The Characteristics of the Motion of a Spinning Flying Disc

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Abstract:- This paper is an investigation into the motion of a flying disc. It starts out by considering the basic concepts of force and pressure acting on the disc during flight and uses intuition to derive differential equations for the motion. It then goes on to consider a variety of factors that may affect the motion of the disc through the air. The aim of the paper is to find a meaningful connection between how uncertain quantities and variations can qualitatively affect the motion of the disc, making use of mathematical techniques along the way to prove proportions and relations.

Keywords:- *Differential Equations, Drag, Fluid Dynamics, Lift, Projectile Motion*

I. BACKGROUND AND QUICK NOTES

This paper is possible because of the contributions made to the fields of Physics and Mathematics by pioneers like Isaac Newton, Ludwig Boltzmann, and James Clerk Maxwell. It was written with the aim of breaking the individual parts and probabilities apart, understanding their qualitative properties, identifying the outcomes of an in-deterministic system and the reasoning behind it.

This paper, most importantly, has been made possible by the guidance and support of my mentor Prof. Soumya Bhattacharya. He has not only continually helped me through my learning process, teaching me advanced trigonometry, calculus, linear algebra, and other aerodynamics related concepts, but also cultivated and nurtured my interest in physics and mathematics over multiple years. This paper is, therefore, a product of the contributions he has made to my growth as a learner and hopeful researcher. It is only through discussions and the critical analysis of my own writing, with him, have I been able to reach the reasoned ideas and conclusions that I have.

The field of study used to analyze the motion of an object starts out as a kinematic issue. Using integration one can determine the equations of motion for an object undergoing ideal projectile motion. It is the purpose of this paper to try and add another layer of realism into the approach considering the lift and drag experienced by the object as well, transitioning into physics.

In no way does this paper, however, contain the most realistic model possible. All across certain assumptions have been made. Those assumptions do not tend to hold true in realistic conditions but must be made. If not, the problem of statistical dynamics and statistical thermodynamics arises. The scope transitions from an analysis of external factors to an analysis of internal variations. I have, as far as possible highlighted the assumptions where they are made, to improve clarity.

Throughout the paper multiple constants are defined whereas some other constant quantities are not included with the definitions of those constants. That is because at a later stage all variables left in the main equation will be used to simulate changing conditions.

I attempt to make use of as novel and original of an approach as possible. I have referred to a few sources for formulae and values for certain constants which have been cited appropriately.

II. DERIVATION FOR THE EQUATIONS OF MOTION

➢ *Simple Projectile Motion*

Simple projectile motion is something that students study as their first interaction with kinematics. It is commonly portrayed as the parametric solution to the two equations below (where $g = 9.81\text{ms}^{-2}$ representing the acceleration due to gravity) taking into account the initial conditions: u (initial velocity) and θ (angle of launch).

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$

The parametric solution is given by the equations below where t stands for the time and T stands for the time period/time of flight.

$$\begin{cases} s_x = ut\cos\theta \\ s_y = ut\sin\theta - \frac{1}{2}gt^2 \end{cases} \quad t \in [0, T]$$

However, as the complexity of motion increases, it becomes more and more difficult to define the underlying "differential equations" as shown above because every instance of motion is affected by the one before it. More formally, the position s at time $(\text{t} + \Delta\text{t})$ is affected not only explicitly by t but also by the position and forces on the object at time t. Therefore, we must switch techniques to consider forces acting on the object as a starting point rather than directly determining the equations of motion.

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➢ *Pressure Forces*

Pressure Forces are forces induced on the object due to differences in pressure distribution around the body. Most objects tend to have complex shapes such as rockets, while flying disks have rather simple shapes, however, in all cases pressure forces act normal to the surface of the object at that point. This can be denoted by using a normal unit vector that changes across the surface of the flying disc, denoted, $\hat{n}$. The common formula for pressure can then be used to derive an equation for force, as shown below.

$$P = \frac{F}{A}$$

$$\vec{F} = \sum (P_i \hat{n} \cdot A_i)$$

To find the actual force, we must take a Closed Surface Integral across every sliver of area and the force acting there to finally end up with the equation below. This is because for an object in motion through a fluid, the local velocity of the fluid about that point changes across the surface, as a result of which, the pressure also varies continuously, thus warranting the use of an integral; specifically, a closed surface integral.

