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On Null Controllability of Nonlinear Neutral-Type Fractional-Order Differential Systems with State Delays and Distributed Delays in the Control, and Impulsive Effects

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Abstract:- In this work, Nonlinear Neutral Type Fractional Order Differential Systems with State Delays and Distributed Delays in the Control and Impulsive Effects in Banach Space is presented for Controllability analysis. From the analysis, we established that the System is Null Controllable, and a set of necessary and sufficient conditions for such systems to be Null Controllable were established. Uses were made of the uniformly asymptotically Stability of its Linear Free Base System and the Properness of its Linear Control Base System. We also established that the System admits the Solution pair (x, u) and hence the form of the null control of the system was obtained, using the Riesz theorem and Schauder's Fixed Point theorem.

Keywords:- *Null Controllability, Asymptotically Stability, Properness, Controllability, Orthogonality.*

I. INTRODUCTION

Recently the study of fractional differential systems have emerged as a new area of research in the field of applied mathematics which have been used to model any practical systems in science and engineering (Sheen and Cao,2012) and (Huang,2017).

The fractional differential expressions have been used in engineering since 1930's to describe the viscoelastic materials, electric circuits and fractal geometry involving non integer spatial dimensions. Also fractional derivatives and integrals can be applied to real systems characterized by power laws, critical phenomena and scale free processes. Moreover, controllability is one of the fundamental concepts in mathematics and fractional control theory and is the generalization of the classical control theory (Ammour, 2009).It is noted in (Oraekie,2012) that any control systems is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some control signals.

Furthermore, it's well known that neutral differential equation is a very special class of ordinary differential equation and it arises in compartmental models in which the system can be divided into separate compartments, marking the path ways of material flow between compartments and the possible outflow into the inflow from the environment of the system; (Gyori and Wu,1991) as it is contained in (Oraekie,2012;Oraekie,2014). Such models are usually used in theoretical epidemiology, physiology and population dynamics to describe the evolution of systems.

The above said models can be remodified as a neutral fractional differential equation or neutral fractional volterra integrodifferential equation. At the same time, time delay is very commonly experienced in diverse scientific systems such as electric, pneumatic and hydraulic networks, chemical process, long transmission lines etc. Because the subsistence of pure time delay, nevertheless if it is available either in the control or the state may result in unacceptable system momentary response or even instability. Also, time delay is one of the inevitable problems in practical engineering applications, which has an important effect on the stability and performance of the system (Li and Song, 2017).

With the interest from the above fact, in the last few years, several studies have been done on the fractional delay differential systems. (Chen and Zhou, 2011) (Kaslik and Sivabundaram, 2012), (Oraekie,2018). Generally, most of the dynamical systems are analyzed in either continuous or discrete time domain, many real systems in physics, biology, chemistry, engineering and information science may experience abrupt changes as certain instants during the continuous dynamical systems (Sundara,2018) and (Li and Wu 2016).According to (Sundara,2018),there has been a somewhat new category of dynamical system; which is neither purely continuous time nor purely discrete time ones, these are called impulsive control system.

This 3rd category of system displays a combination of characteristics of both the continuous and discrete-time systems.

The significance of this system is to control the hasty changes in nature adversity.

Numerous advancements procedure are subject to short term perturbations which act instantaneously in the form of impulses.

For instance, the existence of impulses can be seen in the biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics.

Consequently the impulsive differential equations provide a natural description of observed advancement procedure of several real world problems. Recently (Zhang,2013) have derived the controllability criteria for linear fractional order differential systems with state delay and impulses.

(Sundara,2018),studied the controllability problem of nonlinear neutral type fractional differential systems with state delay and impulsive effects and established a new set of sufficient conditions for the system to be controllable using the controllability grammian and laplace transformation.

To the best of our knowledge there are no relevant reports on the Null Controllability of Nonlinear neutral-type fractional- order differential systems with State delays and distributed delays in the control, and impulsive effects in Banach spaces in the existing literature. Hence the research.

II. VARIATION OF CONSTANT FORMULA/PRELIMINARIES

We reemphasize that we denote $C_\rho([O.T], E^n)$ the space of all piecewise left continuous functions mapping the interval $[O.T]$ $into\ E^n$.

$$Let\ \alpha; \beta > 0\ with\ n-1 < \alpha, \beta < n\ and$$

$n \in N,\ D$ is the usual differential operator; E^m $is\ the$ m-dimensional Euclidean space,

$R^+ = [0, \infty)$ and suppose $f \in L_1(R^+.)$ The following definitions, properties and theorems are familiar and helpful in establishing our main results.

