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Estimation of Input Vector Pair from Embedded Space Vector Corresponding to the Framework Based on Spacer Component Matrices: A Constrained Optimization based Approach

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Abstract:- The research study involves estimation of an input pair of vectors $(x_{m\times 1} \in R^m, y_{n\times 1} \in R^n)$ corresponding to an embedded space vector $b_{s\times 1} \in R^s$ where " s " is the embedding dimension corresponding to the input dimension pair (m,n) .The estimation of the vector pair involves solving a euclidean norm minimization problem, constrained over a convex hull generated by a finite subset of solutions of an associated linear system of equations. The research initiative presents the mathematical formulation of the estimation framework and illustrates the presented methodology through appropriately chosen numerical case study examples.

Keywords:- *Spacer Matrix Components, Embedding Dimension, Embedding Matrices, Constrained Optimization, Convex Hull, Least Squares Estimation*

I. INTRODUCTION

The vector embedding methodology based on the framework of Spacer component matrices invokes compatibility among vectors belonging to coordinate spaces of non-compatible dimensions [6, 7, 8, 9, 10, 11, 12 and 13]. The compatibility is introduced by mapping the input coordinate vectors into the embedding coordinate space R^s, where " s " denotes the embedding dimension corresponding to the input dimension pair (m,n). The mapping into the embedded coordinate space using the Embedding matrices $G_{s\times m}$ and $W_{s\times n}$ preserves the Euclidean vector norm under the embedding transformations.

The primary objective of the present research initiative has been to address the inverse problem corresponding to the embedding approach based on spacer component matrices methodology, namely, given an arbitrary vector belonging to the embedded coordinate space, denoted as $b_{s\times 1} \in R^s$, estimate an input vector pair $(x_{m\times 1}, y_{n\times 1})$ where $x_{m\times 1} \in R^m$ and $y_{n\times 1} \in R^n$

The estimated vector pair is obtained from solution to a convex optimization problem [1, 3, and 17] which essentially involves minimizing the euclidean norm over a convex hull formed from a finite subset of solutions to an associated linear system of equations; the linear system associated with the estimation problem involves a Coefficient matrix $H_{s\times t}$ which is constituted by the spacer component matrices $P_{n\times s}$ and $Q_{s\times m}$ and utilizes the orthogonal projection approach associated with the least squares methodology [2, 4, 5, 16, 18, 19, 20 and 21] , this results in projection of the embedded vector $b_{s\times 1}$ onto $Csp(Q_{s\times m})$ when $m > n$ and onto $Csp((P_{n\times s})^T)$ when $n > m$.

The cardinality of the finite solution subset is determined by the critical dimension: $d = \min(m,n)$, the block-partitioning framework utilized in the present mathematical formulation, results in a linearly independent set and hence a full column rank matrix $\hat{Z}_{t\times d}$, the optimal solution vector $\omega^0{}_{t\times 1}$, which is appropriately block partitioned into the optimal input pair $(x^0{}_{m\times 1}, y^0{}_{n\times 1})$, is a convex combination of the columns of the matrix $\hat{Z}_{t\times d}$ which minimizes $\|\omega_{t\times 1}\|_2$ over the convex hull $\Omega(d) \subseteq R^t$.

The mathematical framework developed in the present research study has been numerically illustrated using suitable case study examples: the input dimension pairs considered are $(m=2, n=3)$ and $(m=3, n=2)$ for both of which the embedding dimension: $s=4$. The research study concludes with a section on discussion of insights obtained from case studies and possible directions for follow up research.

