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From the Constitution of Knowledge to Playing the Game Slice Fractions

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Abstract:- This text presents aspects of a research project in which the possibilities for teaching fractions when playing the game Slice Fractions - Math Makers are questioned. The aim is to get to know the game in order to identify what it opens up for teaching fractions and to analyze the way in which meanings are made possible for the students who play it. The constitution of knowledge in the act of "playing the game" favors the signification of mathematical objects due to our presentiality, situated as an element of playing in this movement, involving an active methodology for understanding fractions. By playing Slice Fractions, we can outline a methodology for games in the classroom and, with this, explore the updates resulting from the experience with games. As a result of the research, we can see the meanings of fractions such as: number; ordering; part-whole; proportion; quotient; representations and ideas relating to operations with fractions. We point to a path with Digital Game Based Learning - DGBL - in the school context, as well as explaining the demands that the presence of the game in the classroom implies for the way of being a professor and the possibilities of activities resulting from this process.

Keywords:- *Games; Meanings of Fractions; Math Education; Phenomenology.*

I. INTRODUCTION

Digital information and communication technologies have changed the way we communicate and the way we learn. Instant messaging apps, video sharing, news feed notifications and financial transactions, among others, are in the "palm of our hands".

However, education has not always kept up with these rapid changes and extensions that digital technologies have brought about in the world of life[1]. Seymour Papert [1], with the story of a time traveler, illustrates this distance still felt today. The time traveler lends his machine to a 19th century surgeon who, when he arrives in the operating rooms of the 20th century, doesn't know what to do. However, when he lends the same machine to a professor from the 19th century, when he enters a classroom in the 20th century, he just wants to know where his professional colleague had "stopped" the subject in order to continue the lesson.

This allegory is intriguing, to say the least, because considering that people live in a world that is constantly being updated, failure to maintain teaching methods makes the school an unpleasant and obsolete space, as Professor Ubiratan D'Ambrosio [3] pointed out. The presence of digital technologies in the educational environment denotes a possible updating of these spaces, although it does not characterize a change in actions.

Among the various digital technologies, we highlight digital games or videogames for the purposes of this text. The history of digital games is not recent. "The history of the computer game is, in parts, a history of technology" [4] and Mark Wolf [5] provides an overview of the history of videogames (or digital games – we call simply "game") that begins in 1962. He presents the creation in 1958 of "tennis for two", on an oscilloscope, with no intention of "being a game"[2] and, in 1962, the mainframe game Spacewar! was developed at MIT.

Emblematically, we can say that games demarcate a "technological advance" from the 20th century to the 21st century by presupposing factors such as immersion and interactivity that give rise to potentialities that are different from those experienced in traditional games. With games there are new experiences that, when treated with didactic objectives, can favor the process of meaning in different areas of knowledge, such as mathematics.

However, even for games with educational purposes, it is essential to maintain the characteristics of fun, because according to Stuart Brown [6], a psychiatrist and researcher in games and play, the action of playing is an essential factor in creative development, so the absence of time and space for playful activities, especially in childhood, can compromise the human capacity to learn, empathy and what we call happiness.

[1] Reference [2] translates the German term *Lebenswelt* as the "world of life", understood as the world of cultures; whenever we address the world, we are addressing the world of life. And the inseparable movement of "being-with" takes place in the world of life.

[2] The intention of the developer, physicist William Higinbotham, was to use the oscilloscope as a way of drawing the attention of visitors to his stand at an annual event open to the public at Brookhaven National Laboratory.

It can be said that, from computer rooms[3] to mobiles, we are connected to a digital universe in which technologies cannot be supporting or mere didactic materials for teaching or learning, as they modify the school environment and the subjects involved in it [7].

Attentive to the potential of this paradigm shift and educational policies, we see that curriculum documents are encouraging actions with new digital technologies, from digital whiteboards to hybrid teaching, in an attempt to promote updates for contemporary education. In line with this trend, Brazil incorporated the topic of technology into the National Common Core Curriculum (NCCC) published in 2017. The challenge ahead is to "be-with" digital technology [7], seeking to overcome a vision of a "simple change of media" in order to understand the processes of constituting knowledge in this new scenario.

The constitution of knowledge "[...] refers to the articulation of sensory data, senses that make up the body-incarnate and that are articulated in the very carnality of this body, and in the movement of this articulation, it defines units, perceives phenomena and articulates and expresses understandings" [9]. It is therefore a process of the person who knows, of the person who gets involved with what they are doing in order to understand what they are doing. What is perceived and understood by the person is articulated and expressed, opening up to dialog and intersubjectivity, thus moving towards the production of knowledge.

We assume an openness to the constitution of knowledge by focusing on the specificities of games, interested in the various ways in which people learn or signify the (mathematical) object they encounter in the act of playing the game. Games in basic education may produce knowledge when they create a movement to solve problems. To the NCCC, games as technological resources provide an opportunity to use and develop digital information and communication technologies in a critical, meaningful, reflective, and ethical way in different contexts of social living, including school practices. [10].

The NCCC's proposal for the Early Years of Primary School is the progression of multiple learning, which requires the articulation of classroom work with experiences, valuing situations that are meaningful to children, including play. To highlight learning with games, we consider that the student/player is also a producer of digital technology, being able to organize images (screenshots) and record/edit videos (gameplay). We are thus breaking with the idea of "technology consumers" and moving towards what is produced with texts, images, videos, etc., that is, towards a vision of being-with digital technologies [7].

Educational spaces can and should foster creativity and authorship, enabling people to be trained in the exercise of "being-with" digital technologies. The mobile game Slice Fractions is an invitation to take on the challenge of helping a small mammoth find its herd. To meet this challenge, it offers the player mechanics and gameplay in which pieces of ice or lava are "sliced", producing parts and creating quantities. With the game, the theme of fractions and their operations are understood by "sliding your fingers" across the mobile screen. This game leads the player through a process of discovery in which "[...] the development of observation, abstraction, generalization and simulation skills are intrinsically linked to problem solving" [11].

We chose the game Slice Fraction - Math makers - for discussion in this text because we believe that the theme of "fractions" permeates the development of mathematical thinking and is present throughout basic education. By exploring it, we can identify the potential for teaching fractions by understanding the way in which meanings are made possible for the students who play the game.

II. METHODOLOGICAL PROCEDURES AND PARTICIPANTS

The game Slice Fractions - Math Makers - (Fig.1) for mobile (Android and iOS)[4] was developed by the Ululab studio (Canada) in partnership with education experts from the Université du Québec à Montréal (UQAM) with the aim of creating a game to introduce the subject of fractions. The mechanics of the game are based on "cutting pieces" by sliding your finger over geometric shapes and generating the parts needed to clear the path of the little mammoth. Solving puzzles, which involve cutting shapes and popping bubbles, allows you to create strategies to clear the path of obstacles and allow the mammoth to continue its journey.

[3] In Brazil, in the 1980s and 1990s, the Ministry of Education and Culture (MEC) also began to worry about linking education and information technology [8].

[4] The app is free and can be installed on Android versions 8 or higher and iOS versions 10 or higher. When you register as a school, you can play on up to 23 devices simultaneously

Fig 1 Levels Examples in Slice Fractions.
Source: Author's composition.

The potential of the digital game [12] can be analyzed by considering three equally relevant and interconnected aspects. 1. The design of the game, its rules and mechanics; 2. The players playing the game or their reports[5] and opinions; 3. Playing[6] the game, i.e. taking on the role of players. The educational environment with games should be a space for exploration and then a proposal for gamified activities aimed at interpreting what was meant by playing the game [13].