- *Definition 2.1*

The Riemann-Liouville fractional integral operator of order $\propto$>0 with the lower limit zero for a function $f: R^+ \to R^n$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^\alpha (t-s)^{\alpha-1} f(s)\, ds, \qquad t > 0$$

Where $\Gamma(.)$ is the euler gamma function

- *Definition 2.2*

The Riemann-fractional derivative of the order $\alpha > 0$, with the lower limit zero for a function $f;\ R^+ \to R^n, n-1 < \alpha < n, n \in N$ is defined as

$$(D_0^\alpha, f)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds,$$

Where the function f has absolutely continuous derivatives up to order $(n-1)$

- *Definition 2.3*

The caputo fractional derivative of order $\propto > 0, n-1 < \propto < n$ is defined as

$$(\ {}^cD_0^\alpha, f)(t) = \frac{1}{\Gamma(n-\propto)} \int_0^t (t-s)^{n-\propto-1} f^n(s)\, ds,$$

Where the function f has absolutely continuous derivatives up to order $(n-1)$.
$If\ 0 < \propto < 1, then$

$$(\ {}^cD_0^\alpha f)(t) = \frac{1}{\Gamma(1-\propto)} \int_0^t \frac{f'(s)}{(t-s)^\propto} ds,$$

Consider the controllability of nonlinear fractional order type differential systems with state delay and impulses, and distributive delays in the control in Banach spaces as follows:

$$^{C}D^{\propto}\left(x(t)-g\big(t,x(t)\big)\right)=Ax(t)+\int_0^t B(t-s)x(s-h)ds\ +\int_{-h}^0\big(d_\theta H(t,\delta)u(t+\delta)\big)$$

$$t\in[0,T]\ -\ \{t_{1,}t_2,\dots,t_k\}.\ \ \Delta x(t_j)=x(t_i^+)-x(t_i^-)=\ I_i\big(x(t_i)\big),i=1,2,3\dots\ ,k$$

$$x(t)=\ \phi(t),\ t\in[-h,0] \tag{1.1}$$

$Where$ $\ ^{C}D^{\propto}x(t)$ denotes an α order caputo's fractional derivative $of\ x(t)$ $0<\propto<1$,

A is a constant matrix and satisfies $A\ \in E^{n\times n}$ $,where\ B$ is a continuous matrix in their argument with initial condition

$$x(t_0)=x_0=x(0),$$

Where $x\in E^n$ is the state space and $u\in E^m$ is the control function, $H(t,\delta)$ is an $n\times m$ matrix continuous at t and of bounded variation in $\delta\ on\ [-h,0];\ h>0$ for each $t\in[0,T];\ 0<T.$

$\phi\in([-h,0],E^n)$ denotes the initial function while $C([-h,0],E^n)$ denotes the space of all continuous functions mapping the interval $[-h,0]\ into\ E^n$;

$I_i\colon E^n\to E^n$ is continuous for

$$i=1,2,3,4,\dots,k\ \ and$$

$$x(t_i^+)=\lim_{\varepsilon\to0^+}x(I_i+\varepsilon)$$

$$x(t_i^-)=\lim_{\varepsilon\to0^-}x(I_i+\varepsilon) \tag{1.2}$$

Represent the right and left limits of $x(t)\ at\ t=t_i$ and the discontinuous points

$$t_1<t_2<t_3<\cdots<t_k \tag{1.3}$$

Where

$0=t_0<t_1,t_k<t_{k+1}=T<\infty,$ and

$$x(t_i^+)=x(t_i^-)$$

Which implies that the solution of the system (1.1) is left continuous.

- *The Mild Solution*

In order to obtain the mild solution of system (1.1), we first consider the representation of solution for nonlinear fractional delay differential systems without impulses as follows:

$$^{C}D^{\propto}\left(x(t)-g\big(t,x(t)\big)\right)=Ax(t)+\int_0^t B(t-s)x(s-h)ds$$

$$+\ f\left(t,x(t),\int_0^t k(s,x(s)ds)\right),t\ \in[0.T]$$

$$x(t)=\phi(t),t\in[-h,0] \tag{1.4}$$

- *Theorem 2.1 (B. Sundara etal,2018)*

Let $0<\propto<1, if\ f\colon[0,T]\to E^n$ is continuous and exponential bounded then the solution of the system (4.4) can be represented as

$$x(t)=\ \phi(0)-g\big(0,x(0)\big)+g\big(t,x(t)\big)+\int_0^t(t-s)^{\propto-1}E_{\propto},\propto(A(t-s)^{\propto})$$

$$.\left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + f\left(s, x(s), \int_0^s K\big(s, \tau, x(\tau)\big)\right)d\tau\right]ds \,, t\epsilon[0, T]$$

$x(t) = \phi(t), t\epsilon[-h, 0]$ (See B .Sundara, etal, 2018 , for the proof)

