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An Inventory Management Framework Designed for Deteriorating Items with Time Sensitive Demand and Shortages under All Unit Price Discount Policy Incorporating a Fuzzy Approach

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Abstract :- The conventional inventory model, known as the crisp inventory model, assumes that parameters have precise and certain values. However, in real-world scenarios, it is observed that uncertainties or imprecisions in the environment prevent the exact definition of all parameters. To address this uncertainty, fuzzy set theory may be applied, providing decision-makers with mathematical tools to formulate inventory models that reflect real-world conditions. This study explores two inventory models: a traditional crisp model and its corresponding fuzzy counterpart. The focus is on a non-instantaneous deteriorating item with time-sensitive demand under a unit price discount policy, considering fully backlogged shortages and incorporating salvage value for deteriorated units. In the fuzzy environment, parameters such as demand, deterioration rate, salvage, and cost components like ordering cost, holding cost, and shortages cost are represented as triangular fuzzy numbers. The Signed distance method is utilized for defuzzification of fuzzy outputs. Numerical examples are offered to justify and compare the anticipated crisp and fuzzy models. Additionally, a sensitivity analysis for the fuzzy model is conducted to provide managerial implications.

Keywords:- *Crisp Inventory Model, Fuzzy Inventory Model, Non-Instantaneous, Time-Sensitive Demand, Price Discount Policy, Fully Backlogged Shortages, Salvage Value, Triangular Fuzzy Number, Defuzzification, Signed Distance Method.*

I. INTRODUCTION

In the conventional inventory model, also known as the crisp inventory model, it is assumed that all parameters have known and fixed values. This includes constant and known demand, fixed ordering costs, known holding costs, and a zero lead time. However, in reality, client demand fluctuates over the stocking period, making effective stock planning challenging for companies, especially when dealing with perishable products. The uncertainty or fuzziness in demand parameters adds complexity to this scenario.

Moreover, Providers of seasonal food frequently offer rebate programs tied to the volume of purchases, presenting the company with an opportunity to substantially lower overall procurement expenses through increased order quantities. However, due to the perishable nature of the products, excessively significant quantities of purchases may not be lucrativeat all times and can even be detrimental to the business. Consequently, inventory costs such as ordering cost, holding cost, and shortages cost can be considered as fuzzy quantities.

In contrast to traditional inventory preparation, where unit purchasing cost is naturally treated as fixed irrespective of the order amount, the study acknowledges that companies often receive fixed reductions on unit purchasing prices on the basis of order volume, particularly for seasonal food items. Consequently, under a price discount strategy, the unit purchasing price may be considered a parameter with certain values.

The presence of deterioration is a common challenge for companies dealing with food stuffs over stocking durations. The rate of deterioration adds another layer of uncertainty to the inventory planning process. Deterioration poses dual challenges for companies: the loss of investments in deteriorated items and the increased complexity of optimal inventory planning to meet client demand over a reasonable period.

Many fresh food items do not immediately begin to deteriorate upon being stocked; instead, their deterioration becomes noticeable after a certain period due to their inherent physical characteristics. These items are commonly referred to as delayed deteriorated items or non-instantaneous deteriorated items in inventory literature, with many seasonal foods falling into this category. The introduction of the salvage value concept is a relatively recent development in inventory management and has been explored by only a limited number of researchers.

The salvage value, which is directly tied to each deteriorated product and is proportional to the unit purchasing cost, is determined at the point of sale with a discounted price marking the conclusion of each business cycle. The integration of salvage value holds significance in business operations as it plays a crucial role in reducing a company's expenses, impacting both purchasing costs and holding costs.

In this study, our initial focus is on exploring the effects of a discount scheme on the unit purchase price in scenarios where items experience non-instantaneous deterioration, and Their rate of consumption depends on the duration of time the items are stocked, both in crisp and fuzzy contexts. To accomplish this goal, the company calculates different expenses stemming from the inventory's fluctuations across the business cycle. Then, the total cost per unit of time for the company is computed.

Next, theoretical outcomes are presented through theorems, and by amalgamating the analytical results, two solution algorithms are introduced: one for the crisp model and another for the fuzzy model, specifically under the quantity-based discount structure. Finally, the study concludes by delivering key findings that hold relevance for the company's decision-making process.

II. LITERATURE REVIEW

Deterioration is a significant and inevitable characteristic of perishable foods. Because of how they're stored and their physical characteristics, the decay of numerous perishable items often starts at a later stage of storage, classifying them as items that deteriorate gradually rather than instantly. Wu et al. (2006) were the pioneers in describing the optimal inventory planning for non-immediate deteriorating items, marking the beginning of subsequent academic studies on the management of these perishable items under various circumstances.

Building upon Wu et al.'s foundational work, several researchers have explored different aspects of inventory management for non-immediate perishable items. Rabbani et al. (2015) delved into integrated replenishment and marketing strategies, introducing a progressive reduction in the selling price once deterioration sets in, following an exponential pattern.Ai et al. (2017) focused on determining optimal inventory planning for companies replenishing multiple non-immediate decay items with constant consumption rates. R.P. Tripathi (2018) developed an EOQ model considering quadratic time-sensitive demand and a parabolic-time linked holding cost with salvage value.

Addressing the goal of reducing deterioration costs, Li et al. (2019) pinpointed the most effective investment cost for preserving non-instantaneous decaying items by mitigating deterioration rates. Duary et al. (2022) derived optimal selling price decisions for companies facing inventory challenges arise with items that deteriorate gradually, especially when subjected to a prepayment mechanism enforced by suppliers. In recent studies, Khan et al. (2022a) provided analytical insights into the most effective restocking strategy for perishable items where deteriorate takes part gradually. Additionally, Khan et al. (2022b) determined the best inventory strategies for non-immediate decay items in situations where stock levels end positively and in cases of stock-outs, considering the interplay of demand with the items storage duration.

Seasonal food items typically have a limited shelf life, prompting dealers to exercise caution in their early-stage sales. To facilitate the marketing of these items, suppliers of seasonal foods often implement various concession regulations on the unit acquisition cost, contingent on the order quantity. Among these rebate policies, the all-unit concession scheme stands out due to its appeal, particularly because it allows for an increase in the purchased quantity. Taleizadeh and Pentico (2014) investigated the consequences of a universal rebate program on a retailer's inventory choices, under the assumption of a consistent demand rate. Alfares and Ghaithan (2016) determined the best pricing and inventory management tactics for retailers functioning under a universal rebate system.

Conversely, Cunha et al. (2019) established the most effective inventory strategy for a production system by offering discounts exclusively for imperfectly manufactured items, rather than for all products. Shaikh et al. (2019) examined the optimal inventory strategy for a company within a universal rebate framework for perishable items, considering demand fluctuations alongside warehouse stock levels. Accounting for the maximum lifespan of stored items regarding deterioration, Khan et al. (2019) investigated the implications on pricing and inventory decisions under a universal rebate framework. Subsequently, Khan et al. (2021) introduced an integrated scheme based on quantity, involving upfront and deferred payments alongside unit purchase price, to determine the best inventory strategy for a company. In a recent investigation, Khan et al. (2022c) identified optimal pricing and inventory tactics for a deteriorating item within a rebate scheme sensitive to both purchased quantity and unit purchase price.