As a way of exemplifying this "environment" constituted with and for the game, in Chart 1 we present the proposal made to students on the Mathematics degree course at a public University in the state of São Paulo, for the exploration of Slice Fraction. It was also presented to a group of early years professors from the municipal public school system.

Table 1 Organization and Task Proposal with Math Mackers - Slice Fractions

Participants: 10 students from the fourth semester of the Mathematics degree course; 12 elementary school professors from the municipal school system (different groups). Duration: 4 hours - 2 meetings of 2 hours. The game was installed on tablets and, although the participants sat in groups, due to the organization of the physical space, each one played on a tablet.	
First meeting	***Second meeting***
Explore the organization of the game to understand that its rules are metaphors for mathematical rules used in arithmetic operations and to establish comparison relationships.	"Translate" the metaphors and create tasks based on the game.
With a tablet for each student, they were invited to play the game Slice Fractions (Math Makers - Fractions). The students/players played freely, exploring and discovering the rules and mechanics (Part 1).	The game was directed by a proposal from the professor/player. The proposal, as presented below in Part 2 and the extra proposal, aimed to organize the actions to translate and formalize the elements (or metaphors) presented by the game.
Proposed tasks	
Part 1: 1.Play the game Slice Fractions: (30-50 min) 2. Play the game identifying: 2.1. What is the aim of the game? 2.2 What are the rules of the game? 2.3 Identify and present two levels that demonstrate these rules.	**Part 2:** Play the game considering the proposed task (30 min) 1. What representations does the game present to achieve the proposed objective? Give examples. 2. What is the meaning of these representations? 3. Select two levels that represent different meanings of fractions and comment on them. 4. If you reached the level where you had to add fractions, what did you notice? Give an example and your thoughts on the operation "addition of fractions".
Extra proposal Choose a level and record a mini pitch explaining the objective, rule, representations and meanings of fractions that you identify in the chosen level.	

Source: Prepared by the author.

[5] We understand reading, as described by Aarseth (2007), as written (forums, blogs, wikis, etc.) or visual (videos, illustrations, etc.) any records of the player within the game.
[6] The professor/player is the one who knows the game and its possibilities and then organizes the time and activity proposals as a consequence of the act of playing

With the proposed task (Chart I) we sought to contextualize Digital Game Based Learning [14] [15] – DGBL – for classrooms, particularly in the training of professors.

III. THE CONSTITUTION OF MATHEMATICAL KNOWLEDGE AND THE MEANINGS OF FRACTIONS: A PHENOMENOLOGICAL LOOK

The philosopher and Professor Maria Bicudo [16] emphasizes that the aim of the phenomenological approach is not to define "something", but rather to turn attentively to what this "something" can become for those who intentionally question it. Therefore, from the phenomenological perspective, the person constitutes knowledge in the experience, which means that there is a movement of this person who intentionally turns towards something. This "something" shows itself in different ways of "appearing" and is apprehended in different sensory stimuli of what appears (or shows itself).

The synthesis of this diversity or multiplicity of senses takes place in the body-self, so that the constitution of knowledge is a continuous process that begins in sensitive experience (perception) and moves towards the unity of the object (in the Husserlian sense, as content thematized by consciousness).

With this, we can talk about the movement of the constitution of knowledge of the person, a subjective process. However, the object or "something" perceived unfolds in understandings and interpretations that come to expression, being named and predicated, communicated, opening up to intersubjectivity. Through language, this "something perceived" is perpetuated and acquires objectivity.

Mathematical knowledge is revealed in the subject's encounter with mathematical objects which, according to Husserl [17], are timeless, as they remain in temporality, and are ideal, as they are sustained in language and constituted in intentionality. "It is an ideality constituted in the intentionality of subjectivity [...] in the soil in which experiences occur and make sense, both for the subject and for the community of co-subjects" [17].

It is worth noting that, we are considering "knowledge as a movement in search of understanding the world, reality, the way human beings live, being with others in the individual singularity of each one and in the equality of each one with all the others" [18], always present in the world of life.

Therefore, in order to understand mathematical objects in the classroom or constitute mathematical knowledge, moments should be provided for encounters in which, through language, it would be possible to identify ourselves as subjects and co-subjects in a multiple space of understanding and expression.

We understand that by playing the game, by being "body" with the game, by being the protagonist in an adventure in which our actions allow the game to continue, we constitute knowledge, we get involved with sensations and emotions, subjective understandings, but also with language, the different forms of expression and communication. With DGBL [15], people learn from their experiences, receiving immediate feedback while playing the game, identifying and evaluating the "mistakes" made and trying to see where their expectations failed in order to change course and find other "ways out".

Playing Slice Fractions opens up an understanding of the mathematical object “fractions,” given the mechanics of the game.

In school terms, fractions are a subject that has been dealt with since early childhood education (as an idea) and in the early years of basic education, unfolding in terms of systematization and language. With the game, the subject can be explored using digital technology and the meaning that the mathematical object fraction gives them while they play the game.

IV. THE SLICE FRACTIONS GAME AND THE POSSIBILITIES FOR TEACHING FRACTIONS

Reference [19] present a broad summary of research that recounts the challenges of teaching fractions from the end of the 19th century to the 20th century, indicating the importance of a historical look at the teaching of fractions and the developing school culture.

[...] this historical perspective, which considers the epistemological point of view of school mathematics, stimulates a reading of these problems and suggests revisiting the mathematics of teaching fractions at different historical moments [19].

Among the challenges pointed out by [19] for exploring the idea of fractions are the different meanings of fractions. Other researchers [20] [21] [22], among others, argue that these meanings present fractions as "number, part-whole, measure, quotient and multiplicative operator" [22].

- Fraction as number - fractions, like integers, are numbers that don't necessarily have to refer to specific quantities. There are two forms of fraction representation: ordinary and decimal;
- Fraction as a part-whole relationship - the idea is to partition a whole into n equal parts, where each part can be represented as 1/n ;
- Fraction as a measure - some measures involve fraction because they refer to intensive quantities, in which the quantity is measured by the relationship between two variables. For example, probability;
- Fraction as a quotient - indicates a division and its result;
- The fraction as a multiplicative operator - like the whole number, fractions can be seen as the scalar value applied to a quantity. In the case of the integer, for example, we

could say 2 bullets; in the case of the fraction, we could say 3/4 of a set of bullets.

It is possible for the same fraction to be written in different ways, because the equivalence of fraction, ratio and proportion allows rational numbers to be presented on different scales:

$$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}...$$

Assuming that mathematical knowledge is constituted in the subject's encounter with mathematical objects, exploring different problem situations in which different meanings can be observed is a possibility to signify fractions, once the person is willing to look at what is shown with the intention of understanding.

A report on an experience with solving problems with the sum of fractions and in the students' resolution, meanings emerged such as the equivalence of fractions and part-whole, expressed with pictorial (geometric) representation. In the experiment, they observed that "the main results regarding the students' difficulties in the problem-solving stages revealed that the strategy chosen (planning) was unique and that its execution was incorrect" [23]. This type of "task", in general, although it has a problem-solving proposal, does not open up the possibility of exploring fractions in a different way or with different meanings due to the lack or precarious presence of constant feedback so that students are able to "see the error of their hypotheses" and try again in a new way.