- *Theorem* 2.2
  Let $0 < \propto < 1$ and $u \in C_p([o, T], E^m)$, then the state response of the system (1.1) can be represented as follows;

For $t\epsilon[-h, 0], then,$

$$x(t) = \phi(t).$$

For $t\epsilon[0, t_1]$, then we have

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \int_0^t (t-s)^{\propto -1} E_{\propto}, \propto [A(t-s)^{\propto}]$$

$$\times \left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + \int_{-h}^0 d_\delta H(s,\delta)u(t+s)\right]ds$$

For $t\epsilon[t_1, t_2]$, then we have

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + I_1\big(x({t_1}^{-})\big) + \int_0^t (t-s)^{\propto -1} E_{\propto}, \propto [A(t-s)^{\propto}]$$

$$\times \left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + \int_{-h}^0 d_\delta H(s,\delta)u(t+s)\right]ds$$

$For \;\; t\epsilon\big[t_j, t_{j+1}\big], \qquad j = 1,2, \dots, k.$

We have

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) \; + \sum_{j=1}^{i} I_j\left(x\big(t_j^-\big)\right) + \int_0^t (t-s)^{\propto -1} E_{\propto}, \propto [A(t-s)^{\propto}].$$

$$\left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big) + \int_0^s B(s-m)x(m-h)dm \; + \int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\right]ds \qquad (1.5)$$

System (1.5) implies:.

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) \; + \sum_{j=1}^{i} I_j\left(x\big(t_j^-\big)\right)$$

$$+ \int_0^t (t-s)^{\propto -1} E_{\propto}, \propto [A(t-s)^{\propto}]. \left[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big)\right]ds$$

$$+ \int_0^t (t-s)^{\propto -1} E_{\propto}, \propto [A(t-s)^{\propto}]. \int_0^s B(s-m)x(m-h)dmds$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}].\int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\,ds \tag{1.6}$$

(1.6) $\Rightarrow$

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right)$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}].\left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}]\int_0^s B(s-m)x(m-h)dmds$$

$$\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}]\int_{-h}^0 d_\delta H(s,\delta)u(s+\delta)\,ds \tag{1.7}$$

A careful observation of the solution of system (1.1) given as system (1.7) shows that the values of the control $u(t)\ for\ t\epsilon[-h,T]$ enter the definition of the initial complete state, thereby creating the need for an explicit variation of constant formula. The control in the last term of formula (1.7), therefore, has to be separated in the intervals $[-h,0]\ and\ [0,T]$.

To achieve this, that term has to be transformed by applying the method of (klamka,1978) as it is contained in (Oraekie,2019). Finally, we interchange the order of integration using the Unsymmetric Fubini's Theorem to have

$$x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right)$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}].\left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_0^t\left[\int_{-h}^0 d_\delta H\int_{0+\delta}^{t+\delta}(t-s-\delta)^{\propto-1}E_{\propto},\propto [A(t-s-\delta)^{\propto}H(s-\delta,\delta)u(s-\delta+\delta)]\right]ds$$

$$\Rightarrow x(t) = \phi(0) - g\big(0,x(0)\big) + g\big(t,x(t)\big) + \sum_{j=1}^{i} I_j\left(x\left(t_j^-\right)\right)$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}]\left[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\right]ds$$

$$+\int_0^t (t-s)^{\propto-1}E_{\propto},\propto [A(t-s)^{\propto}]\int_0^s B(s-m)x(m-h)dmds$$

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$$+\int_0^t\left[\int_{-h}^0 d_\delta H\int_{0+\delta}^{t+\delta}(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}H(s-\delta,\delta)u(s)]\right]ds \quad (1.8)$$

Simplifying system (1.8), we have

$$x(t)=\phi(0)-g\big(0,x(0)\big)+g\big(t,x(t)\big)+\sum_{j=1}^{i}I_j\left(x\big(t_j^-\big)\right)$$

$$+\int_0^t(t-s)^{\propto-1}E_{\propto},\propto[A(t-s)^{\propto}]\big[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\big]ds$$

$$+\int_0^t(t-s)^{\propto-1}E_{\propto},\propto[A(t-s)^{\propto}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta\int_\delta^0(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}H(s-\delta,\delta)u_0(s)]ds$$

$$+\int_{-h}^0 dH_\delta\int_0^{t+\delta}(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}H(s-\delta,\delta)u(s)]ds \quad (1.9)$$

Using again the Unsymmetric Fubini's Theorem on the change of order of integration and incorporating H^* as defined below

$$H^*(s-\delta,\delta)=\begin{cases}H(s-\delta,\delta)\ for\ s\le t\\ 0\ for\ s\ge t\end{cases} \quad (1.10)$$

