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A Comparison of Finite Difference Methods for a One-Dimensional Hyperbolic Equation with Nonlocal Boundary Conditions

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Abstract:- Many fields of physics and technology use hyperbolic partial differential equations pde with initial conditions as models. Recently, significant effort has been invested in investigating these equations, and they have attracted the curiosity of many mathematicians. In this paper, the finite difference method is used to provide the solution to the one-dimensional hyperbolic problem. The wave equation with the first dimension in space and time is taken as the boundary condition. The numerical results obtained from the examples of the Finite Differences Method formulated are compared with an analytical solution showing good results.

Keywords:- *Hyperbolic Partial Differential Equation, Finite Difference, Wave Equation, Implicit Scheme, Explicit Difference Method.*

I. INTRODUCTION

Many physical events, it has been clear in recent years, may be represented using hyperbolic partial differential equations with a traditional boundary condition [1]. Investigations of thermoelasticity [2,3], plasma physics [4], chemical heterogeneity [5, 6], and other fields use this kind of equation. The invention, analysis, and application of numerical approaches to the solution of these problems is receiving increasing attention. Several researchers have investigated hyperbolic problems in one dimension with initial boundary conditions [1, 7, 8, 9, 10, 11]. There are different types of numerical methods with their respective advantages and disadvantages. The method of finite differences appears to be the most used numerical method for approximating solutions to finite differential equations. These methods are widely used to solve time-dependent partial differential equations numerically. Furthermore, the truncation of terms resulted in errors when we approached derivatives using finite differential methods. The remainder of this article is structured as follows. The related literature review is discussed in Section 2. Section 3 introduces the problem's mathematical models. Section 4 discusses typical finite difference discretization of the wave equation. Section 5 uses numerical examples to show the precision and reliability of this methodology. The conclusion of our study is given in Section 6.

II. LITERATURE REVIEW

In the late 1950s, a Chinese physicist called Feng Kang created the finite difference. He introduced the finite difference as a systematic numerical approach for solving partial differential equations used in dam building calculations. The creation of the finite difference is today seen as a watershed moment in computer science [12]. Several scholars have contributed to the development of the finite difference for solving the wave equation. Oliveira [13] recently utilized the fourth order finite difference to solve the acoustic wave equation on irregular grids. Maupin and Dmowska [14] used the finite difference time domain technique to simulate the propagation of seismic waves. Lamoureux et al. [15] applied the Galerkin techniques to solve acoustic, elastic, and viscoelastic wave models. Chua and Stoffab [16] investigated the non-homogeneous grid implicit spatial finite difference approach for solving the acoustic wave equation. Lines et al. [17] investigated the computational stability of finite difference wave equations. Saarelma [18] used graphics rendering to estimate room acoustics using the finite difference time domain algorithm. The finite difference was used by Antunes et al. [19] to calculate the acoustic wave problem using spatially adjusted time steps. Moczo et al. [20] tested the effectiveness of the finite difference and fe methods in terms of the p-wave to s-wave velocity ratio. Dong et al. [21] solved a two-dimensional wave equation using the finite difference and fe techniques.

III. MATHEMATICAL MODELS

In this paper, we consider two typical model problems that involving a hyperbolic partial differential equation.

- *One-Way Wave Equation from First Order*

$$\begin{cases} \vartheta_t + \delta\vartheta_x = 0, \\ \vartheta(x,0) = \zeta(x), \\ \vartheta(0,t) = \phi_1(t) \quad i \quad \delta \geq 0, \\ \vartheta(1,t) = \phi_2(t) \quad if \quad \delta \leq 0. \end{cases} \quad (1)$$

- *Second-Order Linear Wave Equation*

$$\begin{cases} \vartheta_{tt} - \delta^2\vartheta_{xx} = 0, \\ \vartheta(x,0) = \zeta(x), \quad \vartheta_t(x,0) = \upsilon(x), \\ \vartheta(0,t) = \phi_1(t) \quad if \quad \delta \geq 0, \\ \vartheta(1,t) = \phi_2(t) \quad if \quad \delta \leq 0. \end{cases} \quad (2)$$

Where δ is a positive constant and $\phi_1(t)$, $\phi_2(t)$, $\zeta(x)$ and $\upsilon(x)$ are known functions. We assume that the functions $\phi_1(t)$, $\phi_2(t)$, $\zeta(x)$ and $\upsilon(x)$ satisfy the conditions in order that the solution of this equation exists and is unique. By observation, we can see that the exact solution of the problem (1) is:

ϑ(x,t)=ϑ_0 (x-δt) (3)

We may conclude numerous things from formula (3), First, at any time, the solution is a copy of the original function that has been shifted to the right if δ is positive and to the left if δ is negative with a value $|\delta|t_0$. Second, while problem (1) seems to make sense only if ϑ is differentiable, the solution equation (3) does not need ϑ_0 to be differentiable.