$$\vec{F} = \oiint_S P \cdot \hat{n} \, d\Sigma \tag{1.1}$$

Here $d\Sigma$ represents the tiniest slice of area on the surface. The reason $d\Sigma$ was used and not dA to represent area is because $dA = dxdy$. This means that dA represents a two-dimensional square slice, whereas we require a curved slice following the surface S which in this case represents the surface of the flying disc. $d\Sigma$ is used to emphasize this difference. A closed integral is used because the disc can be considered a close surface that encloses a volume similar to real-life rather than a curved surface with infinitesimal thickness. Simply writing $\oiint_S d\Sigma$ would represent the surface area. Therefore $d\Sigma$ must be multiplied by the pressure vector at each position. Pressure, however, is a scalar and must be assigned a direction, in this case, $\hat{n}$. $\hat{n}$ represents the normal to the surface at any point on the surface and its definition therefore changes with position.

To evaluate this integral however, one would have to know the pressure distribution across the entire surface, and thus by extension the velocity distribution of fluid flow. When considering the actual complexity of the flow and various other factors such as turbulence in lower pressure zones and separation of airflow, it makes it almost impossible to calculate.

The first step it to isolate the net force, $\vec{F}$, into two perpendicular components: lift, which acts perpendicular to the direction of movement (denoted here by $\hat{v}$) and drag, which acts in the $\hat{v}$ direction. For most practical use cases, the object of research is set up in a wind tunnel and the lift and drag forces are measured using sensors. For the purposes of this paper, there is no access to a wind tunnel so certain approximations must be made, facilitated by previous research in the field in the form of lift and drag coefficients (C_l and C_d respectively). These coefficients are, intuitively speaking, measures of how well an object generates lift and drag. A flying disc has a streamlined shape and therefore has a low C_d because it generates less drag, but a higher C_l because it generates greater lift. Both values vary greatly with angle of attack (α). Note that the angle of attack is defined not as the angle the disc forms with the horizontal but rather the angle between the direction of motion ($\hat{v}$, refer to Equations 1.4 for more detail) and the orientation of the disc. A greater angle of attack will create larger differences in pressure across the surface. As these differences increase, the magnitudes of both "expressions" of the pressure force (lift and drag) also increase. The exact variations for these coefficients are given by the equations below.

$$C_d = C_{d0} + C_{d\alpha}\alpha^2$$

$$C_l = C_{l0} + C_{l\alpha}$$

Here the 0 designated constants are those for $\alpha = 0\,\text{rad}$. The α designated constants represent the change in magnitude for every radian change in angle of attack. These values are empirically found to be $C_{d0} = 0.15, C_{d\alpha} = 1.24, C_{l0} = 0.188, C_{l\alpha} = 2.37$. Note that the drag coefficient varies with the square of the angle of attack. Also note that for the remainder of this paper α is considered to be 0 unless otherwise specified because the orientation of the disc is considered to be coinciding with the direction of motion. [6]

The changes in these coefficients will be discussed in greater detail further in the paper. The formulae using these coefficients are as given:

$F_d = \frac{1}{2} C_d \rho A_r v^2$ where A_r is reference area (1.2)

$F_l = \frac{1}{2} C_l \rho A_w v^2$ where A_w is wing area (1.3)

➢ *Resolution of Forces*

There are, at this point, three main forces acting on the flying disc: weight, lift and drag. All these three forces must be combined together. Therefore, the net force ($\overrightarrow{F_{net}}$) is given by the equation below.

$$\overrightarrow{F_{net}} = F_d + F_l + W$$

where W is the weight of the disc

Before resolving the vectors, though, it is important to define directions and what the unit vectors represent in terms of a 2-dimensional coordinate system in x and y each represented by unit vectors $\hat{x}$ and $\hat{y}$ rather than the standard $\hat{\imath}$ and $\hat{\jmath}$ for convenience purposes.

$$\hat{v} = \frac{\dot{x}\hat{x} + \dot{y}\hat{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \qquad \hat{l} = \frac{-\dot{y}\hat{x} + \dot{x}\hat{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \tag{1.4}$$

Here $\hat{v}$ represents the unit vector in the direction of movement of the disc. It is co-linear with but exactly opposite to the direction of drag. $\hat{l}$ represents the direction perpendicular to $\hat{v}$ and the direction of lift.