➢ *Notations*

- N denotes the set of all Natural numbers
- R denotes the set of all Real numbers
- (m,n) denotes the Input pair of dimensions
- 's' denotes the Embedding Dimension , 't' denotes the Sum Dimension and 'd' denotes the Critical Dimension
- $M_{\alpha\times\beta}(R)$ denotes the real matrix space of order 'α' by 'β'
- R^{ρ} denotes the real coordinate space of order 'ρ'
- $1_{\rho\times1}\in R^{\rho}$ such that $1_{\rho\times1}=\begin{bmatrix}1\\1\\.\\.\\1\end{bmatrix}_{\rho\times1}$, $0_{\rho\times1}\in R^{\rho}$ such that $0_{\rho\times1}=\begin{bmatrix}0\\0\\.\\.\\0\end{bmatrix}_{\rho\times1}$
- $u_{\rho\times1}=\begin{bmatrix}u_1\\u_2\\.\\.\\u_{\rho}\end{bmatrix}$, $v_{\rho\times1}=\begin{bmatrix}v_1\\v_2\\.\\.\\v_{\rho}\end{bmatrix}$ where $u_{\rho\times1}\in R^{\rho}$, $v_{\rho\times1}\in R^{\rho}$, therefore $u_{\rho\times1}\geq v_{\rho\times1}$ implies that $u_j\geq v_j$, $\forall j=1,2,....,\rho$
- $Csp(A_{\alpha\times\beta})$ denotes the Column Space of the matrix $A_{\alpha\times\beta}$
- $Nsp(A_{\alpha\times\beta})$ denotes the Null Space of the matrix $A_{\alpha\times\beta}$
- $\dim(V)$ denotes the dimension of the vector space V , where $V\subseteq R^{\rho}$
- $I_{\rho\times\rho}$ denotes the Identity matrix of order 'ρ'
- A^T denotes the Transpose of the matrix A
- $(X_{\rho\times\rho})^{-1}$ denote the Proper Inverse of the Invertible matrix $X_{\rho\times\rho}$, i.e. $(X_{\rho\times\rho})^{-1}X_{\rho\times\rho}=X_{\rho\times\rho}(X_{\rho\times\rho})^{-1}=I_{\rho\times\rho}$
- $X_{\rho\times\rho}\in M_{\rho\times\rho}(R)$ such that $X_{\rho\times\rho}$ is symmetric and positive definite , then $(X_{\rho\times\rho})^{-1/2}\in M_{\rho\times\rho}(R)$, $(X_{\rho\times\rho})^{-1/2}$ is symmetric and positive definite such that: $(X_{\rho\times\rho})^{-1/2}(X_{\rho\times\rho})^{-1/2}=(X_{\rho\times\rho})^{-1}$, we also have: $(X_{\rho\times\rho})^{1/2}\in M_{\rho\times\rho}(R)$, $(X_{\rho\times\rho})^{1/2}$ is symmetric and positive definite such that: $(X_{\rho\times\rho})^{1/2}(X_{\rho\times\rho})^{-1/2}=(X_{\rho\times\rho})^{-1/2}(X_{\rho\times\rho})^{1/2}=I_{\rho\times\rho}$ and $(X_{\rho\times\rho})^{1/2}(X_{\rho\times\rho})^{1/2}=X_{\rho\times\rho}$
- $convh(\overline{V})$ denotes the Convex Hull of the subset $\overline{V}$ of R^{ρ}
- $\left\|v_{\rho\times1}\right\|_2$ denotes the Euclidean norm of the vector $v_{\rho\times1}\in R^{\rho}$
- "LICQ" is the abbreviation for Linear Independence Constraint Qualification
- "p-Inverse" is the abbreviation for Pseudoinverse
- $\max(a,b)$ denotes the maximum of the two inputs 'a' and 'b'
- $|a-b|$ denotes the absolute value of the difference between the two inputs 'a' and 'b'

II. MATHEMATICAL FRAMEWORK

- *Properties of the Spacer Component Matrices and Set of Pertinent Results*

$m \in N$, $n \in N$ and $m \neq n$, we define the following: $s = \max(m,n) + |m-n|$, $t = m+n$ and we have $d = \min(m,n)$, therefore, we have the following implications:

- $s \in N$ *such that* $s > m$ *and* $s > n$, $t \in N$ *such that* $t > \max(m,n)$ *and we have* $d \in N$ $\forall m,n \in N$
  The Spacer Component matrices $P_{n\times s}$ and $Q_{s\times m}$ are formulated as follows:

$$P_{n\times s} = \begin{bmatrix} I_{n\times n} & (\frac{1}{n})1_{n\times 1}(1_{(s-n)\times 1})^T \end{bmatrix} , \quad Q_{s\times m} = \begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})1_{(s-m)\times 1}(1_{m\times 1})^T \end{bmatrix}$$

- *We have the Following set of Associated Results:*

- $rank(P_{n\times s}) = rank((P_{n\times s})^T) = rank(P_{n\times s}(P_{n\times s})^T) = rank((P_{n\times s})^T P_{n\times s}) = n$, therefore
- $Csp(P_{n\times s}) = R^n$, $Nsp((P_{n\times s})^T) = \{0_{n\times 1}\}$
- $Csp((P_{n\times s})^T) \subset R^s$, $Nsp(P_{n\times s}) \subset R^s$ where $\dim(Csp((P_{n\times s})^T)) = n$, $\dim(Nsp(P_{n\times s})) = s-n$
- $rank(Q_{s\times m}) = rank((Q_{s\times m})^T) = rank((Q_{s\times m})^T Q_{s\times m}) = rank(Q_{s\times m}(Q_{s\times m})^T) = m$, therefore
- $Csp((Q_{s\times m})^T) = R^m$, $Nsp(Q_{s\times m}) = \{0_{m\times 1}\}$
- $Csp(Q_{s\times m}) \subset R^s$, $Nsp((Q_{s\times m})^T) \subset R^s$ where $\dim(Csp(Q_{s\times m})) = m$, $\dim(Nsp((Q_{s\times m})^T)) = s-m$

- *The Embedding Matrices* $G_{s\times m}$ *and* $W_{s\times n}$ *are formulated as follows:*