Rahman et al. (2022) extensively analyzed the consequences of a universal unit discount arrangement on the optimal inventory strategy within a time interval setting. Inventory managers may face shortages due to disruptions in supply or fluctuations in demand, prompting customers to either seek alternatives or wait for restocking. When all customers wait for their needs to be fulfilled, this scenario is termed 'complete backordering' in inventory literature. Prasad and Mukherjee (2016) developed a deterministic inventory planning model considering varying demand over time with complete backlogged shortages. San-José et al. (2018) further investigated the complete backordering scenario in the context of demand sensitivity to stocking duration. Recent notable studies on complete backordering include those by San-José et al. (2020), Lin (2021), and De-la-Cruz-Márquez et al. (2021).

Moreover, when some customers opt for alternatives during periods of stock-outs while others wait for restocking, this scenario is termed 'partial backordering' in inventory literature. Taleizadeh and Pentico (2014) investigated the best inventory strategy for a company under an order-based rebate scheme, assuming a fixed backlogging rate. Other noteworthy studies in this area of research include those conducted by Taleizadeh et al. (2015), Lin (2018), Gao et al. (2019), Taleizadeh et al. (2020), and Rahman et al. (2021).

Alternatively, Khan et al. (2019) examined the optimal inventory management approach for a company handling perishable goods with a maximum lifespan under an order-based rebate scheme, taking into account a variable backlogging rate based on clients' waiting times. Relevant research on variable backlogging rates includes studies by Panda et al. (2019), Pakhira et al. (2020), Xu et al. (2021), Alshanbari et al. (2021), and Manna et al. (2022). In a different context, Rukonuzzaman et al. (2023) investigated an inventory model for items that deteriorate gradually under a policy of universal price discounts and identified the optimal strategy for inventory control.

All the aforementioned studies focused on the inventory control system in a crisp sense, where inventory parameters were assumed to have precise values. However, in real-world situations, many inventory parameters are characterized by uncertain numerical values. In the current competitive and ever-changing business landscape, acquiring precise information isn't always achievable. As a result, the information pertaining to inventory systems isn't as clearly defined as presumed in conventional models. One practical strategy to tackle these constraints involves leveraging fuzzy set theory (FST), pioneered by Zadeh in 1965. FST facilitates the conversion of vague or imprecise information into mathematical formulations.

For over half a century, FST has gained significant momentum, finding applications in various fields, including operations research and inventory management. Inventory management necessitates accurate demand forecasts and parameters for inventory-related costs, such as ordering, purchasing, holding, and shortages. Since precise estimations of these model attributes are often challenging in practice, fuzzy techniques can be employed to calibrate inventory-related data. This approach facilitates a more appropriate handling of real-world cases, acknowledging the uncertainties inherent in inventory management.In 1983, H. J. Zimmerman ventured into the utilization of fuzzy sets within operations research. K. S. Park (1987) employed fuzzy set concepts to formulate an Economic Order Quantity (EOQ) model, considering the inventory carrying cost as a trapezoidal fuzzy number. Yao and Lee (1999) developed an EOQ model in a fuzzy context for an inventory system without backorders. Wu and Yao (2003) introduced fuzzification of the order quantity and shortages quantities using triangular fuzzy numbers while formulating an EOQ model.

Arindam Roy and Manas Kumar Maiti (2007) formulated an inventory model for a deteriorating item with demand dependent on stock levels, incorporating fuzzy inflation and time discounting across a stochastic planning horizon. Uttam Kumar Bera and Ajoy Kumar Maiti (2012) introduced a practical multi-item fuzzy inventory model integrating multiple price breakpoints. Dutta and Pavan (2012) explored an inventory model within a fuzzy environment characterized by constant demand and no shortages, employing two fuzzy parameters—holding cost and setup cost—represented by trapezoidal fuzzy numbers. They used the signed distance method for defuzzification of the total fuzzy cost. Chandra K. Jaggi and Sarla Pareek (2012) proposed a fuzzy inventory model for deteriorating items with fluctuating demand over time and shortages. Mohan and Venkateshwarlu (2013) investigated a model featuring quadratic time-varying demand, allowing for variable holding costs and salvage values. Dutta, D. and Pavan Kumar (2013) developed a fuzzy inventory model for deteriorating items with shortages under fully backlogged conditions. Mohan R. (2015) examined a fuzzy inventory model for deteriorating items employing trapezoidal fuzzy numbers, with the supplier offering a price discount to the retailer upon replenishment.Nalini Prava and Pradip Kumar Tripathy (2016) introduced an EOQ model in a fuzzy context for time-deteriorating items, incorporating Penalty Cost. Sujata Saha and Tripti Chakrabarti (2017) established a policy for maximizing manufacturer profits within a supply chain management framework characterized by demand dependent on prices in a fuzzy environment. Saranya and Varajan (2018) introduced a fuzzy inventory model allowing for acceptable shortages that were fully backlogged, incorporating three fuzzy parameters: carrying cost, holding cost, and backorder cost. They utilized triangular and trapezoidal fuzzy numbers to compute the fuzzy total cost. Onyenike K. and Ojarikre, H.I (2022) proposed a fuzzy inventory model featuring fuzzy demand and no shortages, employing pentagonal fuzzy numbers. Rukonuzzaman (2024) devised an inventory management model for non-immediate deteriorating items without shortages under a price discount scheme.

While numerous studies on inventory models have been conducted over the past two decades, introducing concepts of Fuzzy Set Theory (FST) under various conditions, only a limited number have addressed the aspect of price discount policies. Consequently, there exists substantial potential for further research and development in this particular domain. This motivated our initiative to create a novel inventory model within a fuzzy environment.

In this work, firstly we have developed a crisp inventory model for an item under all unit price discount facility with linear time dependent demand, constant holding cost and non-instantaneous deterioration rate and associated salvage value where shortages were allowed in fully backlogged condition. Company’s various costs and the earned salvage value is estimated based on the inventory behavior throughout the business cycle and then, the company’s total cost per unit of time is calculated.Afterward, we derived the corresponding fuzzy inventory model. In fuzzy sense related inventory parameters are considered as triangular fuzzy numbers. For defuzzification of the fuzzy model we have used the Signed distance method. Secondly, thetheoretical results are outlined using theorems, followed by the presentation of two solution algorithms: one for the crisp model and another for the fuzzy model, within the context of a quantity-based discount structure. Ultimately, the significant findings pertinent to the company are presented.

The remaining sections of this paper are structured as follows:

Section 2 presents essential definitionsSection 3 outlines fundamental notations and assumptions, while Section 4 elaborates on the mathematical formulation of both the crisp model and its corresponding fuzzy model.Defuzzification procedures are executed in Section 5.In Section 6, the proof of the existence and uniqueness of the optimal solution is provided for both the crisp model and the defuzzified model, along with necessary conditions.Solution algorithms for the crisp and defuzzified models are expounded in Section 7.Section 8 includes two numerical examples aimed at justifying and comparing the crisp and fuzzy models.Sensitivity analysis for the defuzzified model is conducted in Section 9.Section10 presents managerial implications.Ultimately, conclusions and avenues for further research are deliberated in Section 11.

III. DEFINITIONS AND PRELIMINARIES

A. Definition

Fuzzy subset: (By Pu and Liu ,1980)A fuzzy subset $\tilde{A}$of a universe of discourse **U** is characterized by a membership function $\mu_{\tilde{A}}: U \to [0,1]$ which associate each element x of **U** a number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$.

B. Definition

Fuzzy number: A fuzzy number is a convex normalized fuzzy set $\tilde{A}$ defined on **R** whose membership function is a piece wise continuous function.

