ISSN No:-2456-2165

Integral Solutions of Quadratic Diophantine Equation

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Abstract:- The ternary quadratic Diophantine equation $M^2 + N^2 - 14M + 18N = 130(R^2 - 1)$ is analysed for its non-zero integral solutions.Different patterns of solutions are obtained by employing the methods of factorization, and cross multiplication.

I. INTRODUCTION

An equation that relates integer quantities is called a Diophantine equation. Modular arithmetic and Number Theory are closely related concepts while attempting to solve a Diophantine equation [2].The relation between the variables of an equation is often expressed in parametric form when a Diophantine equation has an unlimited number of solutions[1].The term "Diophantine" alludes to Diophantus of Alexandria, a third-century Hellenistic mathematician who studied these equations and was among the first to employ symbolism in algebra[3].Diophantus introduced the mathematical analysis of Diophantine problems, which is today known as Diophantine analysis.There are numerous varieties in Diophantine equations. Diophantine equations in three unknowns have been the focus of significant investigation, and its theory belongs to the most refined and advanced branches of mathematics. Despite this, some of its mysteries remain unknown to future generations of mathematicians.[4,5,6,7].This paper focusses on the positive integral solutions of the quadratic Diophantine equation :

$$M^2 + N^2 - 14M + 18N = 130(R^2 - 1).$$

II. METHOD OF ANALYSIS

The non-zero distinctive integer solutions of the ternary quadratic Diophantine problem weexamine in this paper is given by $M^2 + N^2 - 14M + 18N = 130(R^2 - 1)$ which can be written as

$$(M-7)^2 + (N-9)^2 = 130R^2 \cdots\cdots(1)$$

Introducing the transformations

$S = M - 7, T = N + 9 \cdots\cdots(2)$ reduces (1)to $S^2 + T^2 = 130R^2 \cdots\cdots(3)$. Different methods of finding (1) are presented below.

A. METHOD:1

The choice $S = 7\alpha + 9\beta, T = 9\alpha - 7\beta$ substituting in(3), reduces to

$R^2 = \alpha^2 + \beta^2$ representing the Pythagorean triple$\{\alpha, \beta, R\}$.

Hence$\{\alpha, \beta, R\} = \{k(a^2 - b^2), 2kab, k(a^2 + b^2)\}$for some integers $k, a$and b with$a > b$. Hence the solution of (1) is

$$\mathrm{M} = \mathrm{k}(7a^2 - 7b^2 + 18ab) + 7$$
$$\mathrm{N} = \mathrm{k}(9a^2 - 9b^2 - 14ab) - 9$$
$$\mathrm{R} = \mathrm{k}(a^2 + b^2)$$

- *NOTE:*

Let$S = 7\alpha - 9\beta$and $T = 9\alpha + 7\beta$

Proceeding as above we obtain,

$$\mathrm{M} = \mathrm{k}(7a^2 - 7b^2 - 18ab) + 7$$
$$\mathrm{N} = \mathrm{k}(9a^2 - 9b^2 + 14ab) - 9$$
$$\mathrm{R} = \mathrm{k}(a^2 + b^2)$$

B. METHOD:2

Consider the choice of 130 as

$130 = (9 + 7i)(9 - 7i) \cdots\cdots(4)$ and assuming the value of R as,

$R = a^2 + b^2 \cdots\cdots(5)$ where $a, b > 0$.

The substitution of (4) and(5) in (3) yields
$S + iT = (9 + 7i)(a + ib)^2 \cdots\cdots(6)$on equating the real and imaginary parts of (6) we get

$$S = 9(a^2 - b^2) - 14ab$$
$$T = 7(a^2 - b^2) + 18ab$$

From these values, we can find the non-zero integral solutions of (1) as

$$M = 9(a^2 - b^2) - 14ab + 7$$
$$\mathrm{N} = 7(a^2 - b^2) + 18ab - 9$$
$$\mathrm{R} = (a^2 + b^2)$$

- *NOTE:* Equation (4) can be also written in the following ways

Algebraic expression for 130	Solutions
$(7+9i)(7-9i)$	$M=7(a^2-b^2)-18ab+7$ $N=9(a^2-b^2)+14ab-9$ $R=(a^2+b^2)$
$(-9+7i)(-9-7i)$	$M=-9a^2+9b^2-14ab+7$ $N=-7a^2+7b^2+18ab-9$ $R=(a^2+b^2)$
$(-7+9i)(-7-9i)$	$M=-7a^2+7b^2-18ab+7$ $N=-9a^2+9b^2+14ab-9$ $R=(a^2+b^2)$

C. *METHOD:3*

Consider (3) as

$S^2-9^2T^2=7^2R^2-T^2 \cdots\cdots (7)$ which can be written in the ratio form

$$\frac{S+9R}{7R+T}=\frac{7R-T}{S-9R}=\frac{A}{B}$$

This equation is equivalent to the following two equations

$$-AS-BT+(9A+7B)R=0$$

$$BS-AT+(9B-7A)R=0$$

By the method of cross multiplication,we get

$$S=9A^2-9B^2+14AB$$
$$T=-7A^2+7B^2+18AB$$
$$R=(A^2+B^2)$$

From these values, we can find the non-zero integral solutions of (1) as

$$M=9A^2-9B^2+14AB+7$$
$$N=-7A^2+7B^2+18AB-9$$
$R=(A^2+B^2)$.