Formula (1.9) becomes

$$x(t)=\phi(0)-g\big(0,x(0)\big)+g\big(t,x(t)\big)+\sum_{j=1}^{i}I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^t(t-s)^{\propto-1}E_{\propto},\propto[A(t-s)^{\propto}]\big[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\big]ds$$

$$+\int_0^t(t-s)^{\propto-1}E_{\propto},\propto[A(t-s)^{\propto}]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta\int_\delta^0(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}H(s-\delta,\delta)u_0(s)]ds$$

$$+\int_0^t\left[\int_{-h}^0(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}]d_\delta H^*(s-\delta,\delta)u(s)\right]ds \quad (1.11)$$

Integration is still in the lebesque stieltjes sense in the variable $\delta \; in \; H$.

For brevity, let

$$\boldsymbol{\beta(t)} = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \sum_{j=1}^{i} I_j\left(x(t_j^-)\right)$$

$$+ \int_0^t (t-s)^{\propto-1} E_{\propto}, \propto [A(t-s)^{\propto}]\big[A\phi(0) - Ag\big(0, x(0)\big) + Ag\big(s, x(s)\big)\big]\, ds$$

$$+ \int_0^t (t-s)^{\propto-1} E_{\propto}, \propto [A(t-s)^{\propto}] \int_0^s B(s-m)x(m-h)dmds \tag{1.12}$$

$$\boldsymbol{\mu(t)} = \int_{-h}^0 dH_\delta \int_\delta^0 (t-s-\delta)^{\propto-1} E_{\propto}, \propto [A(t-s-\delta)^{\propto} H(s-\delta, \delta) u_0(s)] ds \tag{1.13}$$

$$\boldsymbol{z(t,s)} = \int_{-h}^0 (t-s-\delta)^{\propto-1} E_{\propto}, \propto [A(t-s-\delta)^{\propto}] d_\delta H^*(s-\delta, \delta) u(s) \tag{1.14}$$

Substituting (1.12),(1.13) and (1.14) into (1.11),we have a precise variation of constant formula for system (1.1) as:

$$\boldsymbol{x(t, x_0, u)} = \boldsymbol{\beta(t)} + \boldsymbol{\mu(t)} + \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{z(t,s) u(s) ds} \tag{1.15}$$

III. BASIC SET FUNCTIONS AND PROPERTIES WITH APPROPRIATE TERMINOLOGIES

- *Definition 3.2.1 (Complete State)*
  The complete state for the system (1.1) is given by $z(t) = \{x_t, u_t\}$

- *Definition 3.2.2 (Null Controllability)*
  $Let \; n \; and \; m$ be positive integer $R = E \; the \; real \; line \; (-\infty, \infty)$. We denote by $R^n = E^n$, the space of real $n - tuples$ with the Euclidean norm defined by $|.|$.

- *Definition 3.2.3 (Null Controllability)*
  $Let \; n \; and \; m$ be positive integer $R = E \; the \; real \; line \; (-\infty, \infty)$. We denote by $R^n = E^n$, the space of real $n - tuples$ with the Euclidean norm defined by $|.|$.

$If \; J = [t_0, t_1], t_1 > t_0$, is any interval in R, the usual lebesgue space of integrable (equivalences) functions from $J \; to \; R^m$ will be denoted by $L_2(J, E^m)$.

$$Let \; h \geq k \geq 0$$

Be a given real number and let $C = C([-h, 0], R^n)$ be a Banach space of functions which are continuous $on \; [-h, 0] \; with$

$$\|\phi\| = \sup_{-h \leq s \leq 0} |\phi(s)| \;, \phi \in C([-h, 0], R^n).$$

$If \; x$ is a function from $[-h, \infty) \; to \; R^n, let \; x_t \;, t \in [0, \infty)$ to be a function from $[-h, 0]$

$to \; R^n$ defined by

$$\boldsymbol{x_t(s) = x(t+s), \qquad s \in [-h, 0].}$$

Then the system (1.1) is said to be null controllable on the interval $[t_0, T_1]$ if for each

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Function $\phi \in C([-h,0],R^n)$, there exists a time $t>0, U \in L_2(J,R^m)$, such that the solution

$$x(t,0,\phi,u)\ of\ system\ (1.1)\ satisfies$$

$$x_{t_0}(t,0,\phi,u) = x(t,0,\phi,u) = \phi, and x(t,0,\phi,u) = 0.$$

- *Definition 3.2.4 (Linear Base Control System)*

Consider the system (1.1) with its standing hypothesis. Then the linear base control system of the system (1.1) is given as:

$$^{C}D^{\propto}\left(x(t)-g\left(t,x(t)\right)\right) = Ax(t) + \int_{-h}^{0}\left(d_{\delta}H(t,\delta)u(t+\delta)\right) \qquad (1.16)$$