C. Definition

Triangular fuzzy number: A fuzzy number $\tilde{A} = (a, b, c)$ where $a < b < c$ defined on R is called a triangularfuzzy number if its membership function is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a} & , a \le x \le b \\ \frac{c-x}{c-b} & , b \le x \le c \\ 0 , & \text{otherwise} \end{cases}$$

D. Definition

Suppose $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are two triangular fuzzy numbers , then the arithmetic operations are defined as:

- $\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
- $\tilde{A} \times \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3)$
- 3.$\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$
- 4.$\tilde{A}/\tilde{B} = (a_1/b_3, a_2/b_2, a_3/b_1)$
- 5.$k \times \tilde{A} = \begin{cases} ka_1, ka_2, ka_3 , & \alpha \ge 0 \\ ka_3, ka_2, ka_1 , & \alpha < 0 \end{cases}$

E. Definition

Let $\tilde{A} = (a_1, a_2, a_3)$ be a triangular fuzzy number, then the signed distance of $\tilde{A}$ is defined as

$$d(\tilde{A}, \tilde{0}) = \frac{1}{2}\int_0^1 [A_L(\alpha) + A_R(\alpha)]\, d\alpha = \frac{a_1 + 2a_2 + a_3}{4}$$

Where $A_\alpha = [A_L(\alpha), A_R(\alpha)] = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha]$, $\alpha \in [0,1]$, is α −cut of the fuzzy set $\tilde{A}$, which is a closed interval.

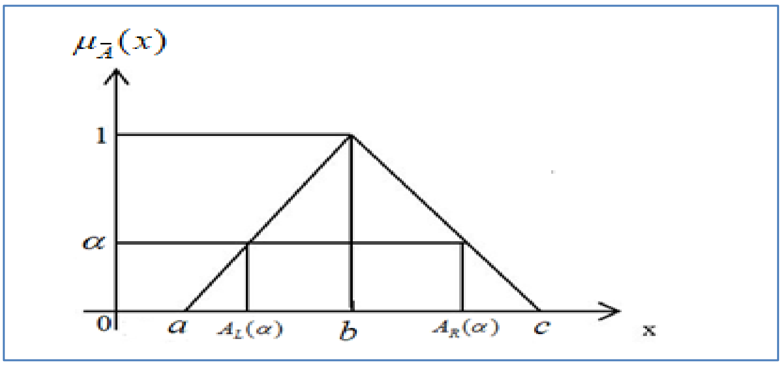

Fig. 1: α-cut of a triangular fuzzy number

IV. NOTATIONSAND ASSUMPTIONS

The mathematical framework presented in this paper is constructed using the designated notations and assumptions outlined herein.

A. Notations

μ :	Time at which deterioration starts (unit of time)
T :	Length of each cycle (unit of time) (a decision variable for the crisp model)
t_1 :	Period for stock available inventory (unit of time) (a decision variable for the crisp model)
θ :	Deterioration rate on positive stock in the warehouse ($0 < \theta << 1$)
γ	Salvage value coefficient
Q :	Purchased amount by the company for each cycle (units)
R :	Backordering quantity (units)
S :	Stock amount at starting moment of each cycle after backordering (units)
V :	Stock amount at time $t = \mu$ (units)
A :	Company's ordering cost ($/order)
h :	Unit carrying cost per unit of time ($/unit/unit of time)
c_j :	Unit acquisition price ($/unit)
c_s :	Shortage cost ($/unit/unit of time)
$D(t)$:	Time varying demand function (units/unit of time)
$I_1(t)$:	Inventory amount in the storage before commencing deterioration at any time $t \in [0, \mu]$
$I_2(t)$:	Inventory amount in the storage at any time $t \in [\mu, t_1]$
$I_3(t)$:	Inventory amount at any time $t \in [t_1, T]$
$TC(t_1, T)$:	Total inventory cost for the company ($/unit of time)
$TC_{\min}(t_1, T)$:	Total minimum inventory cost for the company ($/unit of time)
$\tilde{\theta}$:	Fuzzy deterioration rate on positive stock in the warehouse.
$\tilde{\gamma}$:	Fuzzy Salvage value coefficient
$\tilde{Q}$:	Fuzzy purchased quantity by the company for each cycle (units)
$\tilde{R}$:	Fuzzy backordering quantity (units)
$\tilde{S}$:	Fuzzy stock amount at starting moment of each cycle after backordering (units)
$\tilde{A}$:	Fuzzy ordering cost ($/order)
$\tilde{h}$:	Fuzzy unit holding cost per unit of time ($/unit/unit of time)
$\tilde{c}_s$:	Shortage cost ($/unit/unit of time)
$\tilde{D}(t)$:	Fuzzy time varying demand function (units/unit of time)
$\tilde{T}C(t_1, T)$:	Fuzzy total inventory cost for the company ($/unit of time)
$TC_{sd}(t_1, T)$:	Defuzzified value of the fuzzy number $\tilde{T}C(t_1, T)$
$TC_{sd\min}(t_1, T)$	Defuzzified minimum total inventory cost for the company ($/unit of time)

B. Assumptions

- The rate of consumption, denoted as $D(t)$, for stored foods rises as they ripen with time, but does not escalate once deterioration sets in. The demand function is defined as

$$D(t) = \begin{cases} a + bt, & 0 \le t \le \mu \\ a + b\mu, & \mu \le t \le T \end{cases},$$

where $a > 0, b \ge 0, \mu \ge 0$.

When both parameters for demand, denoted as b and μ, equal zero, the demand remains constant throughout the cycle.

- The quantity of deteriorated items depends uponthe current stock volume and the deterioration rate is θ where $0<\theta<<1$.
- The lead time is negligible and can be disregarded, whereas the replenishment rate is considered to be infinite.
- The problem is examined with regards to an infinite planning horizon.
- The salvage value per deteriorated unit is proportional to the unit purchase cost and estimated as γc_j, where γ is the salvage value parameter.
- Holding cost per unit, per unit time is constant.
- Shortages are allowed and fully backlogged.
- Using $n-$number of quantity breaks (q_i where $i=1, 2, 3, \ldots\ldots, n$ and $q_1<q_2<\ldots\ \ldots<q_n<\infty$), the supplier offers a pricing structure determined by the quantity purchased, wherein the unit price c_j decreases incrementally based on the quantity bought (Q) such that $c_1>c_2>\cdots\cdots>c_n$

Figure 2 presents a visual depiction of this discount policy, featuring four distinct quantity breakpoints.

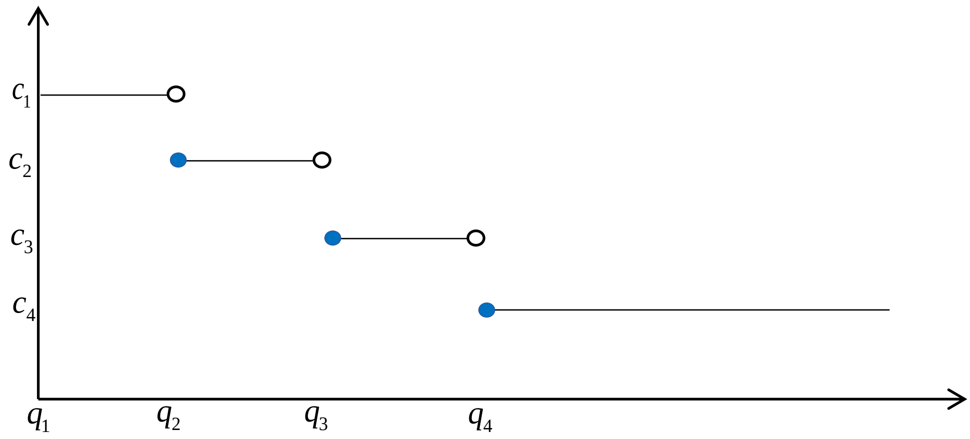

Fig. 2: Visual representation of an all-unit discount policy featuring four quantity thresholds.