To simplify Equation 1.2 and Equation 1.3, quantities that remain constant can be condensed and velocity can be converted into its vector form. Equation 1.2 and Equation 1.3 can therefore be rewritten as Equation 1.5 and Equation 1.6.

$$F_d = -mk_dC_d|\vec{v}||\vec{v}|\hat{v} \qquad \text{where } k_d = \frac{1}{2}\rho A_r\frac{1}{m} \qquad (1.5)$$

$$F_l = mk_lC_l|\vec{v}||\vec{v}|\hat{l} \qquad \text{where } k_l = \frac{1}{2}\rho A_w\frac{1}{m} \qquad (1.6)$$

It is important to note that here the assumption is made that all quantities used to define k_d and k_l will remain constant throughout the entire duration of motion. This is false. Take the density (ρ) for example. The density of the fluid (air in this case) varies not only with altitude but also due to variations in wind patterns. Wind can be considered to have both a laminar and turbulent nature at the same time within different boundaries. The borders between the boundaries will tend to have changes in wind speed and therefore, in terms of a vector field, acts as a relative sink due to the curl of the field representing wind. Similarly, vortexes may create regions of lower density. We cannot therefore safely assume the disc to be a point mass as different regions of the disc will experience different forces due to the above variations.

Another important note is that there is no lift induced drag considered in this case. Lift-induced drag is a phenomenon that occurs when the angle of attack (α) relative to the direction of movement is not 0 rad as the lift acts normal to the orientation but not to the direction of movement. Depending on the angle of attack, there is a component of the lift vector acting opposite to the direction of movement. The magnitude of this force (F_{ld}) is given by.

$$F_{ld} = F_l \sin\alpha$$

The lift-induced drag is not considered because α is considered to be 0 rad. Therefore, there is no component of the lift in the plane of motion.

Combining these equations together we get the final equation for net force as shown.

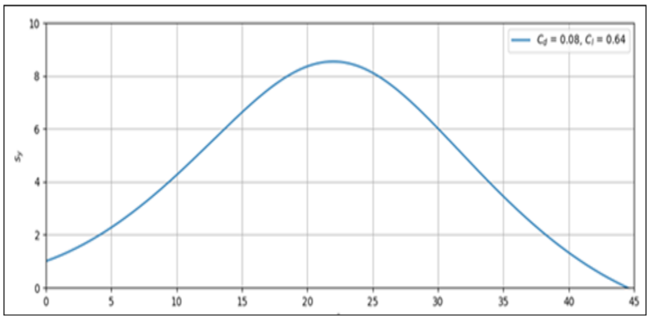

Fig 1 An Exaggerated Display of the Characteristics of the Equation Provided when Solving the System of Differential Equations above.

$$\begin{aligned}\vec{F}_{net} &= -mk_dC_d|\vec{v}||\vec{v}|\hat{v} - mk_lC_l|\vec{v}||\vec{v}|\hat{l} - mg\hat{y} \\ &= m(-k_dC_d\sqrt{\dot{x}^2+\dot{y}^2}\cdot(\dot{x}\hat{x}+\dot{y}\hat{y}) \\ &\quad - k_lC_l\sqrt{\dot{x}^2+\dot{y}^2}\cdot(\dot{y}\hat{x}-\dot{x}\hat{y}) - g\hat{y}) \qquad (1.7)\end{aligned}$$

Using F = ma, we can calculate the acceleration experienced by mass m, as shown below.

$$\begin{aligned}\vec{a} &= \frac{\vec{F}_{net}}{m} \\ &= \begin{bmatrix}\vec{a}\cdot\hat{x} \\ \vec{a}\cdot\hat{y}\end{bmatrix} \\ &= \begin{bmatrix}-k_dC_d\sqrt{\dot{x}^2+\dot{y}^2}\dot{x} - k_lC_l\sqrt{\dot{x}^2+\dot{y}^2}\dot{y} \\ -k_dC_d\sqrt{\dot{x}^2+\dot{y}^2}\dot{y} + k_lC_l\sqrt{\dot{x}^2+\dot{y}^2}\dot{x} - g\end{bmatrix}\end{aligned}$$

This yields the final equation for acceleration as shown below.

$$\vec{a} = \begin{bmatrix}\ddot{x} \\ \ddot{y}\end{bmatrix} = \sqrt{\dot{x}^2+\dot{y}^2}\begin{bmatrix}-k_dC_d\dot{x} - k_lC_l\dot{y} \\ -k_dC_d\dot{y} + k_lC_l\dot{x} - \frac{g}{\sqrt{\dot{x}^2+\dot{y}^2}}\end{bmatrix} \qquad (1.8)$$

