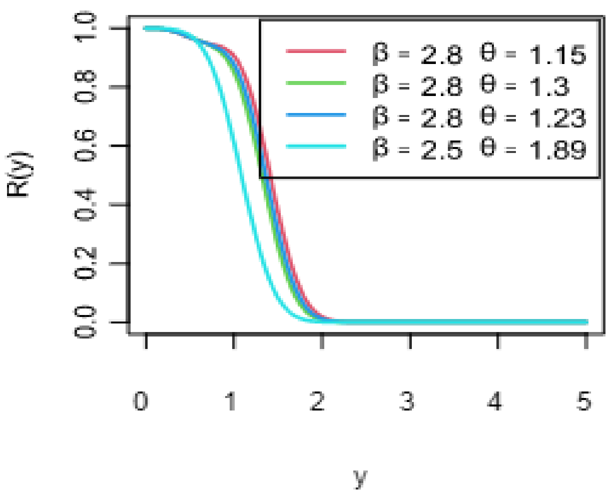

Fig 2 (a) Reliability function of PSBCJ distribution

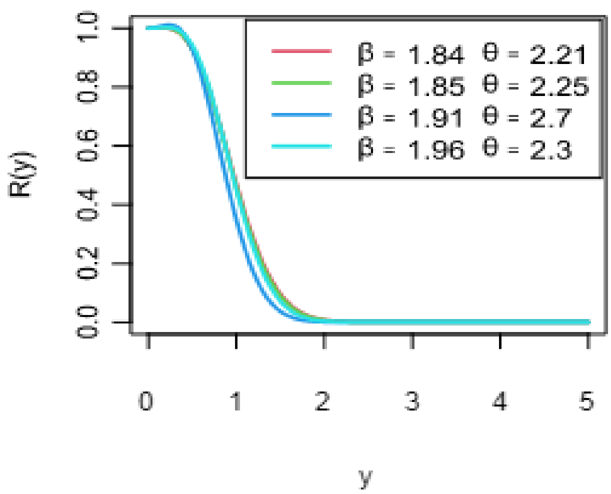

Fig 2 (b) Reliability function of PSBCJ distribution

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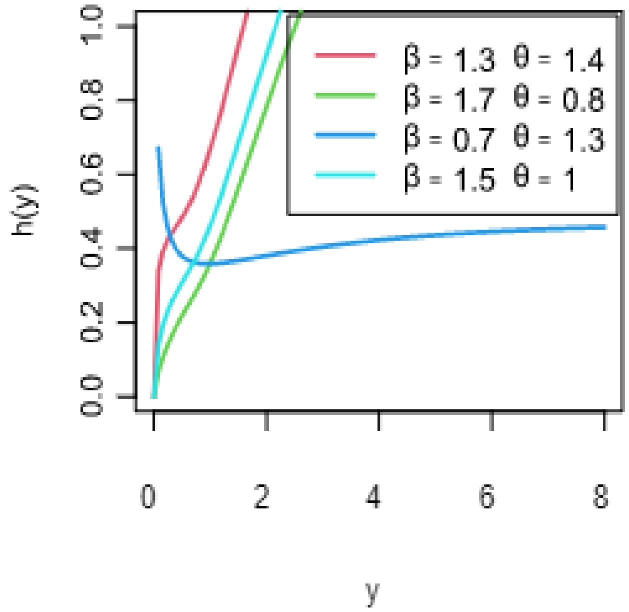

Fig 2 (c) Hazard function of PSBCJ distribution

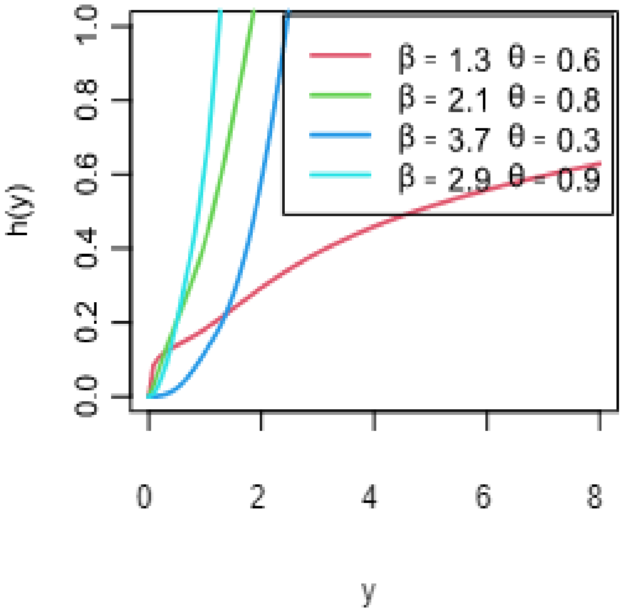

Fig 2 (d) Hazard function of PSBCJ distribution

Fig 2 Reliability and Hazard Rate function of PSBCJ Distribution

The reversed hazard rate function and Odds function of the PSBCJ distributed random variable *X* are respectively as

$$RH_{PSBCJ}(y,\theta,\beta)=\frac{\beta\theta^2(1+\theta y^{2\beta})y^{\beta-1}\mathrm{e}^{-\theta y^\beta}}{\theta\left(1-e^{-\theta y^\beta}\right)+\gamma(3,\theta y^\beta)} \quad (11)$$

And

$$O_{PSBCJ}(y,\theta,\beta)=\frac{\theta\left(1-e^{-\theta y^\beta}\right)+\gamma(3,\theta y^\beta)}{\theta+2-\theta\left(1-e^{-\theta y^\beta}\right)-\gamma(3,\theta y^\beta)} \quad (12)$$