- $G_{s\times m} = Q_{s\times m}[(Q_{s\times m})^T Q_{s\times m}]^{-1/2}$ therefore $(G_{s\times m})^T G_{s\times m} = I_{m\times m}$, $G_{s\times m}(G_{s\times m})^T$ is the Orthogonal Projector onto $Csp(Q_{s\times m})$ and $I_{s\times s} - G_{s\times m}(G_{s\times m})^T$ is the Orthogonal Projector onto $Nsp((Q_{s\times m})^T)$
- $W_{s\times n} = (P_{n\times s})^T[P_{n\times s}(P_{n\times s})^T]^{-1/2}$ therefore $(W_{s\times n})^T W_{s\times n} = I_{n\times n}$, $W_{s\times n}(W_{s\times n})^T$ is the Orthogonal Projector onto $Csp((P_{n\times s})^T)$ and $I_{s\times s} - W_{s\times n}(W_{s\times n})^T$ is the Orthogonal Projector onto $Nsp(P_{n\times s})$

- *The Relationship between Matrices* $P_{n\times s}$ *And* $Q_{s\times m}$ *:*
- Case: $m > n$

- $(P_{n\times s})^T = Q_{s\times m} J_{m\times n}$ where we have $J_{m\times n} = \begin{bmatrix} I_{n\times n} \\ (\frac{1}{n})1_{(m-n)\times 1}(1_{n\times 1})^T \end{bmatrix}$

- Case: $n > m$

- $Q_{s\times m} = (P_{n\times s})^T F_{n\times m}$ where we have $F_{n\times m} = \begin{bmatrix} I_{m\times m} \\ (\frac{1}{m})1_{(n-m)\times 1}(1_{m\times 1})^T \end{bmatrix}$

- *Formulation of the Coefficient Matrix and the Associated Linear System of Equations*

- The Coefficient matrix $H_{s\times t}$ is formulated as follows:
- Case: $m > n$

- $H_{s\times t}=\left[Q_{s\times m}\quad (P_{n\times s})^T\right]=\left[Q_{s\times m}\quad Q_{s\times m}J_{m\times n}\right]=Q_{s\times m}\left[I_{m\times m}\quad J_{m\times n}\right]$, therefore we have, $rank(H_{s\times t})=m$, $Csp(H_{s\times t})=Csp(Q_{s\times m})$, $Nsp(H_{s\times t})\subset R^t$ such that $Nsp(H_{s\times t})=Csp(\begin{bmatrix}-J_{m\times n}\\ I_{n\times n}\end{bmatrix})$, $\dim(Nsp(H_{s\times t}))=n$

- Case: $n>m$

- $H_{s\times t}=\left[Q_{s\times m}\quad (P_{n\times s})^T\right]=\left[(P_{n\times s})^T F_{n\times m}\quad (P_{n\times s})^T\right]=(P_{n\times s})^T\left[F_{n\times m}\quad I_{n\times n}\right]$, therefore we have, $rank(H_{s\times t})=n$, $Csp(H_{s\times t})=Csp((P_{n\times s})^T)$, $Nsp(H_{s\times t})\subset R^t$ such that $Nsp(H_{s\times t})=Csp(\begin{bmatrix}I_{m\times m}\\ -F_{n\times m}\end{bmatrix})$, $\dim(Nsp(H_{s\times t}))=m$

- *The Associated Linear Systems Are Formulated As Follows:*

- Case: $m>n$

- $b_{s\times 1}\in R^s$, we define $\hat{b}_{s\times 1}\in R^s$ such that $\hat{b}_{s\times 1}=G_{s\times m}(G_{s\times m})^T b_{s\times 1}$, we have $x_{m\times 1}\in R^m$, $y_{n\times 1}\in R^n$,
- We define $\omega_{t\times 1}\in R^t$ such that $\omega_{t\times 1}=\begin{bmatrix}x_{m\times 1}\\ y_{n\times 1}\end{bmatrix}$
- Linear system: $H_{s\times t}\omega_{t\times 1}=\hat{b}_{s\times 1}$ which implies $Q_{s\times m}\left[I_{m\times m}\quad J_{m\times n}\right]\begin{bmatrix}x_{m\times 1}\\ y_{n\times 1}\end{bmatrix}=G_{s\times m}(G_{s\times m})^T b_{s\times 1}$
- Case: $n>m$

- $b_{s\times 1}\in R^s$, we define $\hat{b}_{s\times 1}\in R^s$ such that $\hat{b}_{s\times 1}=W_{s\times n}(W_{s\times n})^T b_{s\times 1}$, we have $x_{m\times 1}\in R^m$, $y_{n\times 1}\in R^n$,
  We define $\omega_{t\times 1}\in R^t$ such that $\omega_{t\times 1}=\begin{bmatrix}x_{m\times 1}\\ y_{n\times 1}\end{bmatrix}$

- Linear system: $H_{s\times t}\omega_{t\times 1}=\hat{b}_{s\times 1}$ which implies $(P_{n\times s})^T\left[F_{n\times m}\quad I_{n\times n}\right]\begin{bmatrix}x_{m\times 1}\\ y_{n\times 1}\end{bmatrix}=W_{s\times n}(W_{s\times n})^T b_{s\times 1}$