V. MODEL FORMULATION

Considering the assumptions outlined in the preceding section, we formulate the crisp model for the inventory processwith fully backlogged shortages first and then, the corresponding fuzzy model is developed.

A. Crisp model

Let us assume that a retailer placed an order to a supplier for $Q=S+R$ units of a non –instantaneous deteriorating product under a discount facility. Under the discount policy, the purchase cost per unit item is a decreasing step function based on the amount of order quantity. Immediately after the arrival of the ordered product, R units are shifted to meet up the cumulative backlog of demand and thus the remaining inventory level reaches S. Inventory decreases solely due to customer demand during the subsequent interval $[0,\mu]$ and at time $t=\mu$ it reaches the level *V*. According to the assumption the deterioration of the products starts at $t=\mu$ and throughout the next period $[\mu,t_1]$, the inventory level diminishes due to the combined effect of deterioration along with customer's demand and finally drops to the zero level at $t=t_1$. Afterwards due to persisting customer's demand stockout situation occurs which is gathered throughout the period $[t_1,T]$ and fully backlogged instantaneously at the arrival of the next replenishment. At this moment, a new order is placed, and the entire inventory system restarts.

The entire inventory system can be depicted as in the Figure 3.

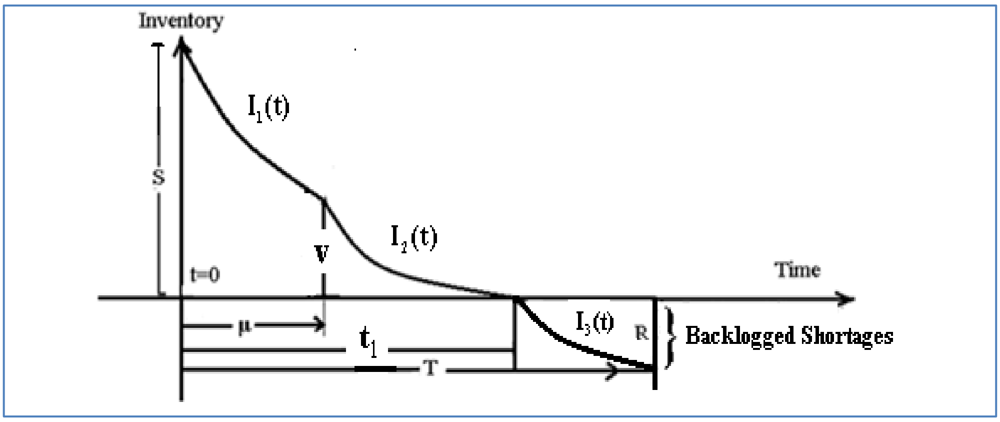

Fig. 3: Visual presentation of the suggested inventory system with complete backlog of shortages

Now at any instant the status of the inventory level can be described by the following differential equations:

$$\frac{dI_1(t)}{dt} = -(a+bt), 0 \le t \le \mu \qquad (1)$$

With $I_1(0)=S$ and $I_1(\mu)=V$

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -(a+b\mu), \mu \le t \le t_1 \qquad (2)$$

With $I_2(\mu)=V$ and $I_2(t_1)=0$

And

$$\frac{dI_3(t)}{dt} = -(a+b\mu) \quad , t_1 \le t \le T \qquad (3)$$

With $I_3(t_1)=0$ and $I_3(T)=-R$

The solution to the differential equations (1) is given by

$$I_1(t) = S - at - \frac{1}{2}bt^2 \qquad (4)$$

Applying boundary condition $I_1(\mu)=V$ in (4) , we get

$$I_1(\mu) = V = S - a\mu - \frac{1}{2}b\mu^2$$

$$S = V + a\mu + \frac{1}{2}b\mu^2 \qquad (5)$$

On solving equation (2) with the boundary condition $I_2(t_1)=0$ we get

$$I_2(t) = \frac{a+b\mu}{\theta}\left[e^{\theta(t_1-t)} - 1\right] = \frac{k}{\theta}\left[e^{\theta(t_1-t)} - 1\right], \quad \text{where } k = a+b\mu \qquad (6)$$

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Applying the boundary condition $I_2(\mu)=V$ in (6), we get

$$I_2(\mu)=V=\frac{k}{\theta}\left[e^{\theta(t_1-\mu)}-1\right]$$

Therefore,

$$V=\frac{k}{\theta}\left[e^{\theta(t_1-\mu)}-1\right]$$

$$=\frac{k}{\theta}\left[\theta(t_1-\mu)+\frac{\theta^2}{2}(t_1-\mu)^2\right]$$

$$=k(t_1-\mu)+\frac{k\theta}{2}(t_1-\mu)^2 \tag{7}$$

Thus, from equation (5) and (7) we have,

$$S=a\mu+\frac{1}{2}b\mu^2+k(t_1-\mu)+\frac{k\theta}{2}(t_1-\mu)^2 \tag{8}$$

Solving equation (3), with boundary condition $I_3(t_1)=0$ we have,

$$I_3(t)=(a+b\mu)(t_1-t)=k(t_1-t) \tag{9}$$

Applying the boundary condition $I_3(T)=-R$ in equation (9), one finds

$$R=k(T-t_1) \tag{10}$$

Consequently, the total number of requisition amount early in a business cycle is given by

$$Q=S+R$$

or, $Q=a\mu+\frac{1}{2}b\mu^2+k(t_1-\mu)+\frac{k\theta}{2}(t_1-\mu)^2+k(T-t_1)$

or, $Q=a\mu+\frac{1}{2}b\mu^2+k(T-\mu)+\frac{k\theta}{2}(t_1-\mu)^2$ (11)

Equation (11) provides an expression for the duration of the business cycle as

$$T=\frac{1}{k}\left[Q-a\mu-\frac{1}{2}b\mu^2-\frac{k\theta}{2}(t_1-\mu)^2\right]+\mu \tag{12}$$

The total cost of the complete inventory system per replenishment cycle comprises the following elements:

- Ordering cost (OC) : A
- Purchasing cost (PC):

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$$c_jQ = c_j(S+R) = c_j\left\{a\mu + \frac{1}{2}b\mu^2 + k(T-\mu) + \frac{k\theta}{2}(t_1-\mu)^2\right\}$$

- Holding cost (HC): Holding cost for the time period $[0,\mu]$ and $[\mu,t_1]$

$$= h\left[\int_0^{\mu} I_1(t)dt + \int_{\mu}^{t_1} I_2(t)dt\right]$$

$$= h\left[\int_0^{\mu}\left(S - at - \frac{1}{2}bt^2\right)dt\right] + h\int_{\mu}^{t_1}\frac{k}{\theta}\left[e^{\theta(t_1-t)} - 1\right]dt \quad .$$

$$= h\left[St - \frac{1}{2}at^2 - \frac{1}{6}bt^3\right]_0^{\mu} + \frac{hk}{\theta}\left[\left(\frac{e^{\theta(t_1-t)}}{(-\theta)} - t\right)\right]_{\mu}^{t_1}$$

$$= h\left[S\mu - \frac{1}{2}a\mu^2 - \frac{1}{6}b\mu^3\right] + \frac{hk}{\theta}\left[\left(\frac{e^{\theta(t_1-\mu)}}{\theta} - \frac{1}{\theta} + \mu - t_1\right)\right]$$

$$= h\left[\frac{k\mu}{\theta}\left(e^{\theta(t_1-\mu)} - 1\right) + a\mu^2 + \frac{1}{2}b\mu^3 - \frac{1}{2}a\mu^2 - \frac{1}{6}b\mu^3\right]$$

$$+ \frac{hk}{\theta}\left[\left(\frac{e^{\theta(t_1-\mu)}}{\theta} - \frac{1}{\theta} + \mu - t_1\right)\right]$$

$$= h\left[\frac{1}{2}a\mu^2 + \frac{1}{3}b\mu^3 + k\mu(t_1-\mu) + \frac{k(1+\theta\mu)}{2}(t_1-\mu)^2 + \frac{k\theta}{6}(t_1-\mu)^3\right]$$

