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Numerical Dual Affine Spaces of Desargues

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Abstract:- Dual affine spaces are geometries with points and lines, lines have three points and, at most, a line passes through two points. Furthermore, we have that its planes are the duals of the affine plane over the field of two elements. If the space is connected, numerical invariants are associated with it. Let n be the number of points in space and k be the number of points that, given a fixed point, are not collinear with it. In this research we characterize the geometric spaces that satisfy the Desargues property "Every pair of non-collinear points has exactly four collinear points." represented by pairs of numbers (n, k) that satisfy certain algebraic properties studied by D. Higman (1964), H. Cárdenas (1999, 2001, 2002) and J. Castañeda (2011, 2020).

Keywords:- *Dual Affine Space, Desargues Configuration, Numerical Dual Affine Spaces, Desargues Spaces.*

I. INTRODUCTION

In 1964, D. Higman introduced some algebraic properties (1-4) that define certain types of finite graphs containing dual affine spaces. These spaces have been investigated by H. Cárdenas, E. Lluís, A. G. Raggi-Cárdenas, R. San Agustín (1999, 2001, 2002). In 2011, J. Castañeda introduces the notion of numerical dual affine space by adding properties (5-6).

- *Let (n, k) a pair of positive integers with n>k. We define the numbers*

$$l = n - k - 1, \mu = \frac{3(k+1)-n}{2}, \lambda = k - 1 - \frac{l\mu}{k}$$

$$h = \frac{l-(k-\lambda-1)}{4}$$

- *Let us consider pairs of numbers (n, k) that satisfy the following conditions:*

- *μ, λ, h, l are positive integers and n>k>μ, λ.*

$$(\lambda-\mu)^2 + 4(k-\mu) = \delta^2$$

Where δ is a integer number.

- *δ divide to D, with*

$$D = 2k + (\lambda-\mu)(l+k)$$

- *If n is an even integer, then 2δ it does not divide D, if n is odd we have 2δ to divide D.*
- *If n is less than or equal to 36, the pair of numbers (n, k) is one of the list: {(15,6), (21,10), (28,15), (36,15)}. These couples are called primitive.*

To each pair of numbers (n, k) that satisfies conditions (1-4) we can associate a pair (n´, k´) of numbers using the function

$$D:\{(n,k)\} \to \{(n',k')\}$$

Defined as

$$D(n,k) = \left(\frac{3(k+1)-n}{2}, \frac{3(\lambda+1)-k}{2}\right)$$

- *If in the pair (n, k) the number n>36 there exists a positive integer q such that $D^q(n,k)$ it is a primitive pair*

- ***Definition 1****. A numerical dual affine space is a pair of numbers (n, k) that satisfies conditions (1-6).*

For brevity, I will henceforth call numerical dual affine spaces, affine spaces.

If (x, y) is an affine space and D(x, y)= (n, k) we will say that (x, y) is the successor of (n, k). In these spaces, the successor is not always unique.

- ***Lemma 1****. If (n, k) is a pair of numbers that satisfies (1), then*

- $lu = k(k-\lambda-1)$
- $l = 2(k-\mu+1)$.
- $l(l-2) = 8hk$.

- **Proof.** It follows directly from the definitions of μ, λ, h, l.

From the above we obtain,

- ***Proposition 1.*** *Let (n, k) be a pair of numbers that satisfies (1) with h=1, then there exists an integer t greater than or equal to 6 such that*

$$n=\binom{t}{2},\quad k=\binom{t-2}{2},\quad \mu=\binom{t-3}{2},\quad \lambda=\binom{t-4}{2}$$

- **Proof.** It follows directly from the relations of lemma (1).

II. DESARGUES SPACES

- ***Definition 2.*** *The pairs (n, k) with h=1 are called Desargues spaces.*
- ***Proposition 2.*** *If (n, k) is a Desargues space, then (n, k) is a numerical dual affine space. Furthermore, it is a subclass.*
- **Proof.** From the previous proposal,

$$n=\binom{t}{2},\quad k=\binom{t-2}{2},\quad \mu=\binom{t-3}{2},\quad \lambda=\binom{t-4}{2}$$

And (1) is fulfilled. The condition (2) follows from that is $\delta=t-2$ a positive integer for t greater than or equal to 3. For t greater than or equal to 5,

$$D=(t-2)(t-3)-(t-4)(n-2)=2k-(t-4)(n-2)$$

and

$$\frac{D}{\delta}=(t-3)-\frac{(t-4)(t+1)}{2}$$

Is a positive integer and (3) is satisfied.

The condition (4), since it $2\delta=2(t-2)$ does not divide D.

(5) The primitive pairs (15,6), (21,10) and (28,15) are Desargues spaces and the successor of a Desargues space with n>36 is another Desargues space.

From condition (6), For a Desargues space (x, y) there exists a positive integer q such that $D^q(x,y)=(28,15)$ or $D^q(x,y)=(21,10)$ or $D^q(x,y)=(15,6)$. Note that these spaces are of the form.

$$\left(\binom{t+3s}{2},\binom{t+3s-2}{2}\right)$$

With s a positive integer, t=8, 7, 6 and it is true that

$$D^s(n,k)=\left(\binom{t}{2},\binom{t-2}{2}\right)$$

III. CONCLUSIONS

Desargues spaces are a subclass of numerical dual affine spaces, since they satisfy conditions (1-6) with h=1.

The subclass of Desargues spaces with the successor operation and the set of primitive pairs {(15,6), (21,10), (28,15)} as elements that are not successors of any other is a closed subset regarding the successor operation. Additionally, for n>36, the successor of a Desargues space is another Desargues space.

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