- *Formulation of Solution subsets and associated Convex Hulls*

- Case: $m>n$

- We define the following set of vectors: $y(1)_{n\times 1}=1_{n\times 1}$, $y(2)_{n\times 1}=\begin{bmatrix}1\\ -1\\ 0_{(n-2)\times 1}\end{bmatrix}$, $y(3)_{n\times 1}=\begin{bmatrix}1\\ 1\\ -2\\ 0_{(n-3)\times 1}\end{bmatrix}$, ... upto $y(n)_{n\times 1}=\begin{bmatrix}1_{(n-1)\times 1}\\ -(n-1)\end{bmatrix}$

- We define $x(\theta)_{m\times 1}=-J_{m\times n}y(\theta)_{n\times 1}+[(Q_{s\times m})^T Q_{s\times m}]^{-1}(Q_{s\times m})^T\hat{b}_{s\times 1}$ for every $\theta\in\{1,2,....,n\}$, therefore we have the solution vector $\omega(\theta)_{t\times 1}\in R^t$ formulated as follows:

$$\omega(\theta)_{t\times 1}=\begin{bmatrix}x(\theta)_{m\times 1}\\ y(\theta)_{n\times 1}\end{bmatrix}=\begin{bmatrix}-J_{m\times n}y(\theta)_{n\times 1}+[(Q_{s\times m})^T Q_{s\times m}]^{-1}(Q_{s\times m})^T G_{s\times m}(G_{s\times m})^T b_{s\times 1}\\ y(\theta)_{n\times 1}\end{bmatrix}$$ Where we have $\theta=1,2,....,n$

- Case: $n > m$

➢ We define the following set of vectors: $x(1)_{m\times 1} = 1_{m\times 1}$, $x(2)_{m\times 1} = \begin{bmatrix} 1 \\ -1 \\ 0_{(m-2)\times 1} \end{bmatrix}$, $x(3)_{m\times 1} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0_{(m-3)\times 1} \end{bmatrix}$,

..... upto $x(m)_{m\times 1} = \begin{bmatrix} 1_{(m-1)\times 1} \\ -(m-1) \end{bmatrix}$

➢ We define $y(\theta)_{n\times 1} = -F_{n\times m} x(\theta)_{m\times 1} + [P_{n\times s}(P_{n\times s})^T]^{-1} P_{n\times s} \hat{b}_{s\times 1}$ for every $\theta \in \{1,2,....,m\}$, therefore we have the solution vector $\omega(\theta)_{t\times 1} \in R^t$ formulated as follows:

$$\omega(\theta)_{t\times 1} = \begin{bmatrix} x(\theta)_{m\times 1} \\ y(\theta)_{n\times 1} \end{bmatrix} = \begin{bmatrix} x(\theta)_{m\times 1} \\ -F_{n\times m} x(\theta)_{m\times 1} + [P_{n\times s}(P_{n\times s})^T]^{-1} P_{n\times s} W_{s\times n} (W_{s\times n})^T b_{s\times 1} \end{bmatrix}$$ Where we have $\theta = 1,2,....,m$

- We define $\hat{Z}_{t\times d} = \begin{bmatrix} \omega(1)_{t\times 1} & \omega(2)_{t\times 1} & . & . & \omega(d)_{t\times 1} \end{bmatrix}$, the associated convex hull, denoted as $\Omega(d)$, is formulated as follows:

$\Omega(d) \subseteq R^t$ such that $\Omega(d) = convh(\omega(1)_{t\times 1}, \omega(2)_{t\times 1}, ..., \omega(d)_{t\times 1})$, this implies

$\Omega(d) = \{\omega_{t\times 1} \in R^t \mid \omega_{t\times 1} = \hat{Z}_{t\times d} w_{d\times 1}, w_{d\times 1} \in R^d, (1_{d\times 1})^T w_{d\times 1} = 1, w_{d\times 1} \geq 0_{d\times 1}\}$

➢ *Properties of the matrix* $\hat{Z}_{t\times d}$ *and the associated matrix* $\hat{\Delta}_{d\times d}$

- Case: $d = m$

➢ $\hat{Z}_{t\times m} = \begin{bmatrix} x(1)_{m\times 1} & x(2)_{m\times 1} & . & . & x(m)_{m\times 1} \\ y(1)_{n\times 1} & y(2)_{n\times 1} & . & . & y(m)_{n\times 1} \end{bmatrix} = \begin{bmatrix} X_{m\times m} \\ Y_{n\times m} \end{bmatrix}$,

Where $X_{m\times m} = \begin{bmatrix} x(1)_{m\times 1} & x(2)_{m\times 1} & . & . & x(m)_{m\times 1} \end{bmatrix}$

The vectors $x(1)_{m\times 1}, x(2)_{m\times 1}, ..., x(m)_{m\times 1}$ forms an orthogonal basis for R^m which implies that $rank(X_{m\times m}) = rank(\hat{Z}_{t\times m}) = m$