- **Shortage cost** (*SC*):

$$-c_s\int_{t_1}^{T} I_3(t)dt = -c_s\int_{t_1}^{T} k(t_1-t)dt = \frac{1}{2}c_sk(T-t_1)^2$$

- **Salvage value** (*SV*):

$$\gamma c_j\left[V - \int_{\mu}^{t_1}(a+b\mu)dt\right]$$

$$= \gamma c_j\left[V - \int_{\mu}^{t_1} k\,dt\right] = \gamma c_j\left[V - k(t_1-\mu)\right]$$

$$= \gamma c_j\left[\frac{k}{\theta}\left\{e^{\theta(t_1-\mu)} - 1\right\} - k(t_1-\mu)\right]$$

$$= \gamma c_j\left\{\frac{k}{2}\theta(t_1-\mu)^2\right\}$$

$$= \gamma c_j\left\{\frac{k}{2}\theta(t_1-\mu)^2\right\}$$

Therefore, the total cost function for the company per unit of time is:

$$TC(t_1,T)=\frac{1}{T}\left[OC+PC+HC+SC-SV\right]$$

$$=\frac{1}{T}\left[\begin{array}{l} A+c_j\left\{a\mu+\frac{1}{2}b\mu^2+k(T-\mu)+\frac{1}{2}k\theta(t_1-\mu)^2\right\} \\ +h\left\{\frac{1}{2}a\mu^2+\frac{1}{3}b\mu^3+k\mu(t_1-\mu)+\ \frac{1}{2}k(1+\theta\mu)(t_1-\mu)^2+\frac{1}{6}k\theta(t_1-\mu)^3\right\} \\ \qquad +\frac{1}{2}c_sk(T-t_1)^2-\gamma c_j\left\{\frac{k}{2}\theta(t_1-\mu)^2\right\} \end{array}\right]$$

$$=\frac{1}{T}\left[\begin{array}{l} A+c_j\left\{a\mu+\frac{1}{2}b\mu^2+k(T-\mu)+\frac{1}{2}k\theta(t_1-\mu)^2\right\} \\ +h\left\{\frac{1}{2}a\mu^2+\frac{1}{3}b\mu^3+k\mu(t_1-\mu)+\ \frac{1}{2}k(1+\theta\mu)(t_1-\mu)^2+\frac{1}{6}k\theta(t_1-\mu)^3\right\} \\ \qquad +\frac{1}{2}c_sk(T-t_1)^2-\gamma c_j\left\{\frac{k}{2}\theta(t_1-\mu)^2\right\} \end{array}\right] \quad (13)$$

The objective now is to determine the optimal duration t_1^* for maintaining stock availability and the optimal length T^* of the replenishment cycle to minimize the company's cumulative cost per unit of time. The fuzzy model corresponding to this crisp model is derived in the subsequent subsection.

B. Fuzzy model

Here we consider the inventory model in fuzzy environment. For this we assume that the parameters $A, a, b, \theta, h, \gamma, k$ and c_S may change within some limits.

Let $\tilde{A}=(A_1,A_2,A_3),\ \tilde{a}=(a_1,a_2,a_3), \tilde{b}=(b_1,b_2,b_3), \tilde{\theta}=(\theta_1,\theta_2,\theta_3), \tilde{h}=(h_1,h_2,h_3), \tilde{\gamma}=(\gamma_1,\gamma_2,\gamma_3),$ $\tilde{c}_s=(g_1,g_2,g_3), \tilde{k}=\tilde{a}+\tilde{b}\mu=(k_1,k_2,k_3)$ be triangular fuzzy numbers.

Then in the fuzzy sense, we have:

Fuzzy initial inventory level, $\tilde{S}=\tilde{a}\mu+\frac{1}{2}\tilde{b}\mu^2+\tilde{k}(t_1-\mu)+\frac{\tilde{k}\tilde{\theta}}{2}(t_1-\mu)^2=(S_1,S_2,S_3)$ (say)

Where ,

$$S_1=a_1\mu+\frac{1}{2}b_1\mu^2+k_1(t_1-\mu)+\frac{k_1\theta_1}{2}(t_1-\mu)^2$$

$$S_2=a_2\mu+\frac{1}{2}b_2\mu^2+k_2(t_1-\mu)+\frac{k_2\theta_2}{2}(t_1-\mu)^2$$

$$S_3=a_3\mu+\frac{1}{2}b_3\mu^2+k_3(t_1-\mu)+\frac{k_3\theta_3}{2}(t_1-\mu)^2$$

Fuzzy back ordered quantity, $\tilde{R}=\tilde{k}(T-t_1)=(R_1,R_2,R_3)$ (say)

Where, $R_1=k_1(T-t_1)$, $R_2=k_2(T-t_1)$, $R_3=k_3(T-t_1)$

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Fuzzy total order quantity of the system per cycle,

$$\tilde{Q}=\tilde{S}+\tilde{R}=\tilde{a}\mu+\frac{1}{2}\tilde{b}\mu^2+\tilde{k}(T-\mu)+\frac{\tilde{k}\tilde{\theta}}{2}(t_1-\mu)^2=(Q_1,Q_2,Q_3) \quad \text{(say)}$$

Where ,

$$\left.\begin{array}{l} Q_1=a_1\mu+\frac{1}{2}b_1\mu^2+k_1(T-\mu)+\frac{k_1\theta_1}{2}(t_1-\mu)^2 \\ Q_2=a_2\mu+\frac{1}{2}b_2\mu^2+k_2(T-\mu)+\frac{k_2\theta_2}{2}(t_1-\mu)^2 \\ Q_3=a_3\mu+\frac{1}{2}b_3\mu^2+k_3(T-\mu)+\frac{k_3\theta_3}{2}(t_1-\mu)^2 \end{array}\right\} \quad (14)$$