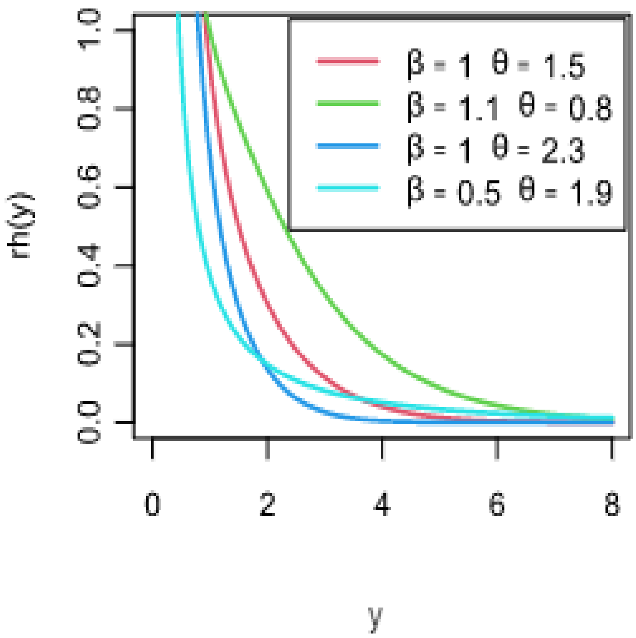

Fig 3 (a) Reversed Hazard Rate function of PSBCJ distribution

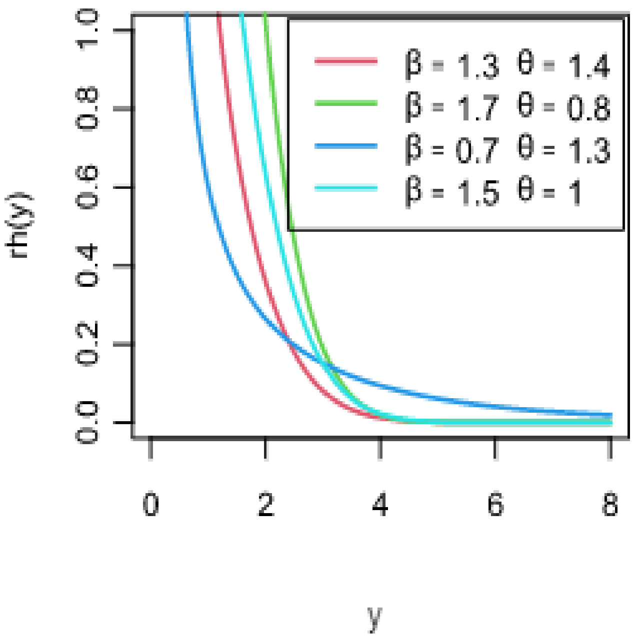

Fig 3 (b) Reversed Hazard function of PSBCJ distribution

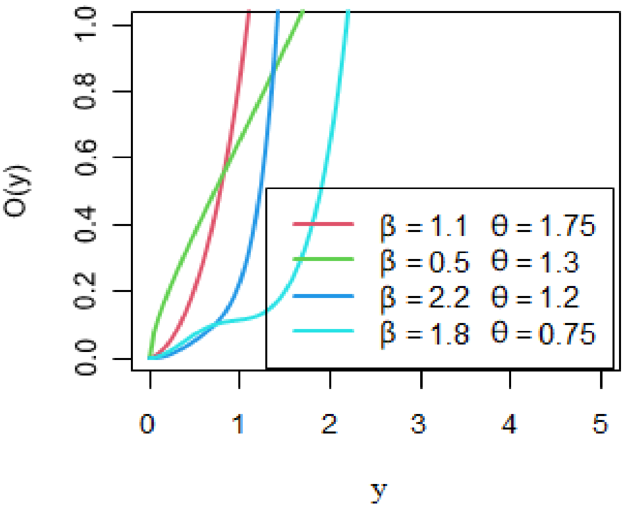

Fig 3 (c) Odds function of PSBCJ distribution

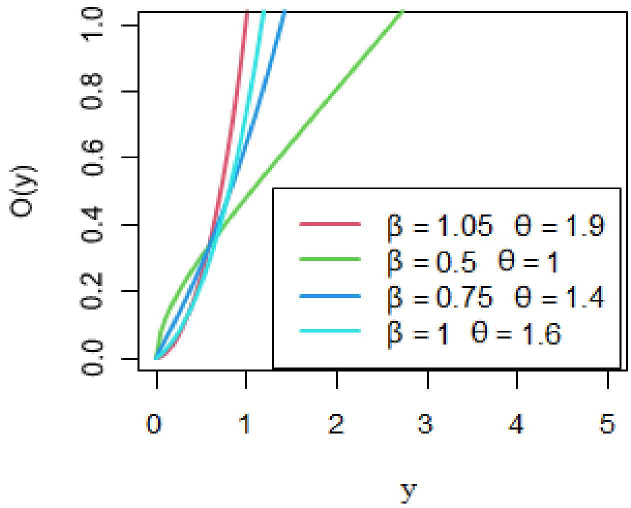

Fig 3 (d) Odds function of PSBCJ distribution

Fig 3 Reversed Hazard Rate and Odds functions of PSBCJ Distribution

- *Moment of the PSBCJ Distribution*

The r^{th} non central moment of the proposed PSBCJ distribution is obtained as follows