On the other hand, the analytical solution of the problem (2) is given by D'Alembert's formula [22] as follows:

$$\vartheta(x,t) = \frac{1}{2}\big(\vartheta_0(x-\delta t) + \vartheta_0(x+\delta t)\big) + \frac{1}{2\delta}\int_{x-\delta t}^{x+\delta t} \upsilon(s)ds \quad (4)$$

Consequently, if $\vartheta_t(x,0) = 0$, the solution is:

$$\vartheta(x,t) = \frac{1}{2}(\vartheta(x-\delta t,0) + \vartheta(x+\delta t,0)). \quad (5)$$

IV. THE FINITE DIFFERENCE METHOD

The primary concept behind finite difference approaches for solving partial differential equations is to approximate the derivatives occurring in the equation by a collection of function values at a specified number of points. Taylor series have been the most common method for generating these approximations. The methods introduced here are mainly based on analogous partial differential equations, which have been thoroughly detailed in[23]. This method provides a convenient measurement of the hypothetical order of accuracy, enabling the methods to be compared. It is also possible to eliminate the dominating error components associated with the finite-difference equations that include free parameters using the truncation error of the adjusted equivalent equation, producing more accurate approaches.

Our methodology, like some other numerical methods, starts with a partition of the domain of the independent variables x and t. The region $[0,\ell] \times [0,\Im]$ is divided into a $\lambda \times \mu$ mesh having spatial step size $h = 1/\lambda$ in direction x and time step size $k = \Im/\mu$. Throughout the content of this article, the symbols (x_i, t_n) will be used, where

$$x_i = ih, \quad i = 0,1,2,\dots,\lambda,$$

$$t_n = nk, \quad n = 0,1,2,\dots,\mu.$$

Where λ and μ are integers.

A. *One Way Wave Equation in Terms of Finite Difference Schemes*

In this subsection, the one-dimensional explicit numerical schemes with Lax-Friedrichs and Leapfrog schemes are presented to approximate the spatial and temporal partial derivatives of the advection equation (1).

- *The Lax-Friedrichs Scheme:*

Lax and Friedrichs presented a solution to the stability problem highlighted by the Forward-Time-Centered-Space FTCS scheme [22]. The fundamental concept is to replace the term ϑ_j^n in the FTCS formula $\vartheta_j^{n+1} = \vartheta_j^n - \frac{\alpha}{2}\left(\vartheta_{j+1}^n - \vartheta_{j-1}^n\right) + \mathcal{O}(\Delta t^2, \Delta x^2 \Delta t)$ with its spatial average, i.e., $\vartheta_j^n = \left(\vartheta_{j+1}^n + \vartheta_{j-1}^n\right)/2$, in order to obtain an advection equation.

$$\vartheta_j^{n+1} = \frac{1}{2}\left(\vartheta_{j+1}^n + \vartheta_{j-1}^n\right) - \frac{\alpha}{2}\left(\vartheta_{j+1}^n - \vartheta_{j-1}^n\right) + \mathcal{O}(\Delta x^2) \quad (6)$$

Where $\alpha = (\delta\frac{(\Delta t)}{(\Delta x)})^2$.