The characteristic of the graph solving the differential equations provided seems to follow the logical path. Figure 1 below demonstrates this general shape. Note that the shape of the graph has been achieved by a $\frac{C_l}{C_d}$ of 25 to exaggerate the characteristics. Under normal conditions, the graphs look similar to the one of simple projectile motion as shown in Figure 2

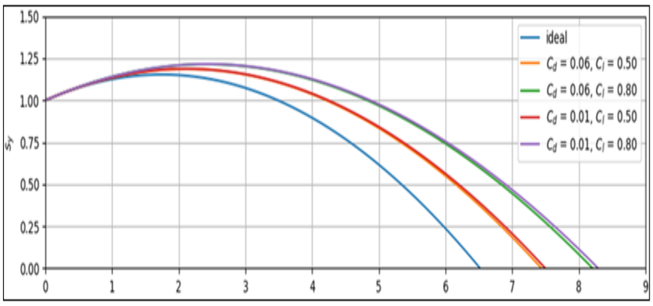

Fig 2 Difference between Simple Projectile Motion and that of the Equation Derived above using a Combination of Drag and Lift Coefficients for Demonstration. Note that Logical Ranges for these Coefficients are from [6].

The graphs above serve the purpose of visually expressing the difference between the two models (simple projectile motion and that expressed in Equation 1.8). Note importantly that all values have conditions expressed below.

$$u = 10ms^{-1}$$

$$\theta = 10°$$

$$k_d, k_l \approxeq 0.03657$$

$$y_0 = 1m$$

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Table 1 shows that for an about 25 percent change in the Drag Coefficient there is no major change in range. This is not however the case for the Lift coefficient because it has a higher absolute value. Further analysis of values in the table reveals some interesting information. There seems to be a linear correlation between the Range (r) and the C_d. Specifically, as the C_d increases by 0.02, r seems to decrease by 0.02 ~ 0.03. When looking at the later decimal places, one can realize that the difference is changing with the C_l. Through mathematical manipulation, it seems to be that the range can be modeled by Equation 1.9 below by creating a linear relation between r and C_d, where the gradient (and therefore the y-intercept) varies with C_l.

$$r = m(C_l)C_d + c(C_l) \tag{1.9}$$

Here the functions $m(C_l)$ and $c(C_l)$ are defined in such an implicit manner because within the scope of my evaluation, there is no linear, polynomial, exponential, or logarithmic correlation that defines the function. Therefore, within the scope of the paper we will consider quadratic regression for $m(C_l)$ and $c(C_l)$ which is fairly accurate with $R^2 = 0.9997$. Thus, both functions can be defined as below (values accurate to six significant figures). A trick is to take the functions as having discrete definitions for specified values but that doesn't really work unless one wants to replicate values with exactly the same conditions as those in the table.

Table 1 The table contains data relevant to but in more detail than Figure 2 for a clearer interpretation of trends and differences. The data has been derived from a computerized differential equation program written by me (using python and the SciPy and NumPy libraries), by finding the x-intercepts of the trajectories.

Table 1 Contains Data Relevant

C_d/C_l	0.50	0.60	0.70	0.80
0.06	7.44	7.68	7.92	8.21
0.08	7.42	7.66	7.89	7.18
0.10	7.40	7.64	7.87	7.16

$$m(x) = -0.406071x^2 - 0.14235x - 0.831949 \tag{1.10}$$
$$c(x) = 1.24521x^2 + 0.93857x + 6.72608 \tag{1.11}$$

It is important to note that these functions are only valid for the values within the constraints of the table and some values outside it (valid for C_d up to 0.12 and C_l up to 0.90).

Due to statistical inaccuracies, the predicted value of range from the defined functions is off by a maximum of 0.01 m which amounts to a maximum error (in the worst-case scenario within the data) of 0.0954%.

III. VARIATIONS DUE TO EXTERNAL CONDITIONS AND ROTATIONAL MOTION

➤ *Reynolds Number and Knudsen Number*

The Reynolds Number (Re) and the Knudsen Number (K_n) are two important quantities that have major effects on the C_d and C_l values because they define the flow regimes around the object, in this case a rotating flying disc.