$$\left.\begin{aligned} \mu_r' &= \int_0^\infty y^r f(x)dx = \frac{\beta\theta^2}{\theta+2}\int_0^\infty \left(1+\theta y^{2\beta}\right) y^{\beta+r-1} e^{-\theta y^\beta} dy \\ &= \frac{\beta\theta^2}{\theta+2}\left[\int_0^\infty y^{\beta+r-1} e^{-\theta y^\beta} dy + \theta\int_0^\infty y^{3\beta+r-1} e^{-\theta y^\beta} dy\right] \\ &\qquad \Rightarrow x = \theta y^\beta; \quad dx = \theta\beta y^{\beta-1} dy \\ &= \frac{\beta\theta^2}{\theta+2}\left[\frac{1}{\theta\beta}\int_0^\infty \left(\frac{x}{\theta}\right)^{\frac{r}{\beta}} e^{-x}dx + \frac{1}{\beta}\int_0^\infty \left(\frac{x}{\theta}\right)^{\frac{r}{\beta}+2} e^{-x}dx\right] \\ &= \frac{1}{\theta+2}\left[\frac{\Gamma\left(\frac{r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}-1}} + \frac{\Gamma\left(\frac{r}{\beta}+3\right)}{\theta^{\frac{r}{\beta}}}\right] \end{aligned}\right\} \tag{13}$$

Definition 3.1. Let $X \sim$ PSBCJ (θ,β), the mean is obtained by replacing r with 1 in equation (13)

$$\mu = \frac{1}{\theta+2}\left[\frac{\Gamma\left(\frac{1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}-1}} + \frac{\Gamma\left(\frac{1}{\beta}+3\right)}{\theta^{\frac{1}{\beta}}}\right] \tag{14}$$

Definition 3.2. Let $X \sim$ PSBCJ (θ,β), the $2^{nd}, 3^{rd}$ and 4^{th} crude moments are obtained by replacing r with 2,3 and 4 respectively in equation (13)

$$\left.\begin{aligned} \mu_2' &= \frac{1}{\theta+2}\left[\frac{\Gamma\left(\frac{2}{\beta}+1\right)}{\theta^{\frac{2}{\beta}-1}} + \frac{\Gamma\left(\frac{2}{\beta}+3\right)}{\theta^{\frac{2}{\beta}}}\right] \\ \mu_3' &= \frac{1}{\theta+2}\left[\frac{\Gamma\left(\frac{3}{\beta}+1\right)}{\theta^{\frac{3}{\beta}-1}} + \frac{\Gamma\left(\frac{3}{\beta}+3\right)}{\theta^{\frac{3}{\beta}}}\right] \\ \mu_4' &= \frac{1}{\theta+2}\left[\frac{\Gamma\left(\frac{4}{\beta}+1\right)}{\theta^{\frac{4}{\beta}-1}} + \frac{\Gamma\left(\frac{4}{\beta}+3\right)}{\theta^{\frac{4}{\beta}}}\right] \end{aligned}\right\} \tag{15}$$

- *Classical Estimation of PSBCJ Distribution Parameters*

In this section, we examine seven Non-Bayesian methods of parameter estimation to learn their efficiency in estimating PSBCJ distribution parameters.

- *Maximum Likelihood Estimation of the PSBCJ Distribution Parameter*

Let $(y_1, y_2, ..., y_n)$ be n random samples drawn from PSBCJ distribution, then the likelihood function is given as

$$\ell(f_{PSBCJ}(y;\beta,\theta)) = \prod_{i=1}^{n} \frac{\beta\theta^2}{\theta+2}(1+\theta y_i^{2\beta})y_i^{\beta-1}e^{-\theta y_i^{\beta}}$$
$$= \frac{\beta^n\theta^{2n}e^{-\theta\sum_{i=1}^{n} y_i^{\beta}}}{(\theta+2)^n}\prod_{i=1}^{n}(1+\theta y_i^{2\beta})y_i^{\beta-1} \quad (16)$$

Taking the natural log of ℓ and differentiating partially with respect to β and θ yield the following results

$$\psi = \ell(x;\beta,\theta) = n\ln\beta + 2n\ln\theta - n\ln\theta + 2 - \theta\sum_{i=1}^{n} y_i^{\beta} + \sum_{i=1}^{n}\ln(1+\theta y_i^{2\beta} + (\beta-1)\sum_{i=1}^{n}\ln y_i^{\beta-1} \quad (17)$$

$$\frac{\partial\psi}{\partial\hat{\beta}} = \frac{n}{\beta} - \theta\sum_{i=1}^{n} y_i^{\beta}\ln y_i + 2\theta\sum_{i=1}^{n}\frac{y_i^{2\beta}\ln y_i}{1+\theta y_i^{2\beta}} + \sum_{i=1}^{n}\ln y_i^{\beta-1} + (\beta-1)\sum_{i=1}^{n} y_i^{\beta-1}\ln y_i \quad (18)$$

$$\frac{\partial\psi}{\partial\hat{\theta}} = \frac{2n}{\theta} - \frac{n}{\theta+2} - \sum_{i=1}^{n} y_i^{\beta} + \sum_{i=1}^{n}\frac{y_i^{2\beta}}{1+\theta y_i^{2\beta}} \quad (19)$$

Set $\frac{\partial\psi}{\partial\hat{\beta}} = 0$, yields the following quadratic result

$$\frac{n}{\beta} - \theta\sum_{i=1}^{n} y_i^{\beta}\ln y_i + 2\theta\sum_{i=1}^{n}\frac{y_i^{2\beta}\ln y_i}{1+\theta y_i^{2\beta}} + \sum_{i=1}^{n}\ln y_i^{\beta-1} + (\beta-1)\sum_{i=1}^{n} y_i^{\beta-1}\ln y_i = 0$$

Set $\frac{\partial\psi}{\partial\hat{\theta}} = 0$, yields the following quadratic result

$$\frac{2n}{\theta} - \frac{n}{\theta+2} - \sum_{i=1}^{n} y_i^{\beta} + \sum_{i=1}^{n}\frac{y_i^{2\beta}}{1+\theta y_i^{2\beta}} = 0$$

which are solved using iterative method since there is no closed-form solution.