➢ We define $\hat{\Delta}_{m\times m} = (\hat{Z}_{t\times m})^T \hat{Z}_{t\times m}$, therefore $\hat{\Delta}_{m\times m} \in M_{m\times m}(R)$ and $\hat{\Delta}_{m\times m}$ is symmetric, positive definite

- Case: $d = n$

➢ $\hat{Z}_{t\times n} = \begin{bmatrix} x(1)_{m\times 1} & x(2)_{m\times 1} & . & . & x(n)_{m\times 1} \\ y(1)_{n\times 1} & y(2)_{n\times 1} & . & . & y(n)_{n\times 1} \end{bmatrix} = \begin{bmatrix} X_{m\times n} \\ Y_{n\times n} \end{bmatrix}$,

Where $Y_{n\times n} = \begin{bmatrix} y(1)_{n\times 1} & y(2)_{n\times 1} & . & . & y(n)_{n\times 1} \end{bmatrix}$

The vectors $y(1)_{n\times 1}, y(2)_{n\times 1}, ..., y(n)_{n\times 1}$ forms an orthogonal basis for R^n which implies that $rank(Y_{n\times n}) = rank(\hat{Z}_{t\times n}) = n$

➢ We define $\hat{\Delta}_{n\times n} = (\hat{Z}_{t\times n})^T \hat{Z}_{t\times n}$, therefore $\hat{\Delta}_{n\times n} \in M_{n\times n}(R)$ and $\hat{\Delta}_{n\times n}$ is symmetric, positive definite

- *Estimation of the Optimal Input Vector Pair* $(x^0_{m\times 1}, y^0_{n\times 1})$

- The optimal vector $\omega^0_{t\times 1} = \begin{bmatrix} x^0_{m\times 1} \\ y^0_{n\times 1} \end{bmatrix}$ where $\omega^0_{t\times 1} \in R^t$, is estimated from the solution to the following optimization problem, denoted as $Opt.Pb$:

  ($Opt.Pb$) minimize $(\frac{1}{2})(w_{d\times 1})^T \hat{\Delta}_{d\times d} w_{d\times 1}$, subject to the constraints $w_{d\times 1} \in R^d$, $(1_{d\times 1})^T w_{d\times 1} = 1$, $w_{d\times 1} \geq 0_{d\times 1}$

- $w^0_{d\times 1}$ be a solution of $Opt.Pb$, the optimal vector $\omega^0_{t\times 1}$ is formulated as: $\omega^0_{t\times 1} = \begin{bmatrix} x^0_{m\times 1} \\ y^0_{n\times 1} \end{bmatrix} = \hat{Z}_{t\times d} w^0_{d\times 1}$, therefore, the vector pair $(x^0_{m\times 1}, y^0_{n\times 1})$ is the estimated input vector pair corresponding to the Embedded space vector $b_{s\times 1}$

- We define the Polyhedral set $\bar{\Omega}(d)$ as follows: $\bar{\Omega}(d) \subseteq R^d$ such that $\bar{\Omega}(d) = \{w_{d\times 1} \in R^d \mid (1_{d\times 1})^T w_{d\times 1} = 1, w_{d\times 1} \geq 0_{d\times 1}\}$, therefore the optimization problem $Opt.Pb$ can be restated as given below:

  ($Opt.Pb$) minimize $(\frac{1}{2})(w_{d\times 1})^T \hat{\Delta}_{d\times d} w_{d\times 1}$, subject to the constraint: $w_{d\times 1} \in \bar{\Omega}(d)$

- The optimization problem $Opt.Pb$ is a Convex optimization problem, the LICQ conditions are satisfied $\forall w_{d\times 1} \in \bar{\Omega}(d)$

- *The Karush-Kuhn-Tucker Conditions [14, 15] Associated with the Optimization Problem* $Opt.Pb$

- Objective function $\chi(w_{d\times 1}) = (\frac{1}{2})(w_{d\times 1})^T \hat{\Delta}_{d\times d} w_{d\times 1} = (\frac{1}{2})(\|\hat{Z}_{t\times d} w_{d\times 1}\|_2)^2$

  Equality constraint: $g(w_{d\times 1}) = (1_{d\times 1})^T w_{d\times 1} - 1 = 0$

  Inequality constraints: $\Pi_{d\times 1}(w_{d\times 1}) = w_{d\times 1} \geq 0_{d\times 1}$ where $w_{d\times 1} = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ w_d \end{bmatrix}$, $w_{d\times 1} \in R^d$

- Lagrange parameter associated with the equality constraint: μ , where $\mu \in R$

  Lagrange parameters associated with the inequality constraints: $\eta_1, \eta_2, ..., \eta_d$ we define $\eta_{d\times 1} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ . \\ . \\ \eta_d \end{bmatrix}$, $\eta_{d\times 1} \in R^d$