- *NOTE:* Following the above process,we get the different ratio form and integral solutions to (1) and are presented below:

Ration form as (3)	Solutions
$\frac{S-9R}{7R-T}=\frac{7R-T}{S+9R}=\frac{A}{B}$	$M=9A^2-9B^2-14AB+7$ $N=7A^2-7B^2+18AB-9$ $R=(-A^2-B^2)$.
$\frac{S-9R}{7R-T}=\frac{7R+T}{S+9R}=\frac{A}{B}$	$M=-9A^2+9B^2+14AB+7$ $N=7A^2-7B^2+18AB-9$ $R=(A^2+B^2)$.
$\frac{S+9R}{7R-T}=\frac{7R+T}{S-9R}=\frac{A}{B}$	$M=-9A^2+9B^2-14AB+7$ $N=-7A^2+7B^2+18AB-9$ $R=(-A^2-B^2)$.

D. *METHOD:4*

Equation (3) can be written as

$$(9^2+7^2)R^2-T^2=S^2.1 \cdots\cdots (*)$$

Assume $S=(9^2+7^2)a^2-b^2 \cdots\cdots (8)$Write 1 as

$1=\frac{\left(\sqrt{(9^2+7^2)}+9\right)\left(\sqrt{(9^2+7^2)}-9\right)}{7^2}\ldots\ldots(9)$

The substitution of (9) and (8) in (*) and applying the method of factorization

$$\left(\sqrt{(9^2+7^2)}a+b\right)^2\left(\frac{\sqrt{9^2+7^2}+9}{7}\right)=\sqrt{9^2+7^2}R+T$$

Equating the rational and irrational factors, we get:

$$T=\frac{1}{7}[260ab+1170a^2+9b^2]$$

$$R=\frac{1}{7}[18ab+130a^2+b^2]$$

$$S=(9^2+7^2)a^2-b^2$$

As finding integral solutions is what we are interested in, substitute a in the above equations to 7A and b to 7B. We can determine non-zero integral solutions of (1) from these values as

$$M=6370A^2-49B^2+7$$
$$N=8190A^2+63B^2+1820AB-9$$
$$R=910A^2+7B^2+126AB$$

➢ *NOTE:* A few other solutions obtained from various expressions for 1 are presented in the table below:

Algebraic expression for 1	Solutions
$\frac{1}{7^2}\left[\left(-\sqrt{(9^2+7^2)}+9\right)\left(-\sqrt{(9^2+7^2)}-9\right)\right]$	$M=6370A^2-49B^2+7$ $N=-8190A^2-63B^2+1820AB-9$ $R=-910A^2-7B^2+126AB$
$\frac{1}{9^2}\left[\left(\sqrt{(9^2+7^2)}+7\right)\left(\sqrt{(9^2+7^2)}-7\right)\right]$	$M=10530A^2-81B^2+7$ $N=8190A^2+63B^2+2340AB-9$ $R=1170A^2+9B^2+126AB$
$\frac{1}{9^2}\left[\left(-\sqrt{(9^2+7^2)}+7\right)\left(-\sqrt{(9^2+7^2)}-7\right)\right]$	$M=10530A^2-81B^2+7$ $N=-8190A^2-63B^2+2340AB-9$ $R=1170A^2-9B^2+126AB$

III. CONCLUSION

Our goal is to examine various Diophantine equations and solving them to obtain integer solutions. This paper outlines four different methods to determine the solvability of Diophantine equations.

REFERENCES

[1]. Andre Weil, Number Theory: An Approach through History, FromHammurapito to Legendre, Bikahsuser, Boston, 1987.
[2]. Manju Somanath, K. Raja, J.Kannanand M. Mahalakshmi, " On A Class Of Solutions For A Quadratic Diophantine Equation", Advances and Applications in Mathematical Sciences, 19 (11) (2020), pp.1097-1103.
[3]. M.A. Gopalan, J. Kannan, Manju Somanath AndK. Raja , "Integral Solutions of an Infinite Elliptic Cone", International Journal of Innovative Research in Science, Engineering and Technology, (IJIRSET), 5(10) (2016),pp. 17551-17557.
[4]. Manju Somanath,T.Anupreethi and V.Sangeetha, "Lattice Points for the Quadratic Diophantine Equation21(x^2+y^2)-19xy=84z^2", International Journal ofInnovative Research science, Engineering andTechnology(IJIRSET) Volume 11,Issue 5, May2022pp.5542-5549.
[5]. Manju Somanath, T. Anupreethiand V. Sangeetha, "Integral Solutions of Quadratic Diophantine Equation with two unknowns11(θ^2 + Ω^2) = 2(120Ω − 1)",Research and Reflections on Education ISSN 0974-648X(P) Vol. 20 No. 3A October 2022 pp. 108- 111.
[6]. Manju Somanath and J. Kannan, "Congruum Problem", International Journal of Pure and Applied Mathematical Sciences (IJPAMS), 9(2) (2016), pp.123-131.
[7]. Manju Somanath, J. Kannan and K. Raja, "On the Positive Integer Solutions for a Diophantine Equation", Journal of Mathematics and Informatics, 10 (2017), pp.173-177.