In the Slice Fractions game, when the player "makes a mistake", he starts again by drawing up new strategies (plans), being open to discoveries. They recognize their mistakes and learn from them, seeing them as part of the process. The game gives you the chance to make problems and solve them. The player solves puzzles to move forward in the game.

A question that accompanies the player at each level is "what do I have to do here?." In this way, the subject of the action, who opens himself up to the game by making "discoveries", is looking at meanings and constant resignification.

In the game we are invited to experience what appears to be a challenge, and in the dialog with the other, the co-subject, we see that "something still needs to be given" and opens up to new explorations and meanings. If we consider, for example, equivalence relations (Fig. 2), talking about what happens to us in the action of playing, exposing the meanings that are manifested, is fundamental for the professor to organize questions in the classroom that can lead what is seen towards a mathematical formalization/ systematization.

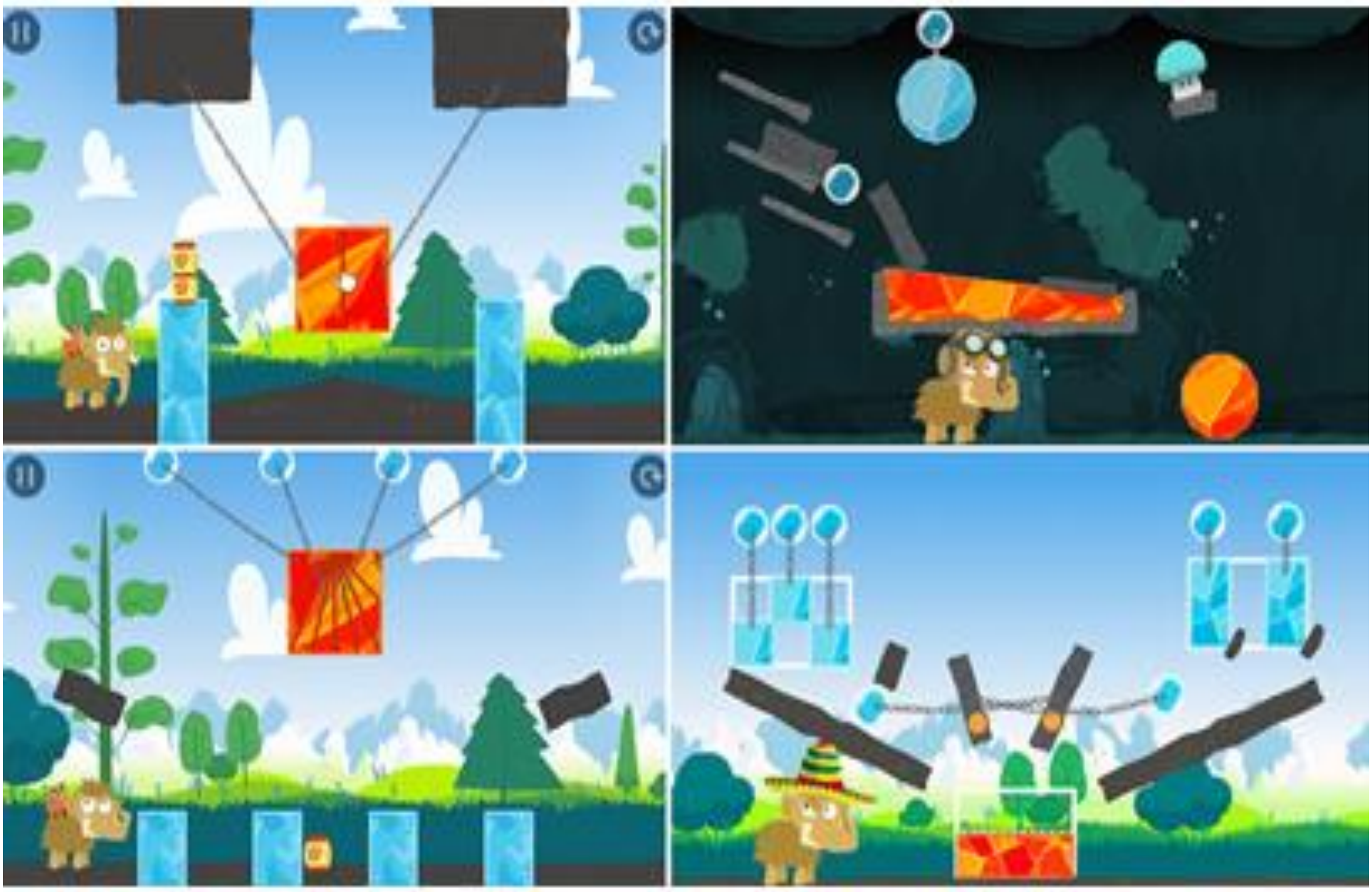

Fig 2 Screenshots from the Slice Fractions Game. Initial Levels with Representations of Fractions with Equality and Equivalence.
Source: Author's composition.

This way of being with the game, open to possibilities, was evident in the students' playing. In the first meeting, the students/players freely played the game Slice Fractions/Math Makers. They played without any intention of looking at the possibilities that the game opened up for the content to be covered in class. Indirectly, they were evaluating the interactivity, engagement and fun provided by the game; evaluating the game design and its level design as well as its mechanics and rules.

These aspects were shared in the dialog with the professor/researcher at the end of the first meeting. Even though there were elements that referred to aspects of the lesson with the game, these dialogues were aimed at the role of student/player, in which the focus was on playing the game (part 1 of the task proposal).

In the second meeting, with the proposed tasks (Chart 1), the undergraduates played in an attempt to identify specific aspects of the game (part 2 of the proposed tasks). They highlighted the game's metaphors in relation to fractions, their representations and how these were revealed in the game.

We have transcribed and organized (Chart 2) two aspects of the dialogue that took place in the second meeting:

Table 2 Dialogues and Organizations.

Dialogues	Math students	Professors
How do fractions appear in the game?	Pictorially (drawings); With mathematical symbology.	
Meanings of fractions	Fraction as a measure, "size", quantity or equivalence; Part-Whole; Equivalence of areas; Value or fraction as a number; Sum of fractions	Understanding fractions as "size" (quantity); Part-Whole; Equivalence and equality; Fractions as division;
Other comments	There is an order to the levels	There is a "logic" to solving each level

Source: Prepared by the author.

The first aspect that caught the participants' attention was the idea of area equivalence expressed by the geometric shapes (Fig. 3). The way in which, during the game, the shapes had to be fractioned (cut) varied in size and appearance, in a move towards the meaning of fractions. When we talk, for example, about the fraction 1/4, its meaning doesn't change regardless of the quantity it refers to. Although 1/4 of 100 is equal to 25 while 1/4 of 1000 is equal to 250, the value 1/4 (the number) remains a fraction-of-quantity.

Reference [24] When we can put a number on a quantity, we call it a magnitude. The number shows the size of the quantity and this comes from measuring it, i.e. the quantity shows how big the amount is.

Therefore, the number 1/4 represents the extent of a magnitude that can be determined from different measurements and even if the magnitude is different, the number (the quantity) remains 1/4. This generalization is neither easy nor obvious, as it expresses the meaning abstracted from what is perceived, in other words, an interpretation is added to what is perceived which is validated and perpetuated, being transmitted by culture over time.