- *Definition 3.2.5 (Free Base Control System)*

Consider the system (1.1) with its standing hypothesis. Then the free base control system of the system (1.1) is given as:

$$^{C}D^{\propto}\left(x(t)-g\left(t,x(t)\right)\right) = Ax(t) + \int_{0}^{t}B(t-s)x(s-h)ds \qquad (1.17)$$

- *3.1.3. Necessary and Sufficient Conditions for the System to be Null Controllable.*

- *Preamble*

Recently,(Oraekie,2018) studied the Euclidean null controllability of Nonlinear infinite Neutral systems with multiple Delays in the control and established sufficient conditions for the system to be null controllable, thereby extended the null controllability concept to the systems with multiple delays in the control.

Not alone,(Oraekie,2019)investigated the Null controllability of Sobolev type Integrodifferential systems in Banach spaces with Distributive Delays in the control of the form:

$$F\dot{x}(t) + Ax(t) = \int_{-h}^{0}[d_{\theta}H(t,\theta)]u(t+\theta) + \int_{0}^{t} g\left(t,s,x(s),\int_{0}^{s}B\left(s,\tau,x(\tau)\right)d\tau\right)ds$$

$$+f\left(t,x(t)\right), \qquad t \in [t_0,t_1],\ t_1 > t_0$$

$$x(0) = x_0$$

He established the Necessary and sufficient conditions for computable criteria for the null Controllability of the system.

In the light of these,we intend to study the Null Controllability of the Nonlinear Neutral-type Fractional-order Differential

Systems with State Delays and Distributed delays in the Control and Impulsive Effects, in Banach spaces of the form:

$$^{C}D^{\propto}\left(x(t)-g\left(t,x(t)\right)\right) = Ax(t) + \int_{0}^{t}B(t-s)x(s-h)ds + \int_{-h}^{0}[d_{\delta}H(t,\delta)]\,u(t+\delta)\,,$$

$$t \in [0,T]-\{t_1,t_2,\dots t_k\}, \qquad \Delta x(t_i) = x(t_i^+) - x(t_i^-) = I_i\left(x(t_i)\right), i = 1,2,3,\dots,k$$

$$x(t) = \phi(t), t \in [-h,0]$$

- *Through:*

- *Its Linear Base Control System given as*

$$^{C}D^{\propto}\left(x(t)-g\left(t,x(t)\right)\right) = Ax(t) + \int_{-h}^{0}[d_{\delta}H(t,\delta)]\,u(t+\delta) \qquad (1.18)$$

- *Its Free Base System given as*

$$^{C}D^{\propto}\left(x(t)-g\left(t,x(t)\right)\right)=Ax(t)+\int_{0}^{t}B(t-s)x(s-h)ds \tag{1.19}$$

➢ *Theorem 3 .2. Necessary and Sufficient Conditions for Null Controllability of System (1.1)*

Assume for the system (1.1) that:

(i).The constraint set U is arbitrarily compact subset of R^n

(ii).The system (1.17) is uniformly asymptotically stable so that the solution of the system (1.17) satisfies

$\|x(t,t_0,\phi,0,0)\| \le Me^{-\lambda(t_1-t_0)}\|\phi\|$ (i.e exponential estimate)

$for\ some\ \lambda>0, M>0\ real\ numbers.$

(iii).The linear control base system (1.16) is proper.

(iv).The continuous function f satisfies

$$\left|f\left(t,x(t)\right),u(t)\right| \le e^{(-\rho t)}\pi\left(x(t),u(t)\right)$$

$$for\ all\ \left(t,x(t),u(t)\right)\in[t_0,\infty)\times C\times L_2,$$

$$where,\int_{t_0}^{\infty}\pi\left(x(s),u(s)\right)ds\le k<\infty\ and\ b-a>0,$$

Then, system (1.1) is Null-Controllable.

- *Proof*
  In system (1.14), we introduce the notation:

$$z(t,s)=\int_{-h}^{0}(t-s-\delta)^{\propto-1}E_{\propto},\propto[A(t-s-\delta)^{\propto}]d_{\delta}H^{*}(s-\delta,\delta);t\ge s\ge t_0\ ,$$

$$and\ define\ the\ controllability\ grammian\ of\ the\ system\ (1.1)\ by$$

$$W(t,0)=\int_{0}^{t}\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^{*}(s-\delta,\delta)\right]$$

$$\times\left[\int_{-h}^{0}(t-s-\delta)^{\alpha-1}E_{\alpha},\alpha[A(t-s-\delta)^{\alpha}]d_{\delta}H^{*}(s-\delta,\delta)\right]^{T}=\int_{0}^{t}Z(t,s)\,Z(t,s)^{T}$$