- *The Leapfrog Scheme:*

The FTCS and the Lax-Friedrichs schemes are both "one-level" schemes, with first-order approximation for the time derivative and second-order approximation for the spatial derivative. In certain cases, $\delta\Delta t$ should be considered to be substantially less than Δx, well below the Courant condition limit. Second-order accuracy in time can be obtained if we insert $\vartheta_j^n = \frac{\vartheta_j^{n+1} - \vartheta_j^{n-1}}{2\Delta t} + \mathcal{O}(\Delta t^2)$, in the FTCS scheme, to find the Leapfrog scheme:

$$\vartheta_j^{n+1} = \vartheta_j^{n-1} - \alpha\left(\vartheta_{j+1}^n - \vartheta_{j-1}^n\right) + \mathcal{O}(\Delta x^2) \quad (7)$$

B. *Second Order Wave Equation in Terms of Finite Difference Schemes*

In this section, two types of finite difference schemes for the problem (2) will be implemented: explicit and implicit schemes with constant velocity $\delta \neq 0$. It is assumed that the functions $\zeta(x)$ and $\upsilon(x)$ are assigned such that

$$\vartheta(x,0) = \zeta(x), \quad \vartheta_t(x,0) = \upsilon(x), \quad 0 \leq x \leq L.$$

➢ *Explicit Method*

First, a grid is defined by subdividing the interval $[0, L]$ into amplitude subintervals $\Delta x = L/(N+1)$ and defining the multiple time instants of a value Δt:

$$x_j = j\Delta x, \quad j = 0,1,2,\dots,N+1, \quad t_n = n\Delta t, \quad n = 0,1,2,\dots.$$

The second partial derivatives are approximated as follows:

$$\vartheta_{xx}(x_j,t_n) \simeq \frac{u_{j+1}^n - 2\vartheta_j^n + \vartheta_{j-1}^n}{(\Delta x)^2}, \quad \vartheta_{tt}(x_j,t_n) \simeq \frac{\vartheta_j^{n+1} - 2\vartheta_j^n + \vartheta_j^{n-1}}{(\Delta t)^2}$$

$$\frac{\vartheta_j^{n+1}-2\vartheta_j^n+\vartheta_j^{n-1}}{(\Delta t)^2} - \delta^2\frac{\vartheta_{j+1}^n-2\vartheta_j^n+\vartheta_{j-1}^n}{(\Delta x)^2} = 0 \qquad (8)$$

$$\vartheta_j^{n+1} - 2\vartheta_j^n + \vartheta_j^{n-1} = \delta^2\frac{(\Delta t)^2}{(\Delta x)^2}(\vartheta_{j+1}^n - 2\vartheta_j^n + \vartheta_{j-1}^n). \quad (9)$$

Using $\alpha = (\delta\frac{(\Delta t)}{(\Delta x)})^2$ in (9), we obtained the following:

$$\vartheta_j^{n+1} = 2\vartheta_j^n - \vartheta_j^{n-1} + \alpha(\vartheta_{j+1}^n - 2\vartheta_j^n + \vartheta_{j-1}^n), \quad j = 1,\dots,N. \qquad (10)$$

➢ *Implicit method*

To solve the wave equation, the second derivative of the spatial type can be discretized in a different way:

$$\vartheta_{xx}(x_j,t_n) \simeq \frac{1}{2}[\vartheta_{xx}(x_j,t_{n+1}) + \vartheta_{xx}(x_j,t_{n-1})] \qquad (11)$$

$$\vartheta_{tt}(x_j,t_n) \simeq \frac{\vartheta_j^{n+1} - 2\vartheta_j^n + \vartheta_j^{n-1}}{(\Delta t)^2}$$

$$\vartheta_{xx}(x_j,t_{n+1}) \simeq \frac{\vartheta_{j+1}^{n+1} - 2\vartheta_j^{n+1} + \vartheta_{j-1}^{n+1}}{(\Delta x)^2}$$

$$\vartheta_{xx}(x_j,t_{n-1}) \simeq \frac{\vartheta_{j+1}^{n-1} - 2\vartheta_j^{n-1} + \vartheta_{j-1}^{n-1}}{(\Delta x)^2}$$

$$\vartheta_{xx}(x_j,t_n) \simeq \frac{1}{2}[\frac{\vartheta_{j+1}^{n+1}-2\vartheta_j^{n+1}+\vartheta_{j-1}^{n+1}}{(\Delta x)^2} + \frac{\vartheta_{j+1}^{n-1}-2\vartheta_j^{n-1}+\vartheta_{j-1}^{n-1}}{(\Delta x)^2}]. \qquad (12)$$

By substituting the approximations into the partial derivative equation (11), we obtain:

$$\frac{\vartheta_j^{n+1} - 2\vartheta_j^n + \vartheta_j^{n-1}}{(\Delta t)^2} = \delta^2[\frac{\vartheta_{j+1}^{n+1} - 2\vartheta_j^{n+1} + \vartheta_{j-1}^{n+1}}{(\Delta x)^2} + \frac{\vartheta_{j+1}^{n-1}-2\vartheta_j^{n-1}+\vartheta_{j-1}^{n-1}}{(\Delta x)^2}]. \qquad (13)$$