As we've said, constituted knowledge belongs to the person who signifies and takes place through experience. In order to understand fractions as numbers, it is important for the person (student) to have the opportunity to appropriate aspects of reality that include, in the perception and gestures of the body, the meaning constituted by the subject-object encounter. The term meaning is presented as an existential orientation founded by the subject in the midst of their experiences [25]. The meaning that something assumes (or has) for a person has pre-linguistic and linguistic instances that start from the materiality of what is perceived by the bodily senses and move towards the complex conjunction of meaning as an existential possibility.

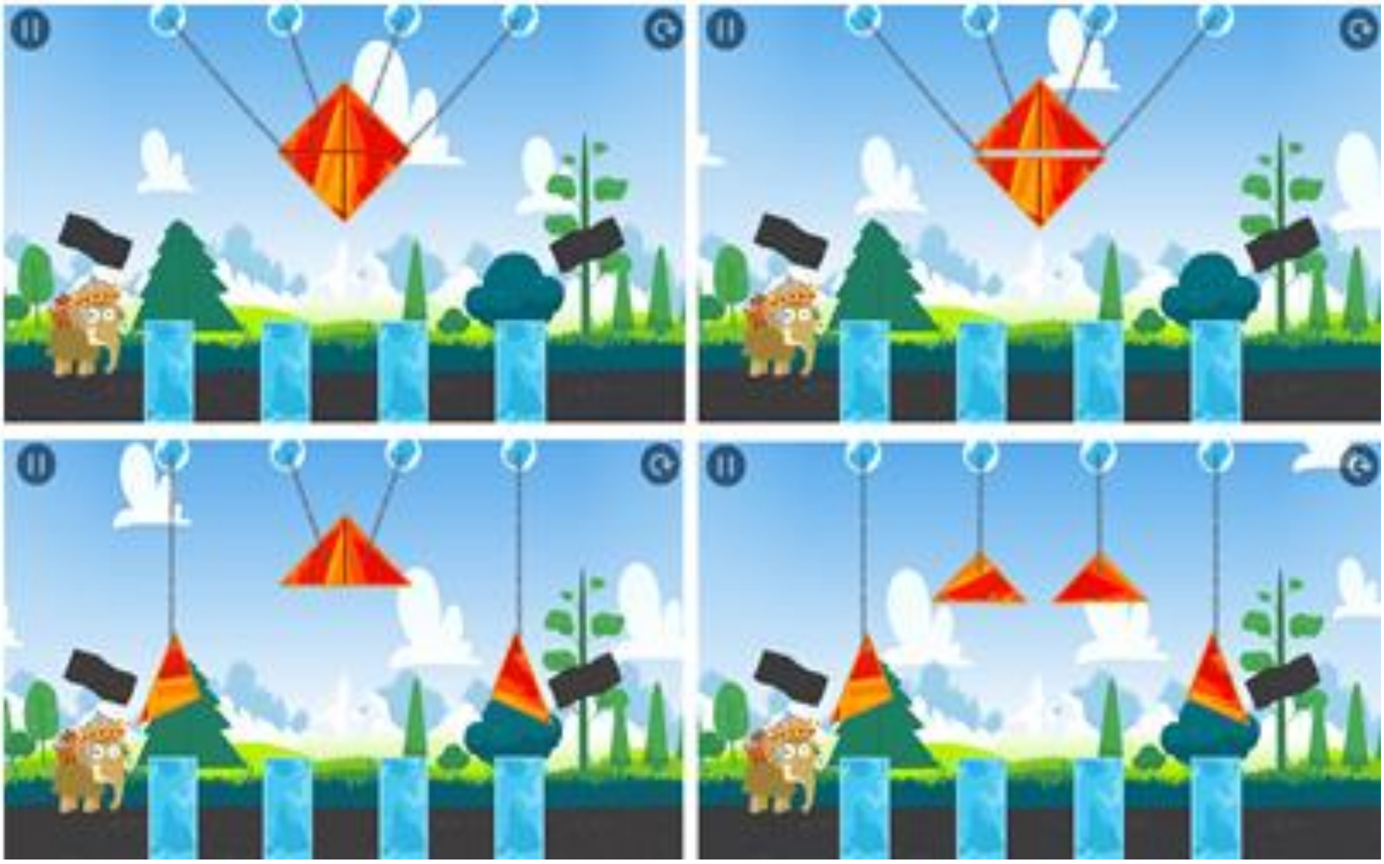

Fig 3 Producing 4.(1/4) to Solve the Level.
Source: Screenshot from the game.

In the game, as we understand it, there is the possibility of an encounter, in which the relationship *noesis* (ways of understanding fractions) *noema* (the perceived fraction) takes place. The participants/players with whom we dialogued in our research evaluated the Slice Fractions game as fun and engaging, with a "refined and well-finished" game design.

In the dialogues about the game and playing it, students and professors identified that "getting it wrong" is not a problem, and even if you get it wrong many times, the "game gives you a hint". This is a positive way of dealing with mistakes, especially when doing mathematics. The pleasure of playing is wrapped up in the clarity that making mistakes is part of the game, as it should also be understood as part of the constitution of knowledge.

In the dialogues with the participants, the importance of transposing this way of dealing with error into the classroom became evident; "I can make mistakes and I can start again! ", they said. The positive effect of actions manifested in playing games is discussed by Jesper Juul [26], who considers that games offer freedom to make mistakes and sometimes "making mistakes" can be more fun than "getting it right the first time", as it involves considerations and hypotheses in the solution and the re-evaluation of strategies.

In the game, there is a gradation in the progression of each level, but at times, the player is "taken by surprise" with emerging challenges that offer a different level that makes them "stop and think". This characteristic in a game doesn't allow the player to jump into the game automatically. This is the concept of the cadence of levels, which is progressive and emergent [27].

In Fig. 4 we illustrate two levels of Slice Fractions that are linked by a theme of skill and challenge[7]. At the initial level, elements such as fraction equivalence (part-whole) and pictorial representation are part of the skill theme. When we move on to another level there is "something familiar", although something new is also inserted: figure and symbol. The skill theme has been extended due to the cadence offered by the game. This structure of the game illustrates what we discussed about what is revealed and not revealed (hidden); what is shown to one subject can be hidden from another. Dialogue is therefore important for articulation, to express what is seen (revealed) and to be able to embark on the adventure of "discovering" what is hidden. It is in this movement of what is perceived, understood, expressed and (re)seen that we have the (subjective) constitution and production (articulated in language) of mathematical knowledge.

[7] Reference [28] indicate for the production of game design described the method known as "challenge, cadence, skill-theme". A challenge is an Errands (short tasks), and a cadence is the way challenges link together, preserving the skill developed and proposing evolutions and extensions to challenges present in the same level or establishing links between levels [29].