The Reynolds Number is an expression of how laminar or turbulent the flow around an object is. It is technically defined as the ratio between inertial and viscous force. It can be calculated for a particular system/situation using the formula below.

$$Re = \frac{v\rho L}{\eta} \tag{2.1}$$

where ρ is fluid density
v is fluid velocity
L is hydraulic diameter
η is fluid viscosity

Re is an important concept to consider because turbulence has multiple effects on lift and drag. In a high viscosity fluid flow, because the vertical pressure distribution is almost completely symmetrical, there is therefore no lift on the object. As the Re increases, flow becomes more turbulent. This creates vortices and pressure differentials. These differences in pressure induce lift and drag as discussed in Section 1.2. The transition Reynolds Number is the set of Reynolds Numbers where the flow is neither laminar nor turbulent. This range varies mainly depending on the fluid, the environment in which the experiment is carried out and the relative velocity of the object in the fluid-stream. It therefore follows that as the velocity of the disc increases, the Reynolds Number must also increase.

The change in flow characteristics and Reynolds Number can be shown through a diagram (Figure 3) that represents the flow visually, but also the how the fluid velocity changes with distance from the surface. [7]

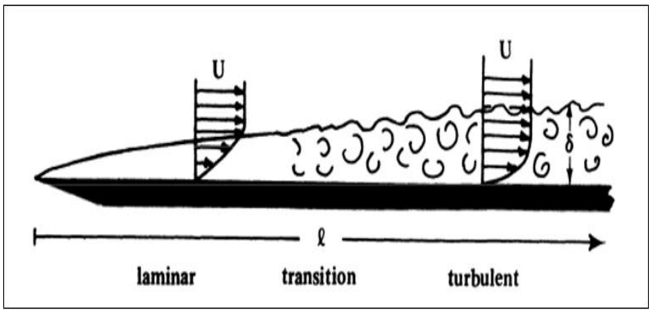

Fig 3 Transition between laminar and turbulent flow along with the relevant fluid velocity changes relative to the initial fluid velocity u_∞. This specific notation can also be interpreted as the fluid velocity at infinite distance from the surface, meaning the surface has no effect on it.

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In the case of a curved surface however, flow separation will occur. The distance and angle at which this will occur from the center of mass of the disc will vary based on the Re for the system. Figure 4 [2] clearly shows how flow separation occurs from turbulent flow. Because the surface curves away from the airflow, the profile of the velocity distribution changes. The fluid velocity vector pointing in the opposite direction represents the flow has been separated from the curved surface. This back flow induces vortices, thus creating a low-pressure zone behind the curved surface. This low-pressure zone is what generates the drag and turbulence.

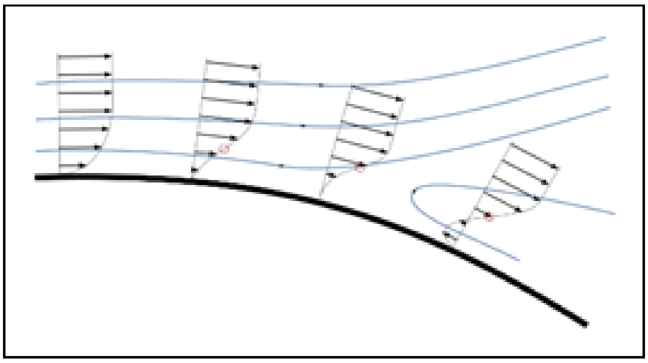

Fig 4 Flow Separation in the Boundary Layer with the Relevant Velocity Vector Distribution.

The relations between the Re and the C_d and C_l are therefore based on these concepts. As the Re increases, C_d decreases because there is less resistance to the motion of the object in turbulent conditions than there is in laminar conditions. The vortices formed in turbulent flow are localized and can easily be disrupted and therefore there is lesser drag. In this case the lift force generated also decreases as the pressure differential decreases. This logic is especially apparent in Formula 1 cars, where they make use of a "slipstream" behind other cars, to reduce downforce and drag on the straights. The same situation can cause the car to lose downforce in the corners and spin off the track. In the case of a disc, it performs the best in laminar conditions so as to generate the maximum lift to drag ratio.

Knudsen Number is given using the formula below. [4]

$$K_n = \frac{\lambda}{L}$$

where $\lambda = \frac{\eta}{\rho}\sqrt{\frac{\pi}{2RT}}$ is molecular mean free path

$R = 8.31\ Jmol^{-1}K^{-1}$ is universal gas constant

T is absolute temperature

L is characteristic length

$$\therefore\ K_n = \frac{\eta}{\rho L}\sqrt{\frac{\pi}{2RT}} \tag{2.2}$$

K_n is analogous to the molecular density. The molecular mean free path (λ) is average distance a particle travels before colliding with another particle. In this case, a higher Knudsen Number represents a greater molecular density. If the K_n is low, then it approaches an ideal condition of continuum flow. However, as the value of K_n increases, the mean free length plays a much greater role in the overall dynamics of the system because of its relative magnitude compared to the L value assigned in the system. The threshold $K_n = 0.01$ is assigned as the maximum Knudsen Number value for continuum flow. After which it is divided into three categories: slip flow ($0.01 < K_n < 0.1$), transition flow ($0.1 < K_n < 10$), and free molecule flow ($K_n > 10$). [3]