Finally, Fuzzy total cost function of the system per unit time is,

$$\tilde{T}C(t_1,T)=\frac{1}{T}\left[\begin{array}{l} \tilde{A}+c_j\left(\tilde{a}\mu+\frac{1}{2}\tilde{b}\mu^2\right)+\tilde{h}\left(\frac{1}{2}\tilde{a}\mu^2+\frac{1}{3}\tilde{b}\mu^3\right)+\tilde{k}\tilde{h}\mu(t_1-\mu) \\ +\frac{1}{2}\tilde{k}\left\{(1-\tilde{\gamma})\tilde{\theta}c_j+\tilde{h}(1+\tilde{\theta}\mu)\right\}(t_1-\mu)^2+\frac{1}{6}\tilde{k}\tilde{h}\tilde{\theta}(t_1-\mu)^3 \\ +c_j\tilde{k}(T-\mu)+\frac{1}{2}\tilde{k}\tilde{c}_s(T-t_1)^2 \end{array}\right] \quad (15)$$

$$=\left(TC_1,TC_2,TC_3\right) \quad \text{(say)}$$

where

$$TC_1=\frac{1}{T}\left[\begin{array}{l} A_1+c_j\left(a_1\mu+\frac{1}{2}b_1\mu^2\right)+h_1\left(\frac{1}{2}a_1\mu^2+\frac{1}{3}b_1\mu^3\right)+k_1h_1\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_1\theta_1-k_3\theta_3\gamma_3)c_j+k_1h_1(1+\theta_1\mu)\right\}(t_1-\mu)^2 \\ +\frac{1}{6}k_1h_1\theta_1(t_1-\mu)^3+c_jk_1(T-\mu)+\frac{1}{2}k_1g_1(T-t_1)^2 \end{array}\right]$$

$$TC_2=\frac{1}{T}\left[\begin{array}{l} A_2+c_j\left(a_2\mu+\frac{1}{2}b_2\mu^2\right)+h_2\left(\frac{1}{2}a_2\mu^2+\frac{1}{3}b_2\mu^3\right)+k_2h_2\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_2\theta_2-k_2\theta_2\gamma_2)c_j+k_2h_2(1+\theta_2\mu)\right\}(t_1-\mu)^2 \\ +\frac{1}{6}k_2h_2\theta_2(t_1-\mu)^3+c_jk_2(T-\mu)+\frac{1}{2}k_2g_2(T-t_1)^2 \end{array}\right]$$

$$TC_3=\frac{1}{T}\left[\begin{array}{l} A_3+c_j\left(a_3\mu+\frac{1}{2}b_3\mu^2\right)+h_3\left(\frac{1}{2}a_3\mu^2+\frac{1}{3}b_3\mu^3\right)+k_3h_3\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_3\theta_3-k_1\theta_1\gamma_1)c_j+k_3h_3(1+\theta_3\mu)\right\}(t_1-\mu)^2 \\ +\frac{1}{6}k_3h_3\theta_3(t_1-\mu)^3+c_jk_3(T-\mu)+\frac{1}{2}k_3g_3(T-t_1)^2 \end{array}\right]$$

In the next section,we defuzzify the fuzzy order quantity $\tilde{Q} = \tilde{S} + \tilde{R}$ and fuzzy total cost function $\tilde{T}C(t_1, T)$ by using Signed distance method.

VI. DEFUZZIFICATION

A. *Defuzzification of* $\tilde{S}$, $\tilde{R}$ and $\tilde{Q}$

By using the Signed distance method, the defuzzified values of the fuzzy number $\tilde{S}$, $\tilde{R}$ and $\tilde{Q}$ are obtained as follows:

$$S_{sd} = \frac{1}{4}(S_1 + 2S_2 + S_3) = \frac{1}{4}\left[\begin{array}{l}(a_1 + 2a_2 + a_3)\mu + \frac{1}{2}(b_1 + 2b_2 + b_3)\mu^2 \\ +(k_1\theta_1 + 2k_2\theta_2 + k_3\theta_3)\frac{(t_1-\mu)^2}{2} + (k_1 + 2k_2 + k_3)(t_1 - \mu)\end{array}\right] \tag{16a}$$

$$R_{sd} = \frac{1}{4}(R_1 + 2R_2 + R_3) = \frac{1}{4}\left[(k_1 + 2k_2 + k_3)(T - t_1)\right] \tag{16b}$$

$$Q_{sd} = \frac{1}{4}(Q_1 + 2Q_2 + Q_3) = \frac{1}{4}\left[\begin{array}{l}(a_1 + 2a_2 + a_3)\mu + \frac{1}{2}(b_1 + 2b_2 + b_3)\mu^2 \\ +(k_1\theta_1 + 2k_2\theta_2 + k_3\theta_3)\frac{(t_1-\mu)^2}{2} + (k_1 + 2k_2 + k_3)(T - \mu)\end{array}\right] \tag{16c}$$

B. *Defuzzification of* $\tilde{TC}(t_1, T)$

Applying Signed distance method, the defuzzified value of the fuzzy number $\tilde{T}C(t_1, T)$ is given by,

$$TC_{sd}(t_1,T)=\frac{1}{4}\left[TC_1+2TC_2+TC_3\right]$$

$$=\frac{1}{4T}\left[\begin{array}{l} A_1+c_j\left(a_1\mu+\frac{1}{2}b_1\mu^2\right)+h_1\left(\frac{1}{2}a_1\mu^2+\frac{1}{3}b_1\mu^3\right)+k_1h_1\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_1\theta_1-k_3\theta_3\gamma_3)c_j+k_1h_1(1+\theta_1\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_1h_1\theta_1(t_1-\mu)^3 \\ \quad +c_jk_1(T-\mu)+\frac{1}{2}k_1g_1(T-t_1)^2 \\ +2\left\{\begin{array}{l} A_2+c_j\left(a_2\mu+\frac{1}{2}b_2\mu^2\right)+h_2\left(\frac{1}{2}a_2\mu^2+\frac{1}{3}b_2\mu^3\right)+k_2h_2\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_2\theta_2-k_2\theta_2\gamma_2)c_j+k_2h_2(1+\theta_2\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_2h_2\theta_2(t_1-\mu)^3 \\ \quad +c_jk_2(T-\mu)+\frac{1}{2}k_2g_2(T-t_1)^2 \end{array}\right\} \\ +A_3+c_j\left(a_3\mu+\frac{1}{2}b_3\mu^2\right)+h_3\left(\frac{1}{2}a_3\mu^2+\frac{1}{3}b_3\mu^3\right)+k_3h_3\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_3\theta_3-k_1\theta_1\gamma_1)c_j+k_3h_3(1+\theta_3\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_3h_3\theta_3(t_1-\mu)^3 \\ \quad +c_jk_3(T-\mu)+\frac{1}{2}k_3g_3(T-t_1)^2 \end{array}\right] \tag{17}$$

Now the objective is to bring out the optimal period $t_1=t_{1sd}^*$ for stock available inventory and the optimal length of the fill-up cycle $T=T_{sd}^*$ to minimize the company's defuzzified total cost function $TC_{sd}(t_1,T)$ per unit of time.

VII. THEORETICAL RESULTS

This section focuses on the theoretical analysis aimed at achieving optimal inventory planning for both models. Initially, the curvature of the company's total cost function in equation (13) for the crisp model is examined, followed by an investigation into the curvature of the defuzzified total cost function in equation (17). To do this, certain results from Theorems 3.2.9 and 3.2.10 in Cambini and Martein (2009) are utilized.

The results state that any function in the subsequent format

$$W(t)=\frac{G_1(t)}{G_2(t)},t=(t_1,t_2,...,t_n)\in R^n \tag{18}$$

is pseudo-convex, when both functions $G_1(t)\geq 0$ and $G_2(t)>0$ exhibit differentiability, with $G_1(t)\geq 0$ satisfying the convexity condition and $G_2(t)>0$ maintaining concavity. Exploiting these crucial findings, we examine the joint pseudo-convexity of the company's total cost function (13) for the crisp model and the defuzzified total cost function (17) for the fuzzy model are investigated.