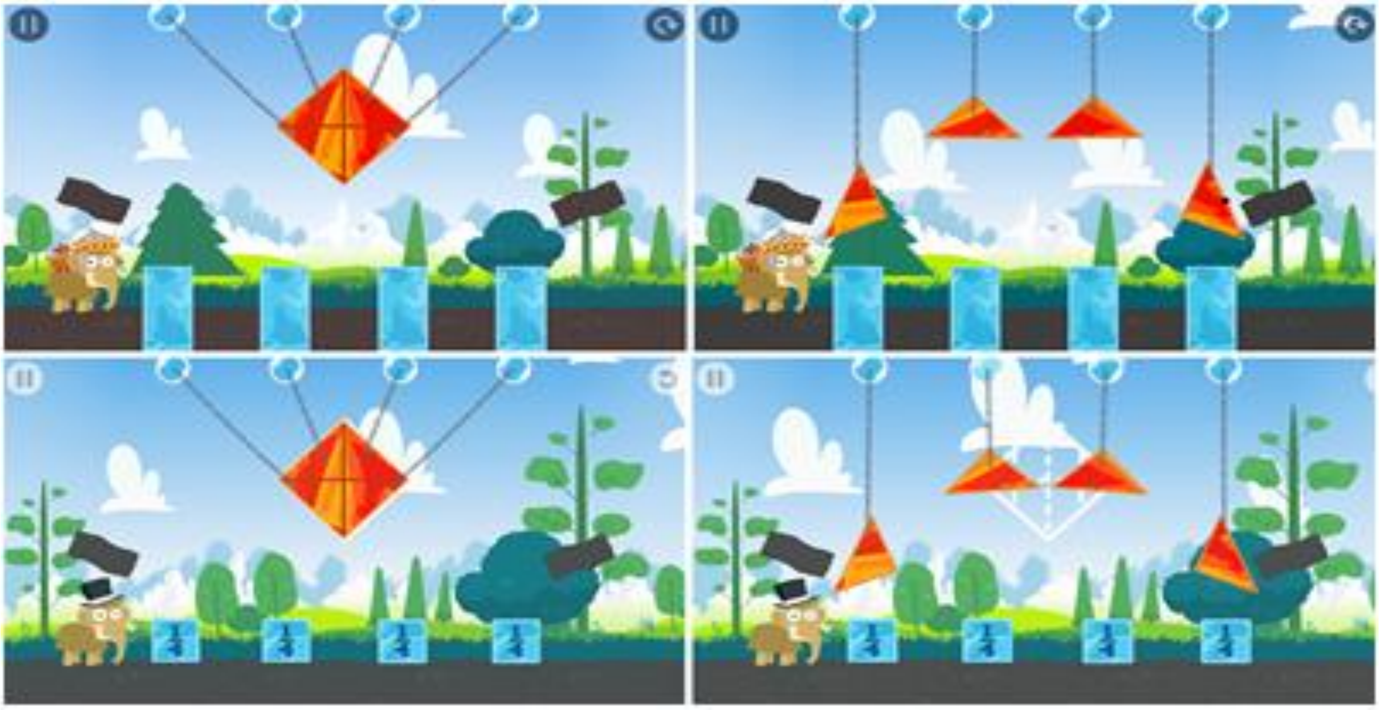

Fig 4 Challenge, Cadence and Skill Theme in Slice Fractions.
Source: Screenshot and composition by the author.

These levels in Fig. 4 are not sequential, we've just organized them in this way to show how the game design technique was applied.

Reference [15], James Paul Gee says that learners need many different opportunities to put their previous experiences into practice and make connections with new and similar situations, in order to improve their interpretations of things.

Slice Fractions offers the opportunity to "debug" and improve interpretations or meanings through dialog. The dialog that emerged from the gamified activity (Proposed tasks, part 2) shows that playing the game is not about solving mathematical operations with fractions, but about solving puzzles that are wrapped up in fractions, in other words, a process of investigation is required when playing.

One of the aspects highlighted by the groups was fractions as numbers, in other words, the possibility of understanding fractions as a mathematical object - numbers - and then delimiting this set and its operations.

The game presents the operation of adding fractions directly and objectively. It allows fractions to be understood as a number, therefore designating a quantity that can be verified in the equivalence of different shapes with the same area (Fig. 4) - in this example the fraction 1/4.

Considering the fraction as a number extends its mathematical representation, given by:

$$Q = \left\{ \frac{a}{b} \; / \; a; b \; \in Z, b \; \neq 0 \right\} \text{ (Set Q of Rational numbers).}$$

When analyzing this "aspect" of the fraction, the undergraduates point out that at school "many understand [the expression] as a number above and another number below, but they don't [identify] the fraction as a number, a single thing!". This observation brings us to a "speech" that is common in the classroom and is often not analyzed.

For example: when writing the fraction 2/3 (read two thirds) it is common to read "two over three". This "reading" is ambiguous, to say the least, if we compare it with another mathematical object expressed by 3^2 (wouldn't we also have "one number above and one below" or "two over three"?). Also, a frequent mistake that students make when adding fractions could be related to this "way of reading", as the phrase refers to "one number above and one below". So why couldn't we add these numbers to get:

$$\frac{1}{2} + \frac{3}{4} = \frac{4}{6}? \text{ (Incorrect way of adding fractions)}$$

Another aspect highlighted was the role played by the professor. The participants understood that the game is a free activity and that playing must be an invitation accepted by the players. However, what should be done if the invitation to play is not accepted by the class? Some experiences were shared, reinforcing the importance of maintaining freedom for the activity, always understood as an action by the student who is willing to do it. You can even associate the activity with the game with others such as those using manipulative materials, or even problem-solving, but what do you do when the invitation isn't accepted? Certainly, the criterion cannot be one of punishment, but the professor must have the ability to "deal with the situation", without despair, because if the task is attractive it will trigger a certain amount of activity from the student.

In the dialogue with the participants, undergraduates and professors, it became clear that the game offers opportunities to plan and carry out many lessons linked to various situations. It was pointed out that approaching and systematizing concepts/ideas through the action of playing requires a large number of lessons, which must be well planned. In addition, you can always return to the game in class and use the players' experiences as an example to solve some levels together. It's also important to give those who want to share their examples an opportunity to do so, so that the professor can use language to present the systematizations required in a formal teaching context.

In this work, in addition to the experience with the group of mathematics undergraduate students and early years professors, we sought to map the meanings of fractions presented in the game Slice Fractions (Math Makers) and analyze how we can translate its metaphors into the classroom. As themes related to the idea of fraction, we identified, in addition to those mentioned by the students, Fraction as Proportion and division; Comparisons between fractions; ordering and operations (addition, subtraction, multiplication and division).

In the levels shown in Fig. 5, it can be seen that explorations can go beyond the initial level. At first, the action requires cutting out parts of the picture that are equivalent to 1/4, so the idea that emerges is one of equivalence. However, in addition to the pictorial and symbolic representation of fractions, the subtraction and addition of fractions can be identified in some situations. Sometimes, the game requires the player to add fractions to overcome the obstacles that the mammoth must overcome. However, when cutting an integer, we may be subtracting from it or dividing it (when the "integer is a part", Fig. 8).