The idea to include the Knudsen Number came from a microscopic analysis of the disc interacting with the airflow and the boundary layer. The random motion of air molecules means that they tend to collide with the disc in all direction. This causes them to rebound into the other air molecules in the layer behind it. This is more apparent in laminar flow because the layers of molecules are more defined. This collision momentarily increases the density of air around the disc. The greater density increases the pressure exerted by the airflow on the disc all round which increases the magnitude of forces experienced by the disc as shown in Equation 1.1.

The concept expressed above is close, in form, to Langevin Dynamics. It represents the set of techniques used to model molecular systems. It makes use of a concept called Random Walk, which is the "process for determining the probable location of a point subject to random motions, given the probabilities (the same at each step) of moving some distance in some direction." [1]. Langevin found the approximate (Root Mean Square Value) distance a particle could move in a given direction x ($\overline{\Delta_x}$) [5]. It is not possible for us to consider such motion with the same level of mathematical rigor here, but similarly to how pressure forces were approximated using C_d and C_l, we try to approximate the effects of the above happenings using Re and K_n.

Considering the Equations 2.1 and 2.2, we can see the correspondence of quantities. Rearranging 2.1, we get

$$\frac{\eta}{\rho L} = \frac{v}{Re}$$

Which can be substituted into 2.2 to give

$$K_n = \frac{v}{Re}\sqrt{\frac{\pi}{2RT}}$$

And then further rearranged to give the final equation, considering $K_n Re = \phi$ as shown below.

$$\phi = v\sqrt{\frac{\pi}{2RT}} \tag{2.3}$$

This equation isolates constants that represent the nature of a closed system and provide a representation of their product in terms of the relative velocity of the object in the fluid. Therefore, this formula can be used to determine minute variations in fluid conditions with velocity. To further determine the nature of the proportionality, we can use dimensional analysis. This process yields the below proportion.

$$\phi \propto \sqrt{\frac{n}{m}}$$

where n is the moles of fluid within the system

where m is the mass of the fluid

This can be further simplified to

$$\phi \propto \sqrt{\frac{1}{M_r}}$$

where M_r is the molecular mass of the fluid

Note that in the case of mixtures like air, one can consider the M_r to be the mean for that specific mixture in the closed system. This is also obviously not true given the molecular mass of the fluid cannot change as the disc moves through it. Since these results yielded from dimensional analysis, one must be able to derive a more logical conclusion from the appearance of the molecular mass which, therefore, may not have the same mathematical rigor as other quantities.

➤ *Stokes' Law*

The thought behind Stokes' Law was to find a more accurate representation for the drag force experienced by an object making use of the viscosity, dimensions of the object and its velocity in the fluid. The equation given by Stokes' Law for a sphere is given below.

$$F_d = 6\pi r\eta v \tag{2.4}$$

An important condition for the equation to be valid is relating to the Reynolds Number of the system. Since this is only true for purely laminar and high viscosity fluids, $Re \ll 1$. This is clearly shown in the Figure 2.3 [7] below where the linear characteristic is only maintained closely for values $Re < 1$. Another thing to note is that the constant, 6π, is a representation of the 3D circular symmetry of the sphere used to derive the equation.

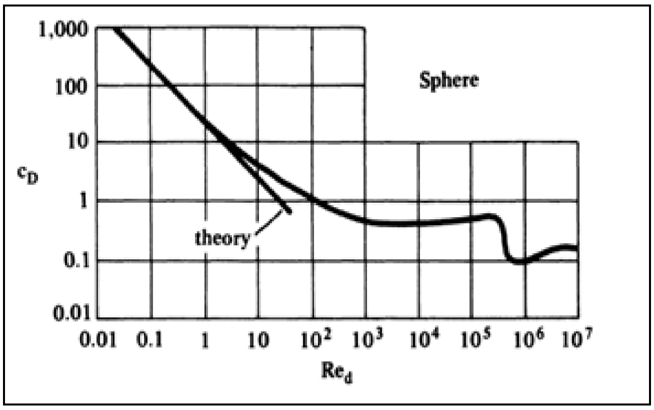

Fig 5 The Experimentally Derived Values of Reynolds Number and Corresponding Drag Coefficient Readings on a Logarithmic Scale.