Where T denotes the matrix transpose by $(iii), W^{-1}(t,0)\ exists\ for\ t>0.$

Now, suppose that pair of functions $(x,u\ )$ form a solution pair to the set of integral equations:

$$u(s)=-z(t_1,s)^{T}W^{-1}(t_1,0)\{\left(\phi(0)-g\left(0,x(0)\right)\right)+g\left(t,x(t)\right)+\sum_{j=1}^{i}I_j\left(x\left(t_j^{-1}\right)\right)$$

$$+\int_{0}^{t}(t-s)^{\alpha-1}E^{\alpha},\alpha[A(t-s)^{\alpha}]\left[A\phi(0)-Ag\left(0,x(0)\right)+Ag\left(s,x(s)\right)\right]$$

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$$+\int_0^t (t-s)^{\alpha-1}E_\alpha,\alpha[A(t-s)^\alpha].\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta\int_\delta^0 (t-s-\delta)^{\alpha-1}\,E_\alpha,\alpha[A(s-\delta)^\alpha]H(s-\delta,\delta)u_0(s)\}$$

$x(t) = \phi(t), t \in [-h, 0].$ (1.20)

For some suitably chosen time $t \in [0, T]$,

$$x(t) = \phi(0) - g\big(0, x(0)\big) + g\big(t, x(t)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^t (t-s)^{\propto-1}E_\propto,\propto[A((t-s)^\propto)]\big[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\big]\,ds$$

$$+\int_0^t (t-s)^{\propto-1}E_\propto,\propto[A((t-s)^\propto)]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta\int_\delta^0 (t-s-\delta)^{\propto-1}E_\propto,\propto[A(t-s-\delta)^\propto H(s-\delta,\delta)u_0(s)]ds$$

$$+\int_0^t\left[\int_{-h}^0 (t-s-\delta)^{\propto-1}E_\propto,\propto[A(t-s-\delta)^\propto]d_\delta H^*(s-\delta,\delta)u(s)\right]ds$$

$x(t) = \phi \quad , \quad t \in [-h, 0].$ (1.21)

Then $u(t)$ is square integrable on the interval $[0, T]$, and $x(t)$ is a solution of system (1.1) corresponding to $u(t)$ *with an initial state*, $x(t_0) = x(0) = \phi$.

Similarly,

$$x(t_1) = \phi(0) - g\big(0, x(0)\big) + g\big(t_1, x(t_1)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^{t_1} (t_1-s)^{\propto-1}E_\propto,\propto[A(t_1-s)^\propto]\big[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\big]\,ds$$

$$+\int_0^{t_1} (t_1-s)^{\propto-1}E_\propto,\propto[A(t_1-s)^\propto]\int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^0 dH_\delta\int_\delta^0 (t_1-s-\delta)^{\propto-1}E_\propto,\propto[A(t_1-s-\delta)^\propto]H(s-\delta,\delta)u_0(s)ds$$

$$+\int_0^{t_1}\left[\int_{-h}^0 (t_1-s-\delta)^{\propto-1}E_\propto,\propto[A(t_1-s-\delta)^\propto]d_\delta H^*(s-\delta,\delta)u(s)\right]ds[-z(t_1,s)^T W^{-1}(t_1,0)]$$

$$.\left\{\phi(0)-g\big(0,x(0)\big)+g\big(t_1,x(t_1)\big)+\sum_{j=1}^{i}I_j\left(x\big(t_j^{-1}\big)\right)\right.$$
$$+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto,}\propto[A(t_1-s)^{\propto}]\big[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\big]$$
$$+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto},\propto[A(t_1-s)^{\propto}]\int_0^{s}B(s-m)x(m-h)dmds$$
$$\left.+\int_{-h}^{0}dH_\delta\int_{\delta}^{0}(t_1-s-\delta)^{\propto-1}E_{\propto},\propto[A(s-\delta)^{\propto}]H(s-\delta,\delta)u_0(s)ds\right\} \tag{1.22}$$

Now Consider

$$\int_0^{t_1}\left[\int_{-h}^{0}(t_1-s-\delta)^{\propto-1}E_{\propto},\propto[A(t_1-s-\delta)^{\propto}]d_\delta H^*(s-\delta,\delta)u(s)\right]ds.\left[-z(t_1,s)^TW^{-1}(t_1,0)\right]$$

$$=-\int_0^{t_1}Z(t_1,s)Z^T(t_1,s)W^{-1}(t_1,s)ds=-\int_0^{t_1}\frac{Z(t_1,s)Z^T(t_1,s)}{W(t_1,0)}ds \tag{1.23}$$

$$=\frac{-\int_0^{t_1}Z(t_1,s)Z^T(t_1,s)}{\int_0^{t_1}Z(t_1,s)Z^T(t_1,s)}=\frac{-\int_0^{t_1}Z(t_1,s)Z^T(t_1,s)ds}{\int_0^{t_1}Z(t_1,s)Z^T(t_1,s)ds}=-1$$