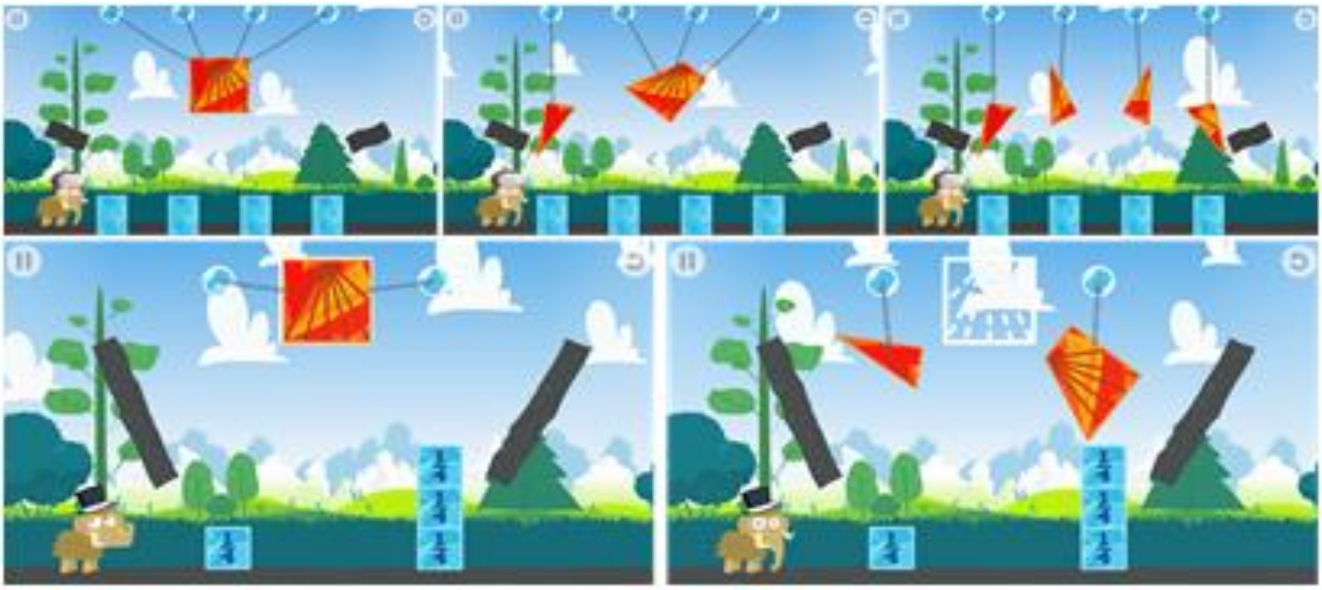

Fig 5 Equivalence and Representations of Fractions, Above Initial Level and Below (Figure and Symbol) and Subtraction and Addition of Fractions.
Source: Screenshot and composition by the author.

The Fig. 5 illustrates some of these possibilities and we organized in Table 3:

Table 3 Operations and Systematizations.

Operations with fractions	Systematizations or class themes
$1-\frac{3}{4}=\frac{4}{4}-\frac{3}{4}=\frac{1}{4}$	Integers as fractions; Subtraction of fractions.
$\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$ Or $3.\frac{1}{4}=\frac{3}{4}$	Sum of fractions; Multiplying a whole number by a Fraction; Fraction as a quotient and multiplicative operator.
If $1=\frac{4}{4}$ and $3=1+1+1$ Then $3=\frac{4}{4}+\frac{4}{4}+\frac{4}{4}=\frac{12}{4}=3$	Fractions as a quotient (apparent fraction).

Source: Prepared by the author.

Let's look at another level in Fig. 6. What does it show?

Fig 6 Solution for Fraction as a Proportion.
Source: Screenshot of the game and author's composition.

From what is shown in this level in Fig. 6, the fraction can be interpreted as Proportion. The solution to the challenge consists of releasing the block composed of the sum of fractions, because although you want the fraction 1/2, even if you try, there is no way of releasing the block with the symbol (½). Therefore, the fraction 1/2 must be obtained by adding fractions 1/6 and the proportionality must be identified.

Adding fractions: $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{3}{6}$

Or

Fraction as a quotient and multiplicative operator: $3.\frac{1}{6}=\frac{3}{6}$

Note that, when writing 3/6 = 1/2, the equality symbol (=) is an extrapolation, as it is being used as an equivalence; formally, we should write: 3/6 ≡ 1/2 (reading three-sixths is equivalent to half).

- *Let's remember that our players - graduate students and professors - said that there was a "logic" to solving each level, referring to the sequence of actions needed to succeed in the challenge. This aspect was classified as the process of solving a puzzle [28], which offers the player a space to test these sequences and hypotheses. In the case of the example in Fig. 6, we will have some steps:*

- Cut "?" so that the block containing the question mark is separated by 1/3;
- Cut the three blocks of fractions 1/6 ;
- Then with 3/6 the path is clear.

In this way, what is perceived is unfolded in other acts of consciousness, such as comprehension, interpretation and expression. The Fig. 7 provides an illustration of the process of perception and the expression of solutions to a problem, or even a problematization of an experience: "the object only determines itself as an identifiable being through an open-ended series of possible experiences" [30]. So the subject knows, including himself, in the midst of the acts he experiences in the world and in the gesture of solidarity of recognizing himself in the acts of those in whom he supports this style[8], as well as through the scope of intentional acts based on this style.

[8] Merleau-Ponty explains that Husserl introduced the notion of style to translate the ways in which the relationship with the world takes place, the means by which the expressive translation of the lived understanding of the world occurs.

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Fig 7 From Perception to Interpretation and Expression of Solutions to a Problem.
Source: composition by the author.

Paul Ricoeur[9] proposes a distinction between figuring and imagining, to clarify the "increase" in the role of the imagination between one context and another. It is not a simple increase, it is an identification of imagination with philosophizing [...], mainly through "imaginative variations"[31]. "It is through the very power of fiction that natural belief is put at a distance and that fact is subjected to the imaginative variations that reveal the eidetic invariant. In both cases, the imaginary is the 'empty house' that allows the play of meaning to begin" [31]. By playing and encountering new challenges, the process of imagination can be favored and in this movement, understanding takes place.

In other situations in the game we come across challenges involving the division of fractions and the product when the "whole is a part" (Fig. 8), i.e. what I want to divide is a fractional value, and these can also be aspects to be dealt with in a lesson topic. We give an example in Fig. 8.