It is also important to note that the coefficient (6π) is only valid for a sphere and therefore a separate coefficient must be derived for a disc like object in 3D rather than 2D. The derivation itself is rather complex involving vector fields, divergence, curl and ∇ and ∇^2 operators. It uses the main assumptions, utilizing pressure and velocity fields, as shown below.

$$\nabla p = \eta \nabla^2 v$$

$$\nabla \cdot v = 0$$

It follows that the gradient of the pressure surface is proportional to the Laplacian of the velocity field (which can be simplified to $p \propto v$). The divergence of the velocity field is also 0 because the kinetic energy of the fluid must be conserved.

Equation 2.4 must be rewritten in terms of an arbitrary constant ψ for a disc. ψ only varies because of the shape of the disc and therefore all other proportions are conserved. Note that in Equation 2.4, $F_d \propto r$ where the reference area varies quadratically with r. In the case of a disc with a rectangular cross-section in the air stream, the reference area varies linearly with r and therefore we must take the square root of r and change the unit of ψ to account for the dimensional change and maintain the homogeneity of the equation.

$$F_d = \psi \eta v \sqrt{r} \tag{2.5}$$

Logically this proportionality seems to work out, but it is possible this is not the case. This is because apart from the differences in whether or not the area is proportional to r or $\sqrt{r}$, the fundamental geometry is different. The rectangular cross-section does not have the 2D symmetry and a disc does not have the 3D symmetry of a sphere. As a result, the viscous drag produced also changes.

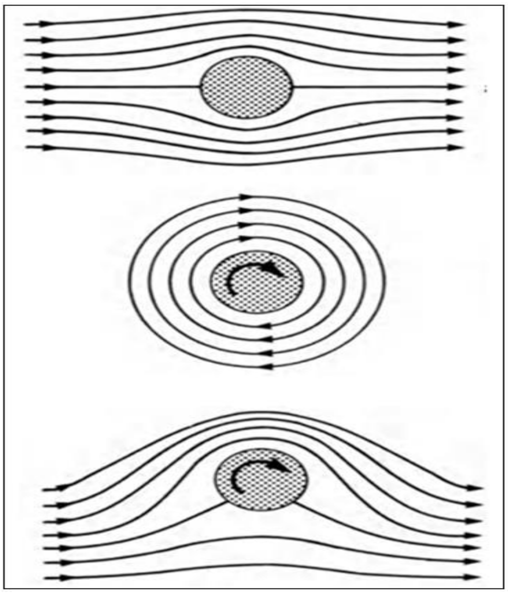

Fig 6 The Third Diagram Represents the Resultant Vector Field Found from the Addition of the First Two.

Realistically, this model would also not work because of a small assumption made in Equations 1.4. While $\hat{v}$ represents the velocity vector, $\hat{l}$ is the lift vector which is perpendicular to the orientation of the disc. Since $\hat{v} \cdot \hat{l} = 0$, the disc is always assumed to be oriented in the $\hat{v}$ direction. This is untrue because variation in motion means there is always the under-body or over-body of the disc exposed to the airflow, thus changing the value of F_d. Another adjustment must also be made. Stokes' Law is used in situations where the flow is high viscosity and $Re \ll 1$, which is untrue for air. The physics of initial contact with the airflow remains the same, but flow separation will result in predictable inaccuracies. It is my belief Equation 2.5 can be empirically adjusted for this as long as the obtuse angle from the equilibrium line at which the flow separates in the horizontal plane, and the similar angle in the vertical plane, remain constant.

➢ *Circulation*

This section addresses the spinning motion of the disc and how this affects the forces experienced by it. Start by considering the flow of air around a circular disc to be represented by a vector field $\overrightarrow{v_f}$. The rotational motion on the disc has an added effect which can be considered to be a separate vector field $\overrightarrow{v_r}$. $\overrightarrow{v_r}$ is a circular vector field where the vectors are arranged tangentially to the disc at the origin. The magnitude of these decreases as the distance from origin increases. The magnitude can be considered to have exponential decay. Both these fields can be added together to given the final field $\vec{v}$. This concept is clearly expressed for ideal flow as shown in Figure 2.4. [7]

Note that the stability of the disc will also depend on the moment of inertia, which will be governed by both the mass of the disc and the frequency of rotation.