- *Least Squares Estimation (LSE)*

The Least Squares Estimation was proposed by Swain et al [12] to estimate the parameters of Beta distribution. Using the deductions from the work of Swain et al [12], we write

$$E[F(x_{i:n}|\beta,\theta)] = \frac{i}{n+1}.$$

$$V[F(x_{i:n}|\beta,\theta)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The least squares estimates $\hat{\beta}_{LSE}$ and $\hat{\theta}_{LSE}$ of the parameters β and θ are obtained by minimizing the function $L(\beta,\theta)$ with respect to β and θ

$$L(\lambda,\theta) = \arg\min_{(\theta)}\sum_{i=1}^{n}\left[F(y_{i:n}|\lambda,\theta) - \frac{i}{n+1}\right]^2$$

The estimates are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left[F(y_{i:n}|\beta,\theta)-\frac{i}{n+1}\right]^2\Delta_1(y_{i:n}|\beta,\theta)=0 \tag{21}$$

$$\sum_{i=1}^{n}\left[F(y_{i:n}|\beta,\theta)-\frac{i}{n+1}\right]^2\Delta_2(x_{i:n}|\beta,\theta)=0 \tag{22}$$

where

$$\Delta_1(y_{i:n}|\beta,\theta)=\frac{\theta y^{\beta}\mathrm{e}^{-\theta y^{\beta}}}{\theta+2}\left[\theta\beta y^{2\beta}-3\theta\ln y-2\ln y-2\beta y^{\beta}\ln y-y^{\beta}+2\theta^2y^{\beta}\ln y\right] \tag{23}$$

$$\Delta_2(y_{i:n}|\beta,\theta)=\frac{y^{\beta}\mathrm{e}^{-\theta y^{\beta}}}{\theta+2}\left[2-3\theta-\beta y^{\beta}-\frac{\theta\beta y^{\beta}}{\theta+2}+\theta\beta y^{2\beta}+\frac{2\theta^2}{\theta+2}+2\theta^2y^{\beta}\right] \tag{24}$$

- *Weighted Least Squares Estimation (WLSE)*

The weighted least squares estimates $\hat{\beta}_{WLSE}$ and $\hat{\theta}_{WLSE}$ of PSBCJ distribution parameters β and θ are obtained by minimizing the function $W(\beta,\theta)$ with respect to β and θ

$$W(\beta,\theta)=\arg\min_{(\beta,\theta)}\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{i:n}|\theta)-\frac{i}{n+1}\right]^2. \tag{25}$$

Solving the following non-linear equation yields the estimate

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_1(y_{i:n}|\theta)=0 \tag{26}$$

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(y_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_2(y_{i:n}|\theta)=0 \tag{27}$$

where $\Delta_1(y.|\lambda,\theta)$ and $\Delta_2(y.|\lambda,\theta)$ is as defined in (23) and (24) respectively.

- *Maximum Product Spacing Estimators (MPSE)*

A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the PSBCJ is given as follows

$$Gs(\beta,\theta|data)=\left(\prod_{i=1}^{n+1}D_l(y_i,\beta,\theta)\right)^{\frac{1}{n+1}}, \tag{28}$$

where $D_l(y_i,\beta,\theta) = F(y_i;\beta,\theta) - F(y_{i-1};\beta,\theta)$, $i = 1,2,3,...,n$

Similarly, one can also choose to maximize the function

$$H(\beta,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\beta,\theta). \tag{29}$$

By taking the first derivative of the function $H(\vartheta)$ with respect to β and θ, and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial\beta}=0$ and $\frac{\partial H(\phi)}{\partial\theta}=0$, where $\phi=(\beta,\theta)$, we obtain the value of the parameter estimates.

- *Cramér-von-Mises Estimation (CVME)*

The Cramér-von-Mises estimates $\hat{\beta}_{CV\,ME}$, and $\hat{\theta}_{CV\,ME}$ of the PSBCJ distribution parameters β, and θ are obtained by minimizing the function $C(\beta,\theta)$ with respect to β, and θ

$$C(\beta,\theta)=\underset{(\beta,\theta)}{\arg\min}\left\{\frac{1}{12n}+\sum_{i=1}^{n}\left[F(y_{i:n}|\beta,\theta)-\frac{2i-1}{2n}\right]^2\right\}. \tag{30}$$

The estimates are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n}\left(F(y_{i:n}|\beta,\theta)-\frac{2i-1}{2n}\right)\Delta_1(y_{i:n}|\beta,\theta)=0$$
$$\sum_{i=1}^{n}\left(F(y_{i:n}|\beta,\theta)-\frac{2i-1}{2n}\right)\Delta_2(y_{i:n}|\beta,\theta)=0 \tag{31}$$

where $\Delta_1(y.|\lambda,\theta)$ and $\Delta_2(y.|\lambda,\theta)$ is as defined in (23) and (24) respectively.