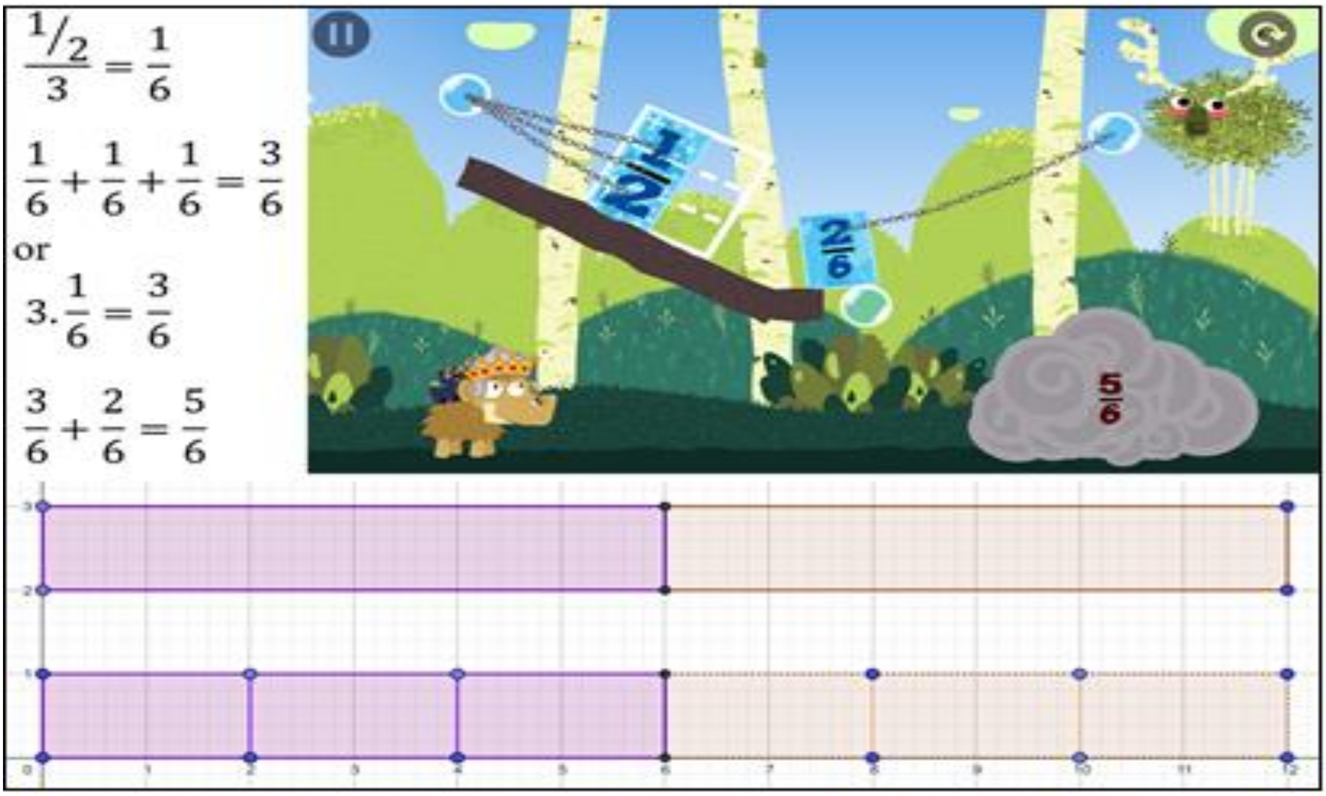

Fig 8 Example of a Level with Division of Fractions and Possible Formalizations and Example of Geometric Resolution Dividing Half into Three Equal Parts Means Dividing the whole into Six Equal Parts.
Source: Screenshot of the game and Geogebra online. Author's composition.

[9] Paul Ricœur was one of the great French philosophers and thinkers of the period following the Second World War. He developed contributions to phenomenology and hermeneutics, in constant dialog with the human and social sciences.

We understand that this way of dealing with fractions, from the early years onwards, provides students with the opportunity to build knowledge, since it gives them the chance to explore, make mistakes, argue, dialog, try again and see what makes sense in the act of playing. We debate that the presence of games in educational spaces and, in particular, the way people interact with digital technologies expands the way knowledge can be produced in the age of technology. [7].

It is important to give students the opportunity for dialogue and to provide space for them to share moments of discovery, since expression is a way of organizing what is perceived and interpreted.

We note that, in formalizing the proposed task, the professor can insert dialogues that denote the translation of the game's metaphors and then explain or make explicit what is done, such as cutting a half into three equal parts.

In the Math Makers module "Fractions - Advanced" we find fractions greater than "1" (Fig. 9), either by representing apparent fractions, when the division (quotient) between the numerator and denominator is equal to a whole number, or mixed fractions, also known as a mixed number, a number represented by a whole part and a fractional part.

Fig 9 Example of a Level with Fractions greater than "1".
Source: Screenshot of the game and author's composition.

A type of "flying narwhal" marks a possible cut, indicating the "size" of the fraction generated by the cut. In the case shown in Fig. 9 we have:

Fraction as a quotient: $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{6}{3}=2$ (apparent fraction)

Representation as a mixed fraction: $1\frac{2}{4}=\frac{4}{4}+\frac{2}{4}=\frac{6}{4}\equiv\frac{3}{2}$ (equivalence)

Therefore, $1\frac{2}{4}\equiv\frac{3}{2}$

The subject of fractions has been part of Brazil's curriculum since the National Curriculum Parameters - NCP [32], which indicate and advise that in elementary school (around age 8-9), three meanings of fractions should be worked on: part-whole, ratio, and quotient; at the end of elementary school (around age 10), the meaning of multiplicative operator could then be introduced, as could equivalent fractions.

Although working with the set of rational numbers is not trivial - if the sequence of natural numbers allows you to choose a number and establish its successor and predecessor, this is not true for rational numbers, because between any two rational numbers you can always find another rational number - fractions offer a fundamental approach to understanding the ordering aspects of the set of rational numbers and, consequently, the premise of continuity or the basis for infinitesimal calculus (limit theory), which shows their importance for mathematical learning.

V. FROM THE RESEARCHER/PLAYER PLAYING PROCESS TO PLAYER ENGAGEMENT AND GAMIFIED ACTIVITIES

When choosing a game for the classroom, the professor needs to become a player. Then, by playing unpretentiously, the professor/player can (indirectly) evaluate their engagement and factors such as fun and immersion until they "finish the game". After this stage, the professor/player can play again, evaluating the elements pertinent to the translations of the metaphors offered or produced by the game. Taking notes on the levels, screenshots and short videos of the gameplay can help the professor in the process of creating gamified activities resulting from the game. In our case, with Slice Fractions, the possibilities are for teaching fractions.

In his researches, Professor Nelson Zagalo [33] delineated a classification of three broad types of 'player profiles' (Chart 4) to see what drives a player to play a game and what conclusions they seek from playing. This process and characteristics of Engagement Design favors the production of games of all kinds, particularly educational games. Understanding the player's motivations when playing indicates an involvement, an acceptance to enter and participate in the game. The greatest complexity faced in developing this model is due to two distinct issues: transdisciplinary and cognitive bias [33].

Table 4 Player Profiles and Elements for Engagement

Player profiles	"Concerns" in the game and context	Artifacts or actions - flows
The Abstractors: concerned with patterns, structures and causes, interested in universal rules or generalizations.	- to know - Progression: moving on to bigger and more difficult challenges;	Puzzles or obstacles to achieve objectives. Flow: Progression
	Context: problem solving.	
The Tinkerers: driven by new sensations and experimentation, with a desire to build, create and make.	- to do - Expression: understand how it works, "disassemble and assemble" understand the connections between the components of a challenge;	Simulations and creation tools including other media such as cardboard, modeling clay. Flow: Expression.
	Context: achieving a new system made up of unknown ideas.	
The Dramatists: attracted to humans and their stories, interested in all forms of socialization.	- to feel - Relationship: the motivation lies in the story of the characters and their relationships, their desires and aspirations.	Human conflicts generate change. Flow: Relationship
	Context: familiar characters and generating empathy.	

Source: Author's interpretation and organization based on Zagalo's model. Prepared by the author.

This proposal extends to the development of games in the most diverse areas and activities, including health; training; education and commercial.

The model consists of three streams linked to the three profiles. However, the core design, like the classification of the streams, are not confined by profiles, the model aimed at defining engagement flows, at understanding and defining the actions of interaction. In other words, each stream presents a pair of interactive elements to support the definition of Profiles, Contexts and Artefacts, which then drives the interaction design through one of the three streams: Progression, Expression or Relation [33].

The flows are linked to the player profiles (Fig. 10) and define the main elements or desirable actions, either for a game or for gamified activities resulting from a game.