Finally, we reach the final expression:

$$-\alpha\vartheta_{j-1}^{n+1} + (2\alpha+1)\vartheta_j^{n+1} - \alpha\vartheta_{j+1}^{n+1} = \alpha\vartheta_{j-1}^{n-1} + (2\alpha - 1)\vartheta_j^{n-1} + \alpha\vartheta_{j+1}^{n-1} + 2\vartheta_j^n. \qquad (14)$$

C. *Stability Criteria*

Consider the following finite difference approximation to the 1D wave equation:

$$\vartheta_j^{n+1} = \alpha^2\vartheta_{j+1}^n + 2(1-\alpha^2)\vartheta_j^n + \alpha^2\vartheta_{j-1}^n - \vartheta_j^{n-1} \qquad (15)$$

We will use the substitution $\vartheta_j^n = \chi_n e^{ij\Delta x\theta}$ into the Equation 15, we conclude:

$$e^{ij\Delta x\theta}\chi_{n+1} = (\alpha^2 e^{i\Delta x\theta} + 2(1-\alpha^2) + \alpha^2 e^{-i\Delta x\theta})e^{ij\Delta x\theta}\chi_n - e^{ij\Delta x\theta}\chi_{k-1} (16)$$

by using the double angle formula we have:

$$\chi_{n+1} = 2(1 + \alpha^2(\cos\Delta x\theta - 1))\chi_n - \chi_{n-1},$$

Thus

$$= 2\left(1 - 2\alpha^2\sin^2\frac{\Delta x\theta}{2}\right)\chi_n - \chi_{n-1} \quad (17)$$

If we now assume that has the following exponential form $\chi_n = G^n$ then Equation 17 reduces to the following quadratic equation:

$$G^2 - 2\gamma G + 1 = 0 \qquad (18)$$

where $\gamma = \left(1 - 2\chi^2\sin^2\frac{\Delta x\theta}{2}\right)$. The solutions of this quadratic equation are given by

$$G_{1,2} = \gamma \pm \sqrt{\gamma^2 - 1} \qquad (19)$$

Now, since G_1 and G_2 are the roots of this quadratic we may conclude that

$$(G - G_1)(G - G_2) = G^2 - (G_1 + G_2)G + G_1G_2 = 0 \quad (20)$$

Comparing the last terms in the two quadratic Equations (18) and (20) we conclude

$$G_1G_2 = 1 \qquad (21)$$

However, for the stability of solutions for the form $\chi_n = G^n$, we require that $|G_1| \le 1$ and $|G_2| \le 1$. Given the constraint (21), the only possibility, if the solutions are to be stable, is that $|G_1| = |G_2| = 1$. Thus G must fall on the unit disk, which implies that

$$|\gamma| \le 1$$

Thus,

$$\left|1 - 2\alpha^2\sin^2\frac{\Delta x\theta}{2}\right| \le 1$$

Or

$$-1 \leq 1 - 2\alpha^2 \sin^2 \tfrac{\Delta x\theta}{2} \leq 1 \qquad (22)$$

So that

$$-2 \leq -2\alpha^2 \sin^2 \tfrac{\Delta x\theta}{2} \leq 0 \quad (23)$$

The second inequality in (23) is satisfied automatically, while the first leads to the condition

$$\alpha^2 \sin^2 \frac{\Delta x\theta}{2} \leq 1$$

Since the maximum value that $\sin^2\left(\frac{\Delta x\theta}{2}\right)$ can achieve is 1 , we conclude that the condition for stability is

$$\alpha = (\delta \Delta t/\Delta x) \leq 1 \qquad (24)$$

V. NUMERICAL EXPERIMENTS AND RESULTS

To complement our theoretical explanation, we present some computational results of numerical experiments employing methodologies based on previous sections. Our approaches may be applied to more generic challenges based on the fundamental approach. However, making them efficient requires more extensive study with future ways anticipated. Since every finite difference formula is indeed a pde estimate, it would not yield the exact solution to such a pde. Rather, it provides a solution to a corresponding pde by performing a Taylor expansion of the terms in the finite difference equation around the (i, n) grid point. Furthermore, with this extension, all time derivatives can be written as spatial derivatives, providing a simplified pde. This method has been discovered to determine the stability of advective-diffusive pdes and develop new extremely accurate numerical methods for the same equation. By sequentially differentiating the relevant pde and applying the results to eliminate undesirable time derivatives, the improved formula is produced. The method's modified equivalent pde was described in earlier parts, and we will use it in the following instances.