Putting the system (1.18) into system (1.17), we have

$$x(t)=\ \phi(0)-g\big(0,x(0)\big)+g\big(t_1,x(t_1)\big)\ +\sum_{j=1}^{i}I_j\left(x\big(t_j^{-1}\big)\right)$$
$$+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto},\propto[A(t_1-s)^{\propto}]\big[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\big]ds$$
$$+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto},\propto[A(t_1-s)^{\propto}]\int_0^{s}B(s-m)x(m-h)dmds$$
$$+\int_{-h}^{0}dH_\delta\int_{\delta}^{0}(t_1-s-\delta)^{\propto-1}E_{\propto},\propto[A(t_1-s-\delta)^{\propto}]H(s-\delta,\delta)u_0(s)ds$$
$$(-1)\{\phi(0)-g\big(0,x(0)\big)+g\big(t_1,x(t_1)\big)\ +\sum_{j=1}^{i}I_j\left(x\big(t_j^{-1}\big)\right)$$
$$+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto},\propto[A(t_1-s)^{\propto}]\big[A\phi(0)-Ag\big(0,x(0)\big)+Ag\big(s,x(s)\big)\big]ds+\int_0^{t_1}(t_1-s)^{\propto-1}E_{\propto},$$
$$\propto[A((t_1-s)^{\propto})]\int_0^{s}B(s-m)x(m-h)dmds$$

$$+\int_{-h}^{0} dH_\delta \int_{\delta}^{0} (t_1 - s - \delta)^{\propto -1} E_{\propto}, \propto [A(t_1 - s - \delta)^{\propto}] H(s-\delta, \delta) u_0(s) ds \ \} = 0.$$

It remains to show that u is an admissible control. That is we need to show that the function $u: [0, T] \longrightarrow U$ is an arbitrary compact subset $of\ R^m\ such\ that$

$$|u| \leq \lambda \ \ for\ some\ \lambda > 0, \qquad \lambda \in R\ and\ U \subset R^m.$$

By the condition (ii) of theorem 3.2, we have

$$|z(t_1, s)^T W^{-1}(t_1, 0)| \leq \eta_1\ for\ some\ \eta_1 > 0,\ \eta_1 \in R, and$$

$$\left|\phi(0) - g\big(0, x(0)\big)\right| \leq \ \eta_2 e^{(-\lambda_1(t_1 - 0))};\ \ \eta_2 > 0;\ \ \eta_2 \in R.$$

Hence

$$|u(t)| \leq \ \eta_1\left[\eta_2 e^{(-\lambda_1(t_1-0))}\right] \int_0^{t_1} \ \eta_3\, e^{(-\lambda_1(t_1-s))} e^{(-\rho s)} \pi\big(x(s), u(s)\big) ds$$

$$i.e\ |u(t)| \leq \ \eta_1\left[\ \eta_2 e^{(-\lambda_1(t_1-0))}\right] \int_0^{t_1} \ \eta_2\, e^{(-\lambda_1(t_1-s))} e^{(-\rho s)} \pi\big(x(s), u(s)\big) ds$$

Thus

$$|u(t)| \leq \ \eta_1\left[\ \eta_2 e^{(-\lambda_1(t_1-0))}\right] + \eta\eta_3 e^{(-\lambda_1 t_1)}\ , since\ \rho - \lambda_1 \geq 0\ and\ s \geq 0.$$

Hence, by taking t_1 sufficiently large we have

$$|u(t_1)| \leq \ \lambda_1\ , \qquad t_1 \in [0, T].$$

Showing that u is an admissible control function.

Secondly, let us show that the solution pair of the integral equations (1.15) and (1.16) exists.

We have to first of all assume that Let Y be a Banach space of all continuous functions

$$(x, u)\ \ from\ [t_0 - h, t_1] \times [t_0 - h, t_1] \longrightarrow R^n \times R^m$$

$$i.e\ (x, u): [-h, t_1] \times [-h, t_1] \longrightarrow R^n \times R^m, \qquad t_0 = 0$$

$where\ x \in Y = C([t_0 - h, t_1], R^n)\ and\ u \in L_2([t_0 - h, t_1], R^m)$ with the norm defined by

$$\|(x, u)\| = \|x\|_2 + \|u\|_2$$

Where

$$\|x\|_2 = \left[\int_0^{t_1} |x(s)|^2 ds\right]^{\frac{1}{2}} and\ \|u\|_2 = \left[\int_0^{t_1} |u(s)|^2 ds\right]^{\frac{1}{2}}$$