- The Lagrange function associated with the problem, denoted as $\Theta(w_{d\times 1}, \mu, \eta_{d\times 1})$, is formulated as following: $\Theta(w_{d\times 1}, \mu, \eta_{d\times 1}) = \chi(w_{d\times 1}) - \mu g(w_{d\times 1}) - (\eta_{d\times 1})^T \Pi_{d\times 1}(w_{d\times 1})$ which implies that

  $\Theta(w_{d\times 1}, \mu, \eta_{d\times 1}) = (\frac{1}{2})(w_{d\times 1})^T \hat{\Delta}_{d\times d} w_{d\times 1} - \mu((1_{d\times 1})^T w_{d\times 1} - 1) - (\eta_{d\times 1})^T w_{d\times 1}$

- The Karush-Kuhn-Tucker conditions associated with $Opt.Pb$ are the following:

- Stationarity condition: $\hat{\Delta}_{d\times d} w_{d\times 1} = \mu 1_{d\times 1} + \eta_{d\times 1}$
- Primal feasibility condition: $(1_{d\times 1})^T w_{d\times 1} = 1$, $w_{d\times 1} \geq 0_{d\times 1}$
- Dual feasibility condition: $\eta_{d\times 1} \geq 0_{d\times 1}$
- Complimentary slackness condition: $\eta_j w_j = 0 \quad \forall j = 1, 2, ..., d$

- *Numerical Case Studies*

- The numerical computations are performed using the Scilab 5.4.1 computational platform
- The Convex Quadratic Programming problem formulations are solved using the Linear Quadratic Programming built-in Solver: qp_solve

❖ $m = 2, n = 3$, therefore $s = 4$, $t = 5$ and $d = 2$

$$P_{3\times 4} = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}, W_{4\times 3} = (\frac{1}{6}) \begin{bmatrix} 4+\sqrt{3} & -2+\sqrt{3} & -2+\sqrt{3} \\ -2+\sqrt{3} & 4+\sqrt{3} & -2+\sqrt{3} \\ -2+\sqrt{3} & -2+\sqrt{3} & 4+\sqrt{3} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \end{bmatrix}, F_{3\times 2} = (\frac{1}{2}) \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix},$$

$$Q_{4\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, G_{4\times 2} = (\frac{1}{2\sqrt{2}}) \begin{bmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1-\sqrt{2} & 1+\sqrt{2} \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \mathrm{H}_{4\times 5} = (\frac{1}{6}) \begin{bmatrix} 6 & 0 & 6 & 0 & 0 \\ 0 & 6 & 0 & 6 & 0 \\ 3 & 3 & 0 & 0 & 6 \\ 3 & 3 & 2 & 2 & 2 \end{bmatrix},$$

$$\bar{\mathrm{H}}_{5\times 4} = (\frac{1}{60}) \begin{bmatrix} 24 & -6 & 9 & 9 \\ -6 & 24 & 9 & 9 \\ 31 & 1 & -14 & 6 \\ 1 & 31 & -14 & 6 \\ -14 & -14 & 46 & 6 \end{bmatrix}$$ Where $\bar{\mathrm{H}}_{5\times 4}$ is the p-Inverse of $\mathrm{H}_{4\times 5}$

We have $\hat{b}_{4\times 1} = W_{4\times 3}(W_{4\times 3})^T b_{4\times 1}$ and $\bar{\omega}_{5\times 1} = \bar{\mathrm{H}}_{5\times 4}\hat{b}_{4\times 1}$ where $\bar{\omega}_{5\times 1}$ is the least Euclidean norm solution

- *Example 1*

$b_{4\times 1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times 1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\hat{Z}_{5\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix}7\\2\end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{9})\begin{bmatrix}9\\5\\0\\4\\2\end{bmatrix}$ which implies $x^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix}9\\5\end{bmatrix}$ and $y^0{}_{3\times1} = (\frac{1}{9})\begin{bmatrix}0\\4\\2\end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{5})\begin{bmatrix}3\\3\\2\\2\\2\end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{6}{5} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{14}{9}$

- *Example 2*

$b_{4\times1} = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1} = (\frac{1}{3})\begin{bmatrix}4\\-2\\-2\\0\end{bmatrix}$, $\hat{Z}_{5\times2} = (\frac{1}{3})\begin{bmatrix}3 & 3\\3 & -3\\1 & 1\\-5 & 1\\-5 & -2\end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix}2\\7\end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{9})\begin{bmatrix}9\\-5\\3\\-1\\-8\end{bmatrix}$ which implies $x^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix}9\\-5\end{bmatrix}$ and $y^0{}_{3\times1} = (\frac{1}{9})\begin{bmatrix}3\\-1\\-8\end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{6})\begin{bmatrix}3\\-3\\5\\-1\\-4\end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{5}{3} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{20}{9}$