- *Anderson-Darling Estimation (ADE)*

The Anderson-Darling estimates $\hat{\beta}_{ADE}$, and $\hat{\theta}_{ADE}$ of the PSBCJ distribution parameters β and θ are obtained by minimizing the function $A(\beta,\theta)$ with respect to β and θ

$$A(\beta,\theta)=\underset{(\beta,\theta)}{\arg\min}\sum_{i=1}^{n}(2i-1)\left\{\ln F(y_{i:n}|\beta,\theta)+\ln\left[1-F(y_{n+1-i:n}|\beta,\theta)\right]\right\} \tag{32}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(y_{i:n}|\beta,\theta)}{F(y_{i:n}|\beta,\theta)}-\frac{\Delta_1(y_{n+1-i:n}|\beta,\theta)}{1-F(y_{n+1-i:n}|\beta,\theta)}\right]=0$$
$$\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(y_{i:n}|\beta,\theta)}{F(y_{i:n}|\beta,\theta)}-\frac{\Delta_2(y_{n+1-i:n}|\beta,\theta)}{1-F(y_{n+1-i:n}|\beta,\theta)}\right]=0 \tag{33}$$

where $\Delta_1(y.|\beta,\theta)$ and $\Delta_2(y.|\beta,\theta)$ is as defined in (23) and (24) respectively.

- *Right-Tailed Anderson-Darling Estimation (RTADE)*

The Right-Tailed Anderson-Darling estimates $\hat{\beta}_{RTADE}$ and $\hat{\theta}_{RTADE}$ of the PSBCJ distribution parameters β and θ are obtained by minimizing the function $R(\beta,\theta)$ with respect to β and θ

$$R(\lambda,\theta)=\underset{(\beta,\theta)}{\arg\min}\left\{\frac{n}{2}-2\sum_{i=1}^{n}F(y_{i:n}|\beta,\theta)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\left[1-F(y_{n+1-i:n}|\beta,\theta)\right]\right\} \tag{34}$$

The estimates can be obtained by solving the following set of non-linear equations

$$-2\sum_{i=1}^{n}\frac{\Delta_1(y_{i:n}|\beta,\theta)}{F(y_{i:n}|\beta,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(y_{n+1-i:n}|\beta,\theta)}{1-F(y_{n+1-i:n}|\beta,\theta)}\right]=0$$
$$-2\sum_{i=1}^{n}\frac{\Delta_2(y_{i:n}|\beta,\theta)}{F(y_{i:n}|\beta,\theta)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(y_{n+1-i:n}|\beta,\theta)}{1-F(y_{n+1-i:n}|\beta,\theta)}\right]=0 \tag{35}$$

where $\Delta_1(y.|\beta,\theta)$ and $\Delta_2(y.|\beta,\theta)$ is as defined in (23) and(24) respectively. The estimates given in (26), (27), (29), (31), (33), (35) are obtained using optim() function in R with the Newton-Raphson iterative algorithm.

- *Bayesian Inference on the PSBCJ Distribution Parameters*

This section deals with the Bayesian Estimate (BE) of the unknown parameters of the PSBCJ distribution. We can consider applying independent gamma priors for the parameters β and θ with pdfs in the parameter prior distributions of PSBCJ as follows;

$$\left.\begin{array}{ll}\pi_1(\beta) \propto \beta^{s_1-1}e^{-q_1\beta} & \beta>0, s_1>0, q_1>0,\\ \pi_2(\theta) \propto \theta^{s_2-1}e^{-q_2\theta} & \theta>0, s_2>0, q_2>0,\end{array}\right\} \quad (36)$$

where the hyper-parameters $s_j,q_j,j = 1,2$ are selected to reflect the prior knowledge about the unknown parameters. The joint prior for $\phi = (\beta,\theta)$ is given by

$$\left.\begin{array}{l}\pi(\phi) = \pi_1(\beta)\pi_2(\theta)\\ \pi(\phi) \propto \beta^{s_1-1}\theta^{s_2-1}e^{\{-q_1\beta-q_2\theta\}}.\end{array}\right\} \quad (37)$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_1,x_2,...,x_n)$ is given by:

$$\pi(\phi \mid \mathbf{x}) = \frac{\pi(\phi)\ell(\phi)}{\int_\phi \pi(\phi)\ell(\phi)d\phi},$$

Which implies that the posterior density function is:

$$\pi(\phi \mid \mathbf{x}) \propto \frac{\beta^{n+s_1-1}\theta^{2n+s_2-1}e^{-q_1\beta-q_2\theta-\theta\sum_{i=1}^n y_i^\beta}}{(\theta+2)^n}\prod_{i=1}^n(1+\theta y_i^{2\beta})y_i^{\beta-1} \quad (38)$$

The gamma prior used here is not in the same distribution family with the derived posterior hence it is not a conjugate of the model. Lack of conjuagte prior makes the estimation procedure mathematically intractable. To obtain estimates of the parameters $\phi = (\beta,\theta)$, we resolve the following expectations;

$$\hat{\beta} = E(\beta|y) = \int_0^\infty \frac{\beta^{n+s_1}\theta^{2n+s_2-1}e^{-q_1\beta-q_2\theta-\theta\sum_{i=1}^n y_i^\beta}}{(\theta+2)^n}\prod_{i=1}^n(1+\theta y_i^{2\beta})y_i^{\beta-1}d\beta \quad (39)$$

And

$$\hat{\theta} = E(\theta|y) = \int_0^\infty \frac{\beta^{n+s_1-1}\theta^{2n+s_2}e^{-q_1\beta-q_2\theta-\theta\sum_{i=1}^n y_i^\beta}}{(\theta+2)^n}\prod_{i=1}^n(1+\theta y_i^{2\beta})y_i^{\beta-1}d\theta \quad (40)$$

Both equations (39) and (40) have no closed-form solution, hence will be implemented with iterative algorithm found in optim() in R.