A. Model for the inventory procedure in Crisp environment

For achieving the best inventory planning for minimizing the total cost $TC(t_1,T)$, we compute the first-order partial derivatives of $TC(t_1,T)$ with respect to t_1 and T, then set equal to zero. We get

$$\frac{\partial}{\partial t_1}\left[TC(t_1,T)\right]=0$$

i.e. $$h\mu+\left\{(1-\gamma)c_j\theta+(1+\theta\mu)h\right\}(t_1-\mu)+\frac{1}{2}h\theta\left(t_1-\mu\right)^2-c_s\left(T-t_1\right)=0 \tag{19}$$

and $\frac{\partial}{\partial T}\left[TC(t_1,T)\right]=0$

i.e.

$$\begin{bmatrix} A+c_j\left(a\mu+\frac{1}{2}b\mu^2\right)+h\left(\frac{1}{2}a\mu^2+\frac{1}{3}b\mu^3\right)-k\mu c_j+kh\mu(t_1-\mu) \\ +\frac{1}{2}k\left\{(1-\gamma)c_j\theta+h(1+\theta\mu)\right\}(t_1-\mu)^2+\frac{1}{6}kh\theta(t_1-\mu)^3 \\ +\frac{1}{2}kc_s(t_1^2-T^2) \end{bmatrix}=0 \tag{20}$$

On solving the equations (19) and (20) simultaneously, the optimal period for stock available inventory t_1^* and the optimal length of the fill-up cycle T^* to minimize total cost per unit of time is achieved. Now, the existence of the point (t_1^*,T^*) and the global optimal value of the company's total cost $TC(t_1,T)$ at the point (t_1^*,T^*) both are examined in the following theorem.

- **Theorem 1.**The total cost function $TC(t_1,T)$ is strictly pseudo-convex function in t_1 and T, and therefore, $TC(t_1,T)$ attains the unique global minimum value at (t_1^*,T^*).
- **Proof**: For computational convenience, define the following supporting functions from equation (13):

$$f_1(t_1,T)=A+c_j\left(a\mu+\frac{1}{2}b\mu^2\right)+h\left(\frac{1}{2}a\mu^2+\frac{1}{3}b\mu^3\right)+kh\mu(t_1-\mu)$$
$$+\frac{1}{2}k\left\{(1-\gamma)c_j\theta+h(1+\theta\mu)\right\}(t_1-\mu)^2+\frac{1}{6}kh\theta(t_1-\mu)^3$$
$$+c_jk(T-\mu)+\frac{1}{2}kc_s(T-t_1)^2 \tag{21}$$

and $\varphi_1(t_1,T)=T>0$ (22)

Subsequently, the company's total cost function $TC(t_1,T)$ becomes $TC(t_1,T)=\frac{f_1(t_1,T)}{\phi_1(t_1,T)}$.

To ensure the joint convexity of $f_1(t_1,T)$, it is necessary to build the Hessian matrix for $f_1(t_1,T)$ with respect to t_1 and T. Thereby, compute the necessary first and second order partial derivatives of $f_1(t_1,T)$ as follows:

$$\frac{\partial f_1(t_1,T)}{\partial t_1}=kh\mu+k\left\{(1-\gamma)c_j\theta+(1+\theta\mu)h\right\}(t_1-\mu)$$
$$+\frac{1}{2}kh\theta\left(t_1-\mu\right)^2-kc_s\left(T-t_1\right) \tag{23}$$

$$\frac{\partial^2 f_1(t_1,T)}{\partial t_1^2}=k\left\{(1-\gamma)c_j\theta+(1+\theta\mu)h+h\theta(t_1-\mu)+c_s\right\}>0 \tag{24}$$

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$$\frac{\partial f_1(t_1,T)}{\partial T} = kc_j + kc_s(T - t_1) \tag{25}$$

$$\frac{\partial^2 f_1(t_1,T)}{\partial T^2} = kc_s > 0 \tag{26}$$

$$\frac{\partial^2 f_1(t_1,T)}{\partial t_1 \partial T} = -kc_s \tag{27}$$

Subsequently, the Hessian matrix for $f_1(t_1,T)$ is given by

$$H_{ii} = \begin{bmatrix} \frac{\partial^2 f_1(t_1,T)}{\partial t_1^2} & \frac{\partial^2 f_1(t_1,T)}{\partial T\, \partial t_1} \\ \frac{\partial^2 f_1(t_1,T)}{\partial t_1\, \partial T} & \frac{\partial^2 f_1(t_1,T)}{\partial T^2} \end{bmatrix}, \quad \forall\; i = 1,\ 2. \tag{28}$$

The first principal minor is

$$H_{11} = \frac{\partial^2 f_1(t_1,T)}{\partial t_1^2} = k\left\{(1-\gamma)c_j\theta + (1+\theta\mu)h + h\theta(t_1-\mu) + c_s\right\} > 0 \quad \text{as since} \quad 0 \le \gamma < 1 .$$

In addition, the second principal minor is

$$\begin{aligned} H_{22} &= \frac{\partial^2 f_1(t_1,T)}{\partial t_1^2} . \frac{\partial^2 f_1(t_1,T)}{\partial T^2} - \left\{\frac{\partial^2 f_1(t_1,T)}{\partial T \partial t_1}\right\}^2 \\ &= \left[k\left\{(1-\gamma)c_j\theta + (1+\theta\mu)h + h\theta(t_1-\mu) + c_s\right\}\right](kc_s) - (kc_s)^2 \\ &= k^2 c_s\left[(1-\gamma)c_j\theta + (1+\theta t_1)h\right] > 0 \end{aligned}$$

Since all the principal minors of the Hessian matrix of $f_1(t_1,T)$ are positive, the Hessian matrix of $f_1(t_1,T)$ is positive definite. Therefore $f_1(t_1,T)$ is a non-negative, differentiable and strictly convex function in t_1 and T simultaneously. In addition, the denominator $\phi_1(t_1,T) = T$ is always positive, differentiable, and affine. As a result, the total cost function per unit time $TC(t_1,T)$ is strictly pseudo-convex function in t_1 and T (by Cambini and Martein ,2009) and has unique minimum value. Consequently, $TC(t_1,T)$ attains the unique global minimum value at (t_1^*, T^*). This finishes the proof.

B. Model for the inventory procedure in fuzzy environment

To determine the best inventory planning for minimizing the defuzzified total cost function $TC_{sd}(t_1,T)$, we compute the first order partial derivatives of $TC_{sd}(t_1,T)$ with respect to t_1 and T, then set equal to zero. We get

$$\frac{\partial}{\partial t_1}\left[TC_{sd}(t_1,T)\right] = 0$$

i.e.

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$$\begin{bmatrix} [k_1h_1\mu+\{(k_1\theta_1-k_3\theta_3\gamma_3)c_j+k_1h_1(1+\theta_1\mu)\}(t_1-\mu)+\frac{1}{2}k_1h_1\theta_1(t_1-\mu)^2-k_1g_1(T-t_1)]+ \\ 2[k_2h_2\mu+\{(k_2\theta_2-k_2\theta_2\gamma_2)c_j+k_2h_2(1+\theta_2\mu)\}(t_1-\mu)+\frac{1}{2}k_2h_2\theta_2(t_1-\mu)^2-k_2g_2(T-t_1)]+ \\ [k_3h_3\mu+\{(k_3\theta_3-k_1\theta_1\gamma_1)c_j+k_3h_3(1+\theta_3\mu)\}(t_1-\mu)+\frac{1}{2}k_3h_3\theta_3(t_1-\mu)^2-k_3g_3(T-t_1)] \end{bmatrix}=0$$

or,

$$\begin{bmatrix} (k_1h_1+2k_2h_2+k_3h_3)\mu+[\{(1-\gamma_1)k_1\theta_1+2(1-\gamma_2)k_2\theta_2+(1-\gamma_3)k_3\theta_3\}c_j \\ +\{k_1h_1(1+\theta_1\mu)+2k_2h_2(1+\theta_2\mu)+k_3h_3(1+\theta_3\mu)\}](t_1-\mu) \\ +\frac{1}{2}(k_1h_1\theta_1+2k_2h_2\theta_2+k_3h_3\theta_3)(t_1-\mu)^2-(k_1g_1+2k_2g_2+k_3g_3)(T-t_1) \end{bmatrix}=0 \qquad (29)$$