- *Example 5.1 As the first example, consider the problem (1) with $\phi_1(t) = 0$, $\phi_2(t) = 0$ and unit function:*

$$\zeta(x) = \begin{cases} 0, & x < 0 \\ 1, & otherwise. \end{cases} \qquad (25)$$

The results for the finite difference schemes (10) and (14) are presented in Figures 1 and 2, respectively, using equation $\zeta(x)$ as an initial condition and input values $\Delta t = 0.02$ and $\Delta x = 0.04$. Figure 1. shows that Lax-Friedrichs generates good inosculating outcomes; although their shock widths are substantially longer, this means that it takes much longer to make the step. The Leapfrogs scheme generates oscillations between $x_0 - ct$ and $x_0 + ct$, the error determined at t's maximum value can be seen in Figure 5.

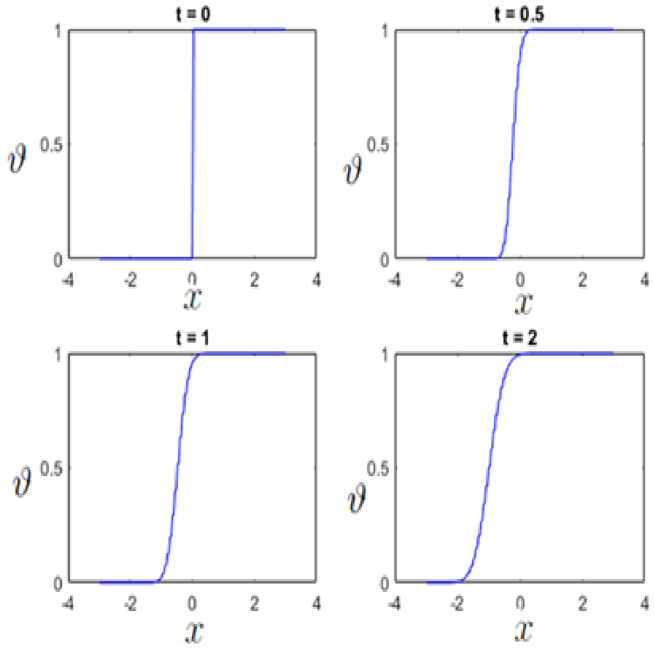

Fig 1 The Result of the Lax-Friedrichs Scheme Using A Unit Function as the Initial Condition.

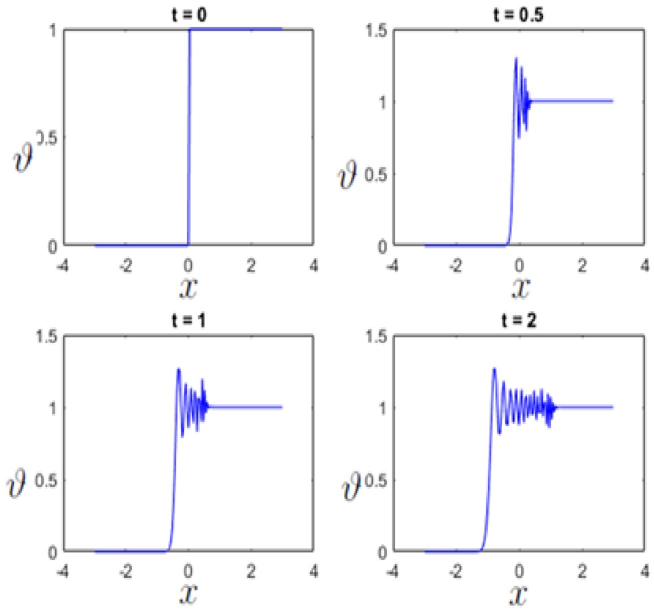

Fig 2 The Results of the Leapfrog Scheme using a unit function as the initial condition

- *Example 5.2 Consider the problem (1) with $\phi_1(t) = 0$, $\phi_2(t) = 0$ and*

$$\zeta(x) = \sin(\pi x) \qquad (26)$$

In the figures 3 and 4, the initial condition 26 produces some oscillations on the left side. Modeling is made easier by the smoothness of the presented condition. Because Lax-Friedrichs had a tendency to decrease or smooth out the wave's size, it caused slightly more error than The Leapfrog schemes. Generally, on this initial condition, almost both schemes gave pretty excellent results.