- *Application*

In the section, we apply the proposed PSBCJ distribution to some real data sets. The first data set is the survival times of guinea pigs injected with different amount of tubercle bacilli studied by [13] and Anabike et al [2] and shown in table 1.

Table 1 The Survival Times of Guinea Pigs Injected with different Amount of Tubercle Bacilli

10	33	44	56	59	72	74	77	92	96	100	100	100	102	105	107	107	108
108	108	109	112	113	115	116	120	121	122	124	124	130	134	136	139	144	146
153	159	160	163	163	168	171	172	176	195	196	196	197	202	213	215	216	222
230	231	240	245	251	253	254	255	278	327	342	342	347	361	402	432	458	5555

Next, we illustrate the proposed PSBCJ distribution by comparing its model performance with those of the Marshall Olkin Sujatha (MOS) distribution, Lindley Distribution (LD), Two-Parameter Lindley (TPL) distribution, Exponential Distribution (ED) and Kumaraswamy-Weibull (KW) distribution using the survival times of Guinea pigs injected with different amounts of tubercle bacilli, as shown in Table 2. The analytical measures of fitness, which include log-likelihood (LL), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and Kolmogorov–Smirnov (K-S) statistics, are such that the model with the smaller values of these analytical measures is best among others. See [14] for relevant modification on model performance criteria namely Bayesian Information Criterion (BIC).

Table 2 The Analytical Measures of Model performance and MLE estimates for the fitted distributions using Guinea Pigs data

Distr	Parameter	Estimates	Std. Error	LL	AIC	BIC	K-S
PSBCJ	β	1.20909	0.07101	-425.58	855.164	857.717	0.50438
	θ	0.01445	0.0055				
MOS	θ	0.01607	0.00355	-425.77	855.540	860.094	0.50639
	β	0.85574	0.54223				
LD	θ	0.01125	0.00093	-429.28	860.570	862.846	0.17004
TPL	θ	0.01130	0.00093	-429.10	862.208	866.761	0.53491
	α	714.1423	8388.61				
ED	θ	0.00568	0.00064	-444.62	891.231	893.508	0.29585
KW	a	0.36088	0.01638	-474.63	957.265	966.372	0.38542
	b	0.05535	0.00654				
	c	0.62883	0.00194				
	λ	0.51970	0.00248				

From table 2, we see that the proposed Power Size Biased Chris-Jerry (PSBCJ) distribution is better than the competing distributions fitted based on the criteria of model performance namely Loglikelihood (LL), Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC).

Table 3 Classical and Bayes Methods Estimates for the PSBCJ Distribution using the Guinea pigs data

Method	Estimates		Std. Error
MLE	$\hat{\beta}$	1.02909	0.07101
	$\hat{\theta}$	0.01445	0.00550
LSE	$\hat{\beta}$	0.69105	0.22826
	$\hat{\theta}$	0.02844	0.01389
WLSE	$\hat{\beta}$	0.66901	0.00592
	$\hat{\theta}$	0.02997	0.00050
CVME	$\hat{\beta}$	0.69178	0.22727
	$\hat{\theta}$	0.02838	0.013914
ADE	$\hat{\beta}$	0.70412	0.11925
	$\hat{\theta}$	0.02747	0.007587
RTADE	$\hat{\beta}$	0.63895	0.13873
	$\hat{\theta}$	0.03340	0.00964
Bayesian	$\hat{\beta}$	1.00436	0.01428
	$\hat{\theta}$	0.01650	0.00072

From the Table (3) the weighted least squares estimate has the least standard error for both $\hat{\beta}$ and $\hat{\theta}$, therefore it is the best method for estimating the parameters of the Power size biased Chris-Jerry distribution.