Define the operator $F: V \longrightarrow Y$ by

$$F(x, u) = (y, v),$$

Where

$$\boldsymbol{v}(\boldsymbol{t}) = -z(t_1,s)^T W^{-1}(t_1,0)\left(\phi(0) - g\big(0,x(0)\big)\right) + g\big(t,x(t)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}\, E^{\alpha},\alpha[A(t-s)^{\alpha}]\big[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\big]$$

$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha},\alpha[A(t-s)^{\alpha}] \times \int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^{0} dH_{\delta} \int_{\delta}^{0} (t-s-\delta)^{\alpha-1}\, E_{\alpha},\alpha[A(s-\delta)^{\alpha}]H(s-\delta,\delta)u_0(s)ds \tag{1.24}$$

$$v(t) = \phi(t);\ \ t \in [-h,t_0], \qquad h > 0, \qquad t_0 = 0$$

And

$$\boldsymbol{y}(\boldsymbol{t}) = \left(\phi(0) - g\big(0,x(0)\big)\right) + g\big(t,x(t)\big) + \sum_{j=1}^{i} I_j\left(x\big(t_j^{-1}\big)\right)$$

$$+\int_0^t (t-s)^{\alpha-1}\, E^{\alpha},\alpha[A(t-s)^{\alpha}]\big[A\phi(0) - Ag\big(0,x(0)\big) + Ag\big(s,x(s)\big)\big]$$

$$+\int_0^t (t-s)^{\alpha-1} E_{\alpha},\alpha[A(t-s)^{\alpha}] \times \int_0^s B(s-m)x(m-h)dmds$$

$$+\int_{-h}^{0} dH_{\delta} \int_{\delta}^{0} (t-s-\delta)^{\alpha-1}\, E_{\alpha},\alpha[A(s-\delta)^{\alpha}]H(s-\delta,\delta)u_0(s)ds$$

$$+\int_0^t \int_{-h}^{0} (t-s-\delta)^{\alpha-1}\, E_{\alpha},\alpha[A(s-\delta)^{\alpha}]d_{\delta}H(s-\delta,\delta)u(s)ds\ for\ \ t \in [o,T], T > 0$$

And

$$y(t) = \phi(t); \tag{1.25}$$

Recall;

We have earlier proved that $|u(t)| \leq \lambda_1, for\ some\ \lambda_1 \in R\ and\ t \in [0,T]\ and$

$v{:}\,[0,T] \longrightarrow U$, We have, $|v(t)| \leq \lambda_1$

Hence,

$\|v(t)\|_2 \leq \lambda_1(t_1 + h - t_0)^{\frac{1}{2}} = M$, and $\|y(t)\| \leq \eta_2 e^{(-\lambda_1(t_1 - t_0))} + \eta_4 \int_0^t |v(s)|ds + \eta_3 e^{-\rho t}$

$\eta_4 = sup|z(t,s)|$, since $\rho > 0\ and\ \ t \geq t_0\ = 0$

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We deduce that

$|y(t)| \leq \eta_1\eta_3\rho(t_1 - t_0) + \eta\eta_2 = M_1, t \in [0,T]$ and $|y(t)| \leq sup|\phi(t)| = M_2;\ t \in [-h, t_0], y > 0$

Hence if

$$\delta = max(M_1, M_2), then\|y\|_2 \leq \delta(t_1 + h - t_0)^{\frac{1}{2}} = M_3 < \infty$$

$Let\ K = max(M, M_3)$. Then, if we let $G(K) = \{(x,u) \in R: \|X\|_2 \leq k, \|y\|_2 \leq k\}$

We have thus shown that $F\ maps\ G\ (k)\ to$

$i.e\ F: G(k)\ to\ G(k).$

Since $\boldsymbol{G}\ (\boldsymbol{k})$ is closed, bounded and convex, by Riesz theorem as contained in (Oraekie,2018),it is relatively compact under the transformation F.

Thus by the schauders fixed point theorem, it implies that F has a fixed point x which is the solution of system (1.1).

Hence system (1.1) is null-controllable.

IV. CONCLUSION

Necessary and sufficient conditions for the null-controllability of the nonlinear neutral-type Fractional-order Differential Systems with State Delay and Distributed Delays in Control, and Impulsive Effects in Banach Spaces have been established.

These conditions are given with respect to the Stability of its Free Linear Base System and the Controllability of its Linear Controllable Base System, with the assumption that it's proper. The form of the control energy function that is capable of steering the system (1.1) from the initial state to the origin in finite time was also established.

The existence of the solution pair (x, u) of the system (1.1) were established.

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