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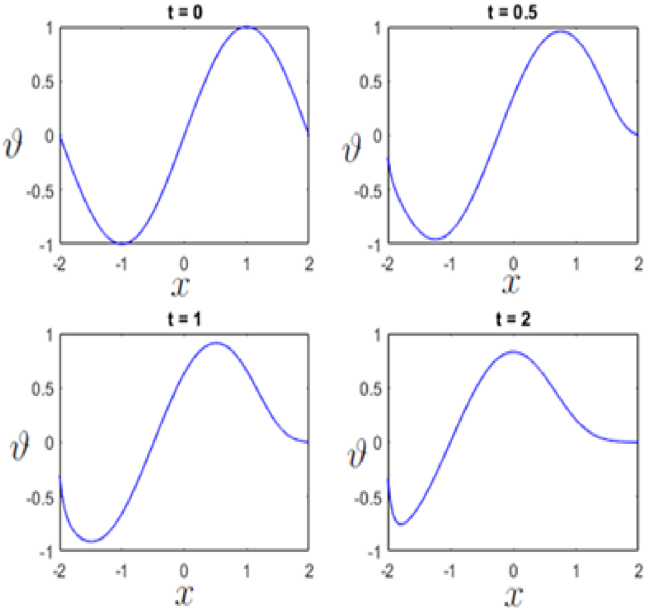

Fig 3 The Results of the Lax-Friedrichs Scheme using smooth sinusoidal function initial condition.

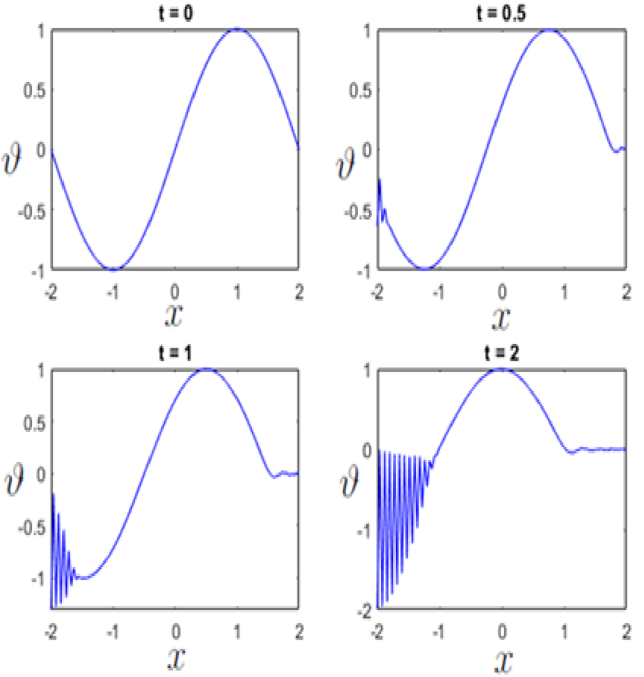

Fig 4 The results of the Leapfrog Scheme using smooth sinusoidal function initial condition.

➢ *Example 5.3 Use explicit and implicit methods to approximate the solution to the initial boundary value problem 2 as $\delta = 2$ with:*

$$\zeta(x) = \sin(2\pi x), \upsilon(x) = 0, 0 \le x \le 1$$

$$\phi_1(t) = 0,\ \phi_2(t) = 0, 0 < t < 0.5.$$

The exact solution of this equation is as follows.