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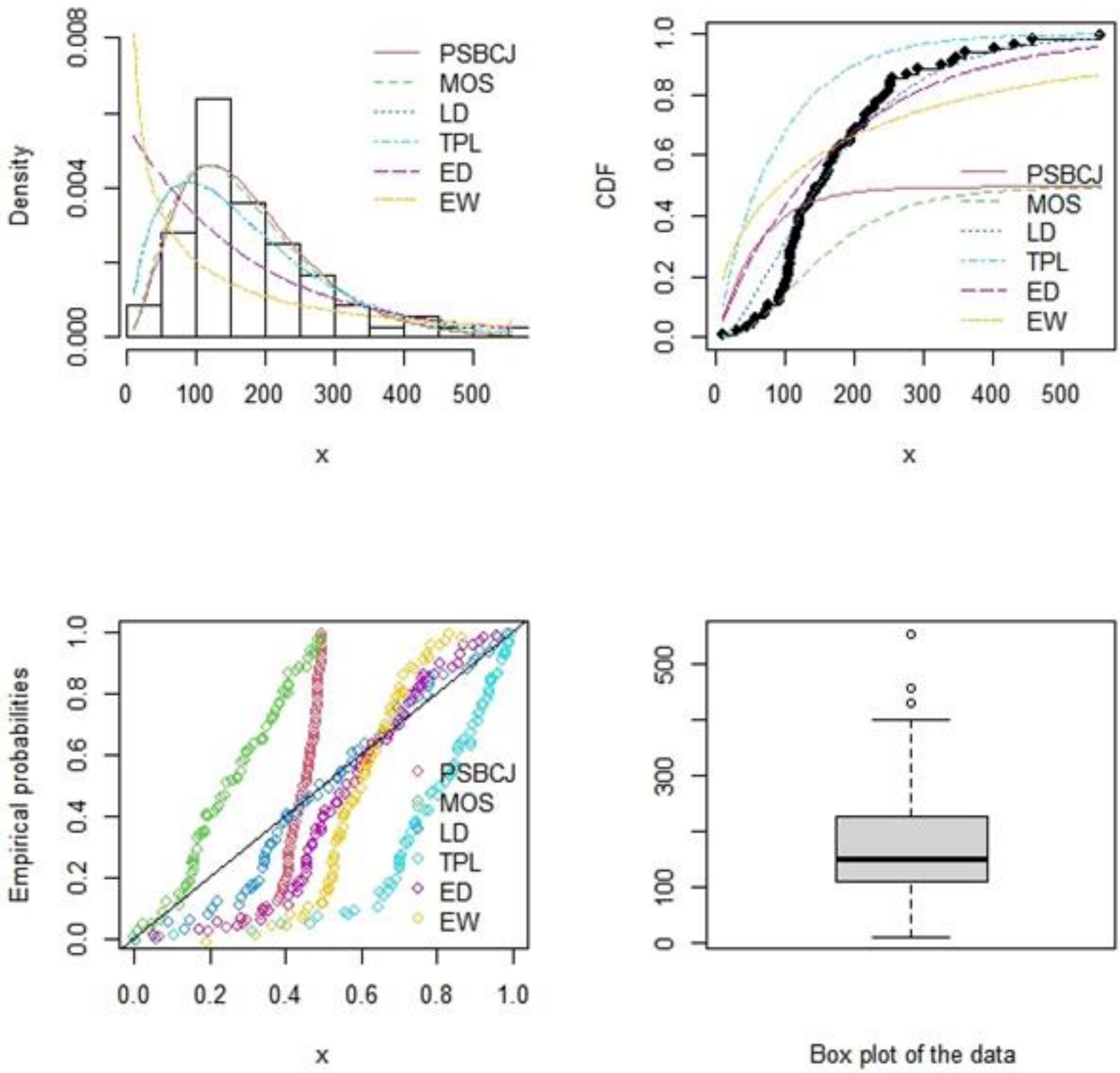

Fig 4 The estimated pdf, cdf, Kaplan-Meier and Box plots of the PSBCJ other fitted distributions using the Guinea Pigs Data

➢ *Second Application*

The following represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm studied by Shanker et al [15].

Table 4 Tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm

1.312	1.314	1.479	1.552	1.7	1.803	1.861	1.865	1.944	1.958	1.966	1.997	2.006
2.021	2.027	2.055	2.063	2.098	2.14	2.179	2.224	2.24	2.253	2.27	2.272	2.274
2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.49	2.511	2.514	2.535
2.554	2.566	2.57	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726	2.77	2.773
2.8	2.809	2.818	2.821	2.848	2.88	2.954	3.012	3.067	3.084	3.09	3.096	3.128
3.233	3.433	3.585	3.858									

Next, we illustrate the proposed PSBCJ distribution by comparing its model performance with those of the ZubairExponential (ZE) distribution, Lindley Distribution (LD), Exponential Distribution (ED), Pareto (P) distribution, Lindley-Lomax (LL) distribution and Lindley-Pareto (LP) distribution using data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm as shown in Table 4.

Table 5 The Analytical Measures of Model performance and MLE estimates for the fitted distributions using data on the Tensile Strength, Measured in GPa, of 69 Carbon Fibers Tested under tension at Gauge Lengths of 20mm

Distr	Parameter	Estimates	Std. Error	LL	AIC	BIC	K-S
PSBCJ	β	3.19536	0.24435	**-51.061**	**106.122**	**110.590**	0.20704
	θ	0.142	0.03529				
ZE	α	46.41	16.6048	-54.7950	113.590	118.058	0.08979
	θ	2.04919	0.17323				
LD	θ	0.65359	0.05795	-119.311	240.622	242.856	0.40044
ED	θ	0.40729	0.04903	-130.979	263.958	266.192	0.44771
P	α	123.0462	168.432	-131.247	266.494	270.962	0.44731
	θ	301.5539	413.904				
LL	a	69.20619	30.9224	-131.456	268.912	276.614	0.44605
	b	0.00589	0.00257				
	t	-0.00256	0.04782				
LP	a	0.45429	0.07725	-50.5038	107.008	113.710	0.05729
	k	3.62024	0.31964				
	t	0.00370	0.00074				

From table 5, we see that the proposed Power Size Biased Chris-Jerry (PSBCJ) distribution is better than the competing distributions fitted based on the criteria of model performance namely Loglikelihood (LL), Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC).

Table 6 Classical and Bayes Methods Estimates for the PSBCJ Distribution using data on the Tensile Strength, Measured in GPa, of 69 Carbon Fibers Tested under tension at gauge lengths of 20mm

Method	Estimates		Std. Error
MLE	$\hat{\beta}$	3.19536	0.24435
	$\hat{\theta}$	0.14200	0.03529
MPSE	$\hat{\beta}$	2.98722	0.01575
	$\hat{\theta}$	0.17577	0.01323
LSE	$\hat{\beta}$	3.03116	0.25681
	$\hat{\theta}$	0.16419	0.06128
WLSE	$\hat{\beta}$	3.00633	0.00711
	$\hat{\theta}$	0.16909	0.00224
CVME	$\hat{\beta}$	3.04168	0.25384
	$\hat{\theta}$	0.16208	0.05994
ADE	$\hat{\beta}$	3.00066	0.01901
	$\hat{\theta}$	0.17069	0.01395
RTADE	$\hat{\beta}$	2.99762	0.03672
	$\hat{\theta}$	0.17449	0.02100
Bayesian	$\hat{\beta}$	3.19137	0.07749
	$\hat{\theta}$	0.14118	0.00220

Again, from Table (6) the weighted least squares estimate has the least standard error for both $\hat{\beta}$ and $\hat{\theta}$, therefore it is the best method for estimating the parameters of the Power size biased Chris-Jerry distribution.