- *Pseudo-Changes in Viscosity*

At the risk of minor ambiguity, consider a pot of honey. The glass rod is placed in the honey and it is stirred, at first, because of the high viscosity of honey a large resistive force is experienced by the rod. However, as the rotation of the rod continues, it becomes easier to stir and increase the velocity of stirring as well. This rotational motion seems to induce a localized reduction in viscosity but this is more likely a consequence of the inertia of the molecules closest to the glass rod.

In the case of a disc, molecules in the boundary layer have a lower velocity compared to those in the air stream. It is possible that the boundary layer acts as a sort of "insulation" reducing the viscous drag on the disc. The greater the angular velocity (ω), the lower the viscous drag, thus reaching the conclusion

$$F_{vd} \propto \frac{1}{\omega}$$

This insulation could be a result of the properties of the boundary layer and has varying effects depending in whether it is turbulent or laminar as shown in Figure 2.1. Logically, this effect seems to be more viable when the boundary layer is laminar because of the sharp change in velocity between the normal flow and the boundary layer. There are two cases where this could be applied. One is where the fluid flow is moving in the direction of rotation of the disc. In this case, the velocity of fluid molecules should increase closer to the surface of the disc, then reduce, and then start to increase again such that the magnitude of velocity has a parabolic character. The second case, where the rotation of the disc is opposite to that of the fluid may cause the velocity to be negative (in the opposite direction) relative to the surface. This, as discussed, leads to flow separation from the disc and turbulent flow, further causing vortices to form in the wake since the airflow combining at the back of the disc is not at the same speed. In this case as well it is possible for the magnitude of velocity to have parabolic character.

Over time though ω is bound to reduce due to air friction. As ω reduces, the very same effect will have the opposite results on the motion of the disc. Because of inertia, the relative speed between the molecules in the boundary layer and the surface of the disc might be greater than or equal to the relative speed between the surface of the disc and u_∞. As a result, this must also be considered when creating a mathematical model to simulate its effect.

Another factor to consider is the presence of "grooves" or "rims". The presence of surface irregularities induces localized turbulence which reduced the drag acting on the disc.

- *Magnus Effect*

A consequence of circulation is that one side of the disc is moving in the direction of the airflow while the other is moving the in the opposite direction. The side of the disc moving in the direction of the airflow experiences less drag than the other. This is because, as discussed in the second case in Section 2.3.1 above, greater surface friction generates more drag, representing the path of greatest resistance. In a more technical sense, the addition of circulation increases the density of the field lines of the velocity vector field on the side where the disc's rotation is along with the airflow. It is simple to conclude how pressure (p) and density (ρ) are related directly with each other. Bernoulli's Equation is as below.

$$P_1 + \frac{1}{2}\rho {v_1}^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho {v_2}^2 + \rho g h_2 \qquad (2.6)$$

According to Bernoulli's Equation, when the velocity increases (in this case the "density" of the field lines in the vector field), the pressure decreases. This pressure differential creates an added lateral force causing the disc to follow a sightly curved trajectory, in the horizontal plane. However, for a disc the Magnus effect is almost negligible over short distances, and is therefore not considered.

IV. CONCLUDING STATEMENTS

It was first determined that it is imperative to break down the forces acting on the disc into two components (drag and lift) to most accurately approximate the Pressure Forces represented in Equation 1.1. Then the forces were resolved to create the final system of equations that represented the motion of the disc through the air. The next chapter attempted to add on qualitative rather than quantitative detail to the model by analyzing the effect of the system (through Re and K_n), understanding Stokes' Law governing viscous drag in more detail, and looking at the effect of the disc rotating.

As mentioned in the Background and Quick Notes section, this is neither the most accurate nor the most complete model. As I continue working upon this project, I feel a few more details are in order. Through this paper I have successfully generalized a collection of forces and have qualitatively considered a few other factors both external and internal. My goal is to continue researching these phenomena in order to gain a more detailed understanding while providing explanations with more mathematical rigor. This is because I have noticed multiple sections which can be further elaborated upon. Also, most results derived from graphs and charts may not be perfectly true. Therefore, the reasoning behind the trends we see in the data is up to interpretation.

Another interesting thing to note is that the conclusion reached in this paper can be applied to all dynamical systems. Given most of the research paper deals in variables and does not assign any values anywhere (except in the end of Section 1.3 to provide graphs), the values themselves can be adjusted to represent any other object, save for a few technical changes such as proportionality changes to quantities similar to those in Section 2.2. This is because the interactions between the object and its environment remain the same.

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