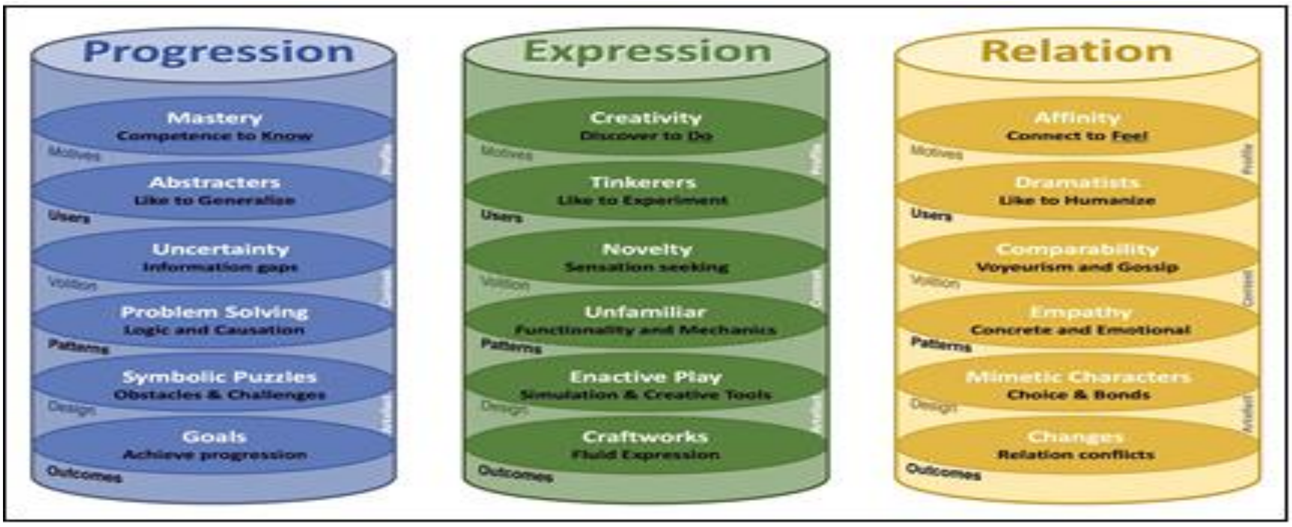

Fig 10 The Flows of the Engagement Design Model [33].

The game Slice Fractions has a subjective narrative, which means that if the player has the profile of an abstractor, they will hardly be concerned with the characters or their relationship with the challenges. On the other hand, a player whose profile is that of a playwright may "tire" of progressive gameplay, as they can't find answers to questions that explain the motivations of the characters and the protagonist. These characteristics allow us to establish relationships with the game and extend it to other gamified activities such as Part 2 (in addition to the one presented in the methodology) with a view to engaging players who had not been contemplated (Chart 5), such as playwrights (Dramatists) and Tinkerers (artisans or craftsmen).

Table 5 Proposals for Extra Activities Involving Engagement Design

Dramatists	Tinkerers
Relation: Story → conflict Focus on the narrative Who is the mammoth? What is his story? Why was it frozen? What is the mammoth looking for? Why does he like hats? Who are the other characters? How do they interact with the mammoth? In advanced fractions: Who is the snail? Why did the snail take the mammoth's cap? Why is the cap important to the mammoth?	*Expression*: *Game* → problem Focus on rules and mechanics Explain how a level works - step by step Can I create a different challenge to the ones I have overcome? Recreate a level using cardboard (or graph paper); Creative mode: create levels to challenge your classmates.
Scriptwriting; Propose the production of short stories - expressed in written text, comics, cartoons, etc. - inspired by the game's characters and setting. You could also choose a secondary character to tell your story. How did he get to mammoth?	Creating challenges; Choose manipulative materials that can help solve the challenges in the game (for example, dividing fractions).

Source: Prepared by the author.

In the game the "storytelling" process must be participatory, i.e. the player is a kind of spectator/participant and the narrative expresses their actions in the game. The events they "stage often have the immediacy of personal experiences" [34]. With this, action in the game has the character of "being with", because the game is a space for experience and "the media is, in this movement, updating itself, mediating formal linguistic instruments and potentializers" [35].

Cinema doesn't need to spend time telling the details that are shown. The world of the story is given, already constructed, to the receiver, providing a direct perception of the visual world that increases emotion and learning. With video games, storytelling activates an even wider range of cognitive skills and learning possibilities. The story is no longer just an act to be told or shown, but rather an integrated set of active participation, in other words, "action" [36].

This narrative experience made possible by the digital environment allows interlocutors to construct their own stories so that, we can "conceive of this environment less as a story and more as a narrative world capable of supporting many possible stories" [34]. Each player, in their actions, tells a story in this narrative world opened up by playing the game and, by proposing gamified activities (Chart 5), there is a dialogue with different learning flows as well as different profiles of players who, by playing, experience their own narratives and create their own "narrative world".

VI. CONCLUSION

We consider that the real and the virtual are not contrary or separate, but that the virtual is one of the modalities of reality; the cyber world is understood as a modality of the lifeworld. As such, "[it is] where we are, being when we carry out actions together with the co-subjects who are with us in space-time materiality, therefore historical" [9].

- *In this modality of the world of life, experience is equally important, which leads us to highlight, for working with games in the classroom, three fundamental and articulated aspects that should be the focus of the professor/player's attention:*

- The time to play the game freely - game in the classroom - in other words, providing the opportunity to meet and invite people to be with the game and explore it freely as a player;
- The game in the classroom is not closed in on itself, but requires connections between the experience of playing and the systematizations planned for educational spaces; the game for the sake of the game does not teach mathematics, it teaches how to play, it teaches the game [13];
- The professor proposes that, when playing the game, some aspects of this activity should be recorded, such as: the objective and the rules of the game (metaphors). These aspects, when shared, open up to discussion, offering the opportunity for meanings that go beyond the

game towards the systematization of lesson themes (development of lesson plans) that are not isolated, but are articulated with the experiences of playing the game.

This work points out that action in digital games demands the potential for problem-solving, problematization of situations and contextualization, and is relevant to mathematical development, since it involves data collection, analysis, conjecture, quantification and hypothesis testing [37].

The freedom provided by the activity of playing and the possibility of sharing "discoveries" can promote engagement and become an opportunity to insert gamified activities that lead actions towards the formal elements of mathematics teaching. In the case of this text, the Slice Fractions game makes it possible to systematize ideas relating to fractions and their operations, "[games] can [...] use the full force of storytelling mechanics, making use of a tripartite combination of telling, showing and doing" [36] makes us understand that these are important activities in the school context.

An offshoot of the research initially proposed - with undergraduates - led us to an activity with professors who teach mathematics in the early years in a municipal public school. The actions were proposed in a similar way to what was done with the Mathematics degree students. The intention was to analyze the feasibility of working with practicing professors in the classroom. As this is a continuing education proposal, in addition to exploring the Slice Fractions game, the intention is to discuss fractions with manipulative materials (such as tangram and two-color cards), opening up the opportunity to discuss continuous and discrete models.

So far, the research has presented a possible path for activities involving games in the classroom. When we play a game is always a "being played" [38], because the game only becomes "fully the game" with the presence of the player. The technologies of digital games or games become this space for a "being played", enabling the constitution of knowledge.

In playing the game, thinking, acting, doing, planning, are all acts of the same movement aimed at solving a problem or overcoming a challenge based on a situation that opens up in perception. The game disposes the subject to... makes them attentive to... In this disposition, he plays, investigates, dialogues and launches himself. The movement of creative thinking that gives meaning to actions is established [7].

When playing games, it is considered that players always learn in specific contexts, because they learn from the embodied experiences they have in the world of life, in which the virtual is just one modality. Regardless of the school year (as a class organization), it is the player, i.e. the subject of the action, who constitutes knowledge in the encounter with mathematical objects. What is constituted, in the subjectivity of the one who plays, who faces the challenges, who overcomes the obstacles with their own strategies, is, through language, articulated and expressed, shared, (re)known, revealed in the act of playing with the other, a companion with whom one dialogues. With this, the game is no longer a "solitary game", it is collective and is being played in an encounter of shared discoveries.

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