$$\vartheta(x,t) = \cos 4\pi t \sin 2\pi x. \quad (27)$$

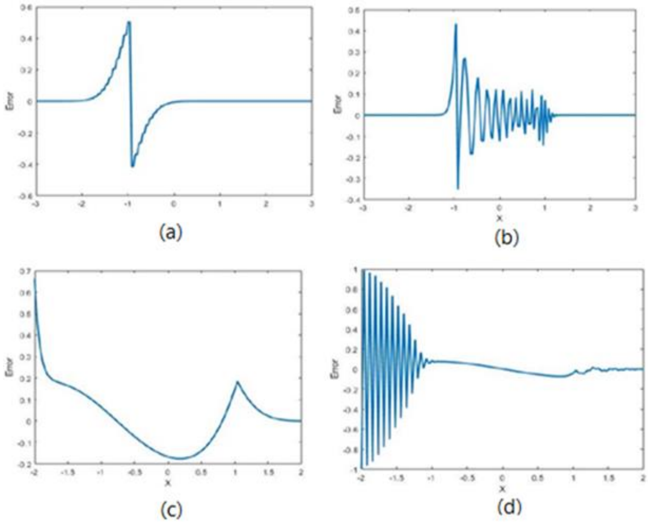

Fig 5 (a)-(d) represent the errors in the previous figures, respectively.

We may examine the absolute error of each finite difference method using the analytical solution equation (27). Table 1. discusses the results for ($h = 0.1$ and $h = 0.01$) using the explicit method provided in formula (10), while Table 2. shows the results for ($h = 0.1$ and $h = 0.01$) using the implicit method developed in formula (14). The absolute error was determined using the MATLAB software using the finite difference approaches mentioned. Both tables illustrate 2 runs, one being stable while the other is unstable. The explicit method solution is numerically better accurate than the implicit method, as seen in the tables 1 and 2.

➢ *Example 5.4 Use the explicit and implicit methodS to approximate the solution to the initial boundary value problem 2 as $\delta = 2$ with:*

$$\zeta(x) = \sin(\pi x) + \sin(2\pi x), \upsilon(x) = 0,\ 0 \le x \le 1$$

$$\phi_1(t) = 0,\ \phi_2(t) = 0,\ 0 < t < 0.5.$$

Tables 3 and 4 show how the computational results derived by comparable finite difference techniques have been compared with the analytical solution at various values of h, as well as the absolute error was reported at $t = 0.5$. As seen in the tables, the results produced using the explicit approach have less accuracy than those in table 1, however, the explicit scheme has better accuracy overall based on the involved schemes.

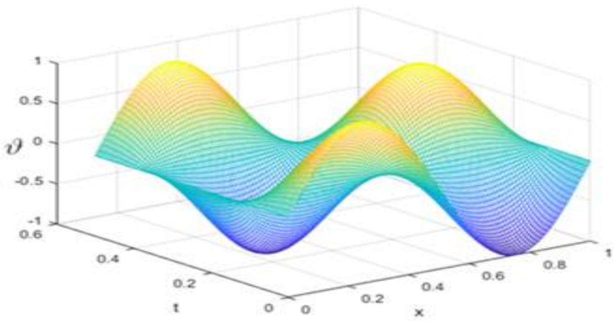

Fig 6 Plot of the exact solution in Example 5.3

Table 1 Results of explicit method

x	Exact value	Absolute error as h = 0.1	Absolute error as h = 0.01
0.00	0.00000000	0.0000e + 000	0.0000e + 000
0.10	0.58778525	1.6110e − 001	5.5430e − 004
0.20	0.95105652	2.6066e − 001	8.9687e − 004
0.30	0.95105652	2.6066e − 001	8.9687e − 004
0.40	0.58778525	1.6110e − 001	5.5430e − 004
0.50	0.00000000	6.3435e − 006	6.2617e − 016
0.60	-0.58778525	1.6109e − 001	5.5430e − 004
0.70	-0.95105652	2.6067e − 001	8.9687e − 004
0.80	-0.95105652	2.6066e − 001	8.9687e − 004
0.90	-0.58778525	1.6110e − 001	5.5430e − 004
1.00	-0.00000000	2.4493e − 016	2.44930e − 016

Table 2 Results of Implicit Method

x	Exact value	Absolute error as h = 0.01	Absolute error as h = 0.05
0.00	0.000000	0.0000e + 000	0.0000e + 000
0.10	0.58778525	2.4724e − 003	2.2237e − 001
0.20	0.95105652	4.0004e − 003	3.5980e − 001
0.30	0.95105652	4.0004e − 003	3.5980e − 001
0.40	0.58778525	2.4724e − 003	2.2237e − 001
0.50	0.00000000	3.6814e − 017	1.5509e − 016
0.60	-0.58778525	2.4724e − 003	2.2237e − 001
0.70	-0.95105652	4.0004e − 003	3.59800e − 001
0.80	-0.95105652	4.00040e − 003	3.5980e − 001
0.90	-0.58778525	2.4724e − 003	2.2237e − 001
1.00	0.000000	2.41930e − 016	2.4493e − 016