- *Example 3*

$b_{4\times1} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1} = (\frac{1}{6})\begin{bmatrix}7\\1\\1\\3\end{bmatrix}$, $\hat{Z}_{5\times2} = (\frac{1}{6})\begin{bmatrix}6 & 6\\6 & -6\\1 & 1\\-5 & 7\\-5 & 1\end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{2})\begin{bmatrix}1\\1\end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{6})\begin{bmatrix}6\\0\\1\\1\\-2\end{bmatrix}$ which implies $x^0{}_{2\times1} = \begin{bmatrix}1\\0\end{bmatrix}$ and $y^0{}_{3\times1} = (\frac{1}{6})\begin{bmatrix}1\\1\\-2\end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{60})\begin{bmatrix}33\\3\\37\\7\\-8\end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{43}{60} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{7}{6}$

- *Example 4*

$b_{4\times1} = \begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1} = (\frac{1}{3})\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}$, $\hat{Z}_{5\times2} = (\frac{1}{3})\begin{bmatrix}3 & 3\\3 & -3\\-1 & -1\\-4 & 2\\-4 & -1\end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{3})\begin{bmatrix}1\\2\end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{3})\begin{bmatrix}3\\-1\\-1\\0\\-2\end{bmatrix}$ which implies $x^0{}_{2\times1} = (\frac{1}{3})\begin{bmatrix}3\\-1\end{bmatrix}$ and $y^0{}_{3\times1} = (\frac{1}{3})\begin{bmatrix}-1\\0\\-2\end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{12})\begin{bmatrix}3\\-3\\5\\-1\\-4\end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{5}{12} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{5}{3}$

❖ $m = 3, n = 2$, therefore $s = 4$, $t = 5$ and $d = 2$

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$$P_{2\times4}=\begin{bmatrix}1&0&\frac{1}{2}&\frac{1}{2}\\0&1&\frac{1}{2}&\frac{1}{2}\end{bmatrix},\quad W_{4\times2}=(\frac{1}{2\sqrt{2}})\begin{bmatrix}1+\sqrt{2}&1-\sqrt{2}\\1-\sqrt{2}&1+\sqrt{2}\\1&1\\1&1\end{bmatrix},\quad J_{3\times2}=(\frac{1}{2})\begin{bmatrix}2&0\\0&2\\1&1\end{bmatrix},$$

$$Q_{4\times3}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{bmatrix},\quad G_{4\times3}=(\frac{1}{6})\begin{bmatrix}4+\sqrt{3}&-2+\sqrt{3}&-2+\sqrt{3}\\-2+\sqrt{3}&4+\sqrt{3}&-2+\sqrt{3}\\-2+\sqrt{3}&-2+\sqrt{3}&4+\sqrt{3}\\\sqrt{3}&\sqrt{3}&\sqrt{3}\end{bmatrix},$$

$$\mathrm{H}_{4\times5}=(\frac{1}{6})\begin{bmatrix}6&0&0&6&0\\0&6&0&0&6\\0&0&6&3&3\\2&2&2&3&3\end{bmatrix},\quad \bar{\mathrm{H}}_{5\times4}=(\frac{1}{60})\begin{bmatrix}31&1&-14&6\\1&31&-14&6\\-14&-14&46&6\\24&-6&9&9\\-6&24&9&9\end{bmatrix}$$

Where $\bar{\mathrm{H}}_{5\times4}$ is the p-Inverse of $\mathrm{H}_{4\times5}$

We have $\hat{b}_{4\times1}=G_{4\times3}(G_{4\times3})^T b_{4\times1}$ and $\bar{\varpi}_{5\times1}=\bar{\mathrm{H}}_{5\times4}\hat{b}_{4\times1}$ where $\bar{\varpi}_{5\times1}$ is the least Euclidean norm solution

- *Example 1*

$b_{4\times1}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1}=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $\hat{Z}_{5\times2}=\begin{bmatrix}0&0\\0&2\\0&1\\1&1\\1&-1\end{bmatrix}$

Therefore:

- $w^0{}_{2\times1}=(\frac{1}{9})\begin{bmatrix}7\\2\end{bmatrix}$, $\omega^0{}_{5\times1}=(\frac{1}{9})\begin{bmatrix}0\\4\\2\\9\\5\end{bmatrix}$ which implies $x^0{}_{3\times1}=(\frac{1}{9})\begin{bmatrix}0\\4\\2\end{bmatrix}$ and $y^0{}_{2\times1}=(\frac{1}{9})\begin{bmatrix}9\\5\end{bmatrix}$
- $\bar{\varpi}_{5\times1}=(\frac{1}{5})\begin{bmatrix}2\\2\\2\\3\\3\end{bmatrix}$, we have $(\|\bar{\varpi}_{5\times1}\|_2)^2=\frac{6}{5}<(\|\omega^0{}_{5\times1}\|_2)^2=\frac{14}{9}$