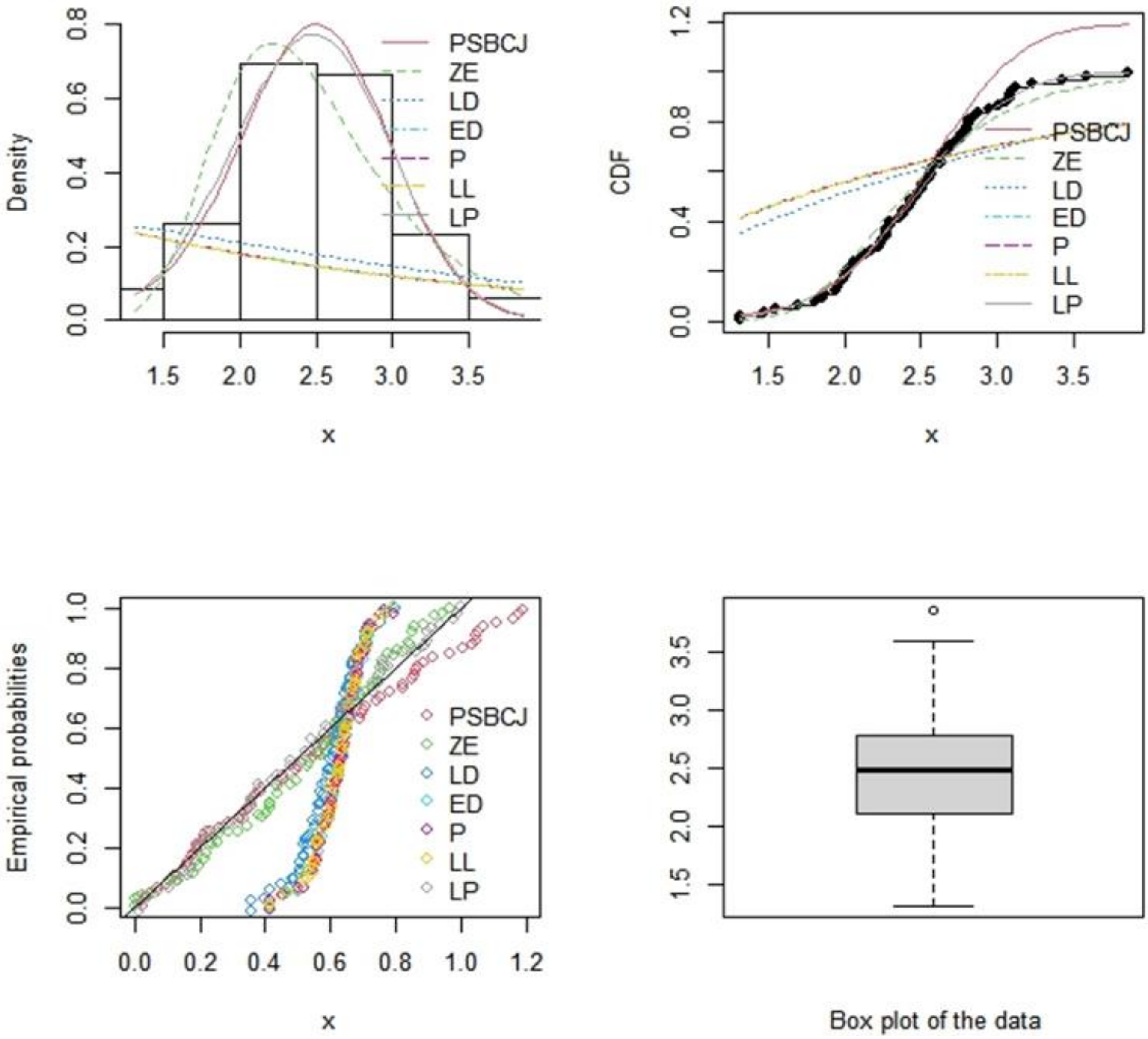

Fig 5 The estimated pdf, cdf, Kaplan-Meier and Box plots of the PSBCJ other fitted distributions using tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm

II. CONCLUSION

In this paper we have extended the one-parameter lifetime distribution named Chris-Jerry distribution. The extension is called Power Size Biased Chris-Jerry distribution having two parameters. We derived the r^{th} moment, some classical estimation procedures were implemented including bayesian estimation of the parameters. Application to tow real data sets namely the survival times of Guinea pigs injected with different amount of tubercle bacilli and data on tensile strength measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm. From the two analytical measures of fitness and performance (LL, AIC, CAIC, BIC, and K-S), the proposed PSBCJ distribution is preferred to the following fitted distributions, Marshall-Olkin Sujatha (MOS) distribution, Two-Parameter Lindley (TPL) distribution, Kumaraswamy-Weibull (KW) distribution, Zubair-Exponential (ZE) distribution, Lindley Distribution (LD), Exponential Distribution (ED), Pareto (P) distribution, Lindley-Lomax (LL) distribution and Lindley-Pareto (LP) distribution.

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