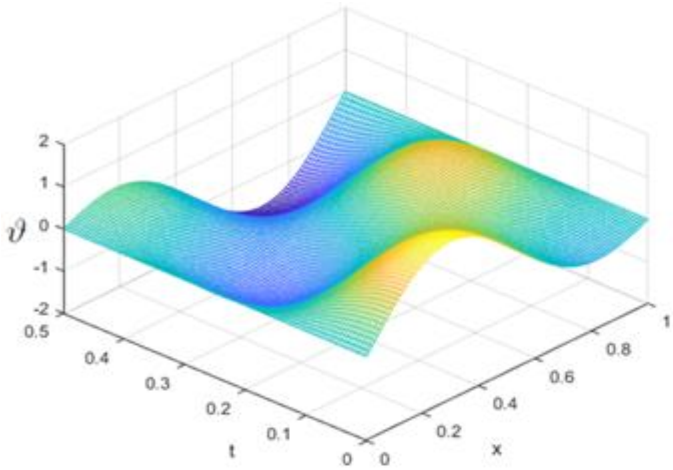

Fig 7 Plot of the exact solution in Example 5.4

Table 3 Results of Explicit Method

x	Exact value	Absolute error as h = 0.05	Absolute error as h = 0.1
0.00	0.00000000	0.0000e + 000	0.0000e + 000
0.10	0.27876826	5.5511e − 017	6.53726e − 002
0.20	0.36327126	1.1102e − 016	1.0566e − 001
0.30	0.14203952	2.2204e − 016	1.0538e − 001
0.40	-0.36327126	2.2204e − 016	6.4546e − 002
0.50	-1.00000000	5.5511e − 016	1.2857e − 003
0.60	-1.53884177	8.8818e − 016	6.6992e − 002
0.70	-1.76007351	2.2204e − 016	1.0746e − 001
0.80	-1.53884177	6.6613e − 016	1.0717e − 001
0.90	-0.89680225	2.2204e − 016	6.6166e − 002
1.00	-0.00000000	3.6739e − 016	3.6739e − 016

Table 4 Results of Implicit Method

x	Exact value	Absolute error as $h = 0.01$	Absolute error as $h = 0.1$
0.00	0.00000000	$\mathbf{0.0000e+000}$	$\mathbf{0.0000e+000}$
0.10	0.27876826	$\mathbf{6.0671e-004}$	$\mathbf{2.4646e-001}$
0.20	0.36327126	$\mathbf{9.8023e-004}$	$\mathbf{3.9761e-001}$
0.30	0.14203952	$\mathbf{9.7659e-004}$	$\mathbf{3.9468e-001}$
0.40	-0.36327126	$\mathbf{5.9613e-004}$	$\mathbf{2.3794e-001}$
0.50	-1.00000000	$\mathbf{1.6470e-005}$	$\mathbf{1.3263e-002}$
0.60	-1.53884177	$\mathbf{6.2746e-004}$	$\mathbf{2.6317e-001}$
0.70	-1.76007351	$\mathbf{1.0032e-003}$	$\mathbf{4.1614e-001}$
0.80	-1.53884177	$\mathbf{9.9959e-004}$	$\mathbf{4.1321e-001}$
0.90	-0.89680225	$\mathbf{6.1689e-004}$	$\mathbf{2.5466e-001}$
1.00	-0.00000000	$\mathbf{3.6739e-016}$	$\mathbf{3.6739e-016}$

VI. CONCLUSION

In this paper, some finite difference schemes for addressing a one-dimensional hyperbolic pde with given initial conditions and under boundary conditions have been proposed. For one-dimensional wave equations with specific conditions, these approaches are carried out effectively. Whenever h is small enough, the numerical tests performed using the methods provided in the article offer an accurate result and predict convergence to the exact solution. Unfortunately, the procedures described in this article are conditionally stable. Although the methodology is only demonstrated here for the one-dimensional case, it is possible to generalize to equivalent problems with just minor processing. Finally, we mention that one topic of future research will be to establish an equivalent approach for solving the two-dimensional wave equation.

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