- *Example 2*

$b_{4\times1} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1} = (\frac{1}{3})\begin{bmatrix} 4 \\ -2 \\ -2 \\ 0 \end{bmatrix}$, $\hat{Z}_{5\times2} = (\frac{1}{3})\begin{bmatrix} 1 & 1 \\ -5 & 1 \\ -5 & -2 \\ 3 & 3 \\ 3 & -3 \end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix} 2 \\ 7 \end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{9})\begin{bmatrix} 3 \\ -1 \\ -8 \\ 9 \\ -5 \end{bmatrix}$ which implies $x^0{}_{3\times1} = (\frac{1}{9})\begin{bmatrix} 3 \\ -1 \\ -8 \end{bmatrix}$ and $y^0{}_{2\times1} = (\frac{1}{9})\begin{bmatrix} 9 \\ -5 \end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{6})\begin{bmatrix} 5 \\ -1 \\ -4 \\ 3 \\ -3 \end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{5}{3} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{20}{9}$

- *Example 3*

$b_{4\times1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1} = (\frac{1}{6})\begin{bmatrix} 7 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, $\hat{Z}_{5\times2} = (\frac{1}{6})\begin{bmatrix} 1 & 1 \\ -5 & 7 \\ -5 & 1 \\ 6 & 6 \\ 6 & -6 \end{bmatrix}$

Therefore:

- $w^0{}_{2\times1} = (\frac{1}{2})\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\omega^0{}_{5\times1} = (\frac{1}{6})\begin{bmatrix} 1 \\ 1 \\ -2 \\ 6 \\ 0 \end{bmatrix}$ which implies $x^0{}_{3\times1} = (\frac{1}{6})\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $y^0{}_{2\times1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $\bar{\omega}_{5\times1} = (\frac{1}{60})\begin{bmatrix} 37 \\ 7 \\ -8 \\ 33 \\ 3 \end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2 = \frac{43}{60} < (\|\omega^0{}_{5\times1}\|_2)^2 = \frac{7}{6}$

- *Example 4*

$b_{4\times1}=\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$, we have the following set of associated results: $\hat{b}_{4\times1}=(\frac{1}{3})\begin{bmatrix}2\\-1\\-1\\0\end{bmatrix}$, $\hat{Z}_{5\times2}=(\frac{1}{3})\begin{bmatrix}-1&-1\\-4&2\\-4&-1\\3&3\\3&-3\end{bmatrix}$

Therefore:

- $w^0{}_{2\times1}=(\frac{1}{3})\begin{bmatrix}1\\2\end{bmatrix}$, $\omega^0{}_{5\times1}=(\frac{1}{3})\begin{bmatrix}-1\\0\\-2\\3\\-1\end{bmatrix}$ which implies $x^0{}_{3\times1}=(\frac{1}{3})\begin{bmatrix}-1\\0\\-2\end{bmatrix}$ and $y^0{}_{2\times1}=(\frac{1}{3})\begin{bmatrix}3\\-1\end{bmatrix}$
- $\bar{\omega}_{5\times1}=(\frac{1}{12})\begin{bmatrix}5\\-1\\-4\\3\\-3\end{bmatrix}$, we have $(\|\bar{\omega}_{5\times1}\|_2)^2=\frac{5}{12}<(\|\omega^0{}_{5\times1}\|_2)^2=\frac{5}{3}$

III. DISCUSSION AND CONCLUDING REMARKS

The least euclidean norm solution to the linear system $H_{s\times t}\omega_{t\times1}=\hat{b}_{s\times1}$ is $\bar{\omega}_{t\times1}=\bar{H}_{t\times s}\hat{b}_{s\times1}$ where $\bar{H}_{t\times s}$ is the p-inverse of the matrix $H_{s\times t}$.Nevertheless the convex hull $\Omega(d)$, which is the feasible set of the associated optimization problem , may not necessarily contain the p-inverse based solution point $\bar{\omega}_{t\times1}$ and hence $\|\bar{\omega}_{t\times1}\|_2\leq\|\omega_{t\times1}\|_2 \quad \forall\omega_{t\times1}\in\Omega(d)$,in the numerical case study examples , we observe strict inequality to hold for all the considered example cases.

The projection of the vector $b_{s\times1}$, denoted as $\hat{b}_{s\times1}\in Csp(H_{s\times t})$, determines the matrix $\hat{Z}_{t\times d}$ and hence the convex hull $\Omega(d)$, therefore variations in choice of the complementary pair of Projection matrices $(\Sigma_{s\times s},I_{s\times s}-\Sigma_{s\times s})$ where $Csp(\Sigma_{s\times s})=Csp(H_{s\times t})$ allows for choices of different $\hat{b}_{s\times1}$ for a given $b_{s\times1}\in R^s$, this approach allows for generalizing the mathematical framework developed in the present research initiative to the situation involving oblique decomposition of the embedded space vector $b_{s\times1}\in R^s$.

In conclusion, it can be emphasized that the presented mathematical framework allows for a constrained optimization approach to estimate an input vector pair from an embedded space vector element under the framework based on the spacer matrices and related matrix components. The approach presented in the research can be appropriately modified, as required, to handle estimation issues pertaining to the associated regularized problems. Research studies along these lines would be addressed through follow up studies.

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