and $\frac{\partial}{\partial T}\left[TC_{sd}(t_1,T)\right]=0$

i.e. $\frac{\partial}{\partial T}\left[\frac{1}{4T}(TC_1+2TC_2+TC_3)\right]=0$

or,

$$\begin{bmatrix} (A_1+2A_2+A_3)+(a_1+2a_2+a_3)\mu c_j+(b_1+2b_2+b_3)\frac{\mu^2c_j}{2}+(a_1h_1+2a_2h_2+a_3h_3)\frac{\mu^2}{2} \\ +(b_1h_1+2b_2h_2+b_3h_3)\frac{\mu^3}{3}-(k_1+2k_2+k_3)\mu c_j+\mu(k_1h_1+2k_2h_2+k_3h_3)(t_1-\mu) \\ +[\{(1-\gamma_1)k_1\theta_1+2(1-\gamma_2)k_2\theta_2+(1-\gamma_3)k_3\theta_3)\}c_j+\{k_1h_1(1+\theta_1\mu)+2k_2h_2(1+\theta_2\mu) \\ +k_3h_3(1+\theta_3\mu)\}]\frac{(t_1-\mu)^2}{2}+(k_1h_1\theta_1+2k_2h_2\theta_2+k_3h_3\theta_3)\frac{(t_1-\mu)^3}{6} \\ +(k_1g_1+2k_2g_2+k_3g_3)\frac{(T-t_1)^2}{2}-T(k_1g_1+2k_2g_2+k_3g_3)(T-t_1) \end{bmatrix}=0 \qquad (30)$$

On solving equations (29) and (30) simultaneously, the optimal period for stock available inventory t_{1sd}^* and the optimal length of the fill-up cycle T_{sd}^* to minimize the company's defuzzified total cost function $TC_{sd}(t_1,T)$ per unit of time is achieved.Now, the existence of the point (t_{1sd}^*,T_{sd}^*) and the global optimal value of the defuzzified total cost function $TC_{sd}(t_1,T)$ at the point (t_{1sd}^*,T_{sd}^*) both are examined in the following theorem.

- **Theorem 2.**The defuzzified total cost function $TC_{sd}(t_1,T)$ is strictly pseudo-convex function in t_1 and T, and therefore, $TC_{sd}(t_1,T)$ attains the unique global minimum value at (t_{1sd}^*,T_{sd}^*).
- **Proof**: From the equation (17) we have,

$$TC_{sd}(t_1,T)=\frac{1}{4}\left[TC_1+2TC_2+TC_3\right]$$

where $$TC_1=\frac{1}{T}\begin{bmatrix} A_1+c_j\left(a_1\mu+\frac{1}{2}b_1\mu^2\right)+h_1\left(\frac{1}{2}a_1\mu^2+\frac{1}{3}b_1\mu^3\right)+k_1h_1\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_1\theta_1-k_3\theta_3\gamma_3)c_j+k_1h_1(1+\theta_1\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_1h_1\theta_1(t_1-\mu)^3 \\ +c_jk_1(T-\mu)+\frac{1}{2}k_1g_1(T-t_1)^2 \end{bmatrix}$$

$$TC_2=\frac{1}{T}\begin{bmatrix} A_2+c_j\left(a_2\mu+\frac{1}{2}b_2\mu^2\right)+h_2\left(\frac{1}{2}a_2\mu^2+\frac{1}{3}b_2\mu^3\right)+k_2h_2\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_2\theta_2-k_2\theta_2\gamma_2)c_j+k_2h_2(1+\theta_2\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_2h_2\theta_2(t_1-\mu)^3 \\ +c_jk_2(T-\mu)+\frac{1}{2}k_2g_2(T-t_1)^2 \end{bmatrix}$$

$$TC_3=\frac{1}{T}\begin{bmatrix} A_3+c_j\left(a_3\mu+\frac{1}{2}b_3\mu^2\right)+h_3\left(\frac{1}{2}a_3\mu^2+\frac{1}{3}b_3\mu^3\right)+k_3h_3\mu(t_1-\mu) \\ +\frac{1}{2}\left\{(k_3\theta_3-k_1\theta_1\gamma_1)c_j+k_3h_3(1+\theta_3\mu)\right\}(t_1-\mu)^2+\frac{1}{6}k_3h_3\theta_3(t_1-\mu)^3 \\ +c_jk_3(T-\mu)+\frac{1}{2}k_3g_3(T-t_1)^2 \end{bmatrix}$$

By the **theorem 1**, it is easily follows that the functions TC_1, TC_2 and TC_3 are strictly pseudo convex functions in t_1 and T. Again $TC_{sd}(t_1,T)$ is a linear combination of the strictly pseudo convex functions TC_1, TC_2 and TC_3. Consequently, the defuzzified total cost function $TC_{sd}(t_1,T)$ is a strictly pseudo convex function in t_1 and T.

Thus, $TC_{sd}(t_1,T)$ possesses the unique global minimum value at (t_{1sd}^*, T_{sd}^*). This finishes the proof.

VIII. SOLUTION ALGORITHM

This section deals the algorithm that was proposed in Shaikh et al. (2019), to achieve the best inventory plan for the company's crisp inventory model. With some modifications the algorithm is being used in case of fuzzy environment. Suppose the supplier of the item offers a rebate scheme on unit acquisition price based on the purchased quantity by using $n-$ number of quantity breaks (q_i where $i=1, 2, 3, \ldots\ldots, n$ and $q_1<q_2<\ldots\ \ldots<q_n<\infty$) where the unit acquisition price (c_j) decreases stepwise based on the purchased quantity (Q) such that $c_1>c_2>\cdots\cdots>c_n$. Under this offered rebate scheme, the purchased quantity of the company may stand at any quantity break $Q=q_i$ or lie between any two successive quantity breaks. When the optimum requisition amount Q lies between any two successive quantity breaks, the best Q^* for the crisp model is achieved from equation (11) with the assistance of t_1^* and T^*, calculated by solving equations (19) and (20) simultaneously. Similarly, the optimal order quantity Q_{sd}^* for the fuzzy model is determined from the equation (16c) with the help of t_{1sd}^* and T_{sd}^* calculated by solving equations (29) and (30) simultaneously. On the other hand, when the purchased quantity of the company Q^* (for the crisp model) or Q_{sd}^* (for the fuzzy model) stands at any quantity break, the lot size becomes fixed and then the duration of the cycle of the company is expressed as a function of t_1 [one can see equation (12) for crisp model and equation (16c) for fuzzy model] . As a result, the equations (19) and (20) (for crisp model) or equations (29) and (30) (for fuzzy model) are not compatible to compute the best inventory plan for the company under these circumstances and it is necessary to derive new condition to achieve the best inventory plan.

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A. New necessary conditions for the crisp model

This part of the subsection determines the necessary condition to obtain the best inventory plan for the company when the best Q^* stands at any quantity break. After substituting the expression of T with the assistance of equation (12), the company's total cost TC in equation (13) involves only one decision variable t_1. Thus, it is necessary to determine the condition for computing t_1^* under which the company's total cost $TC(t_1)$ is optimized.

$$\frac{dT}{dt_1} = -\theta(t_1-\mu) \quad (31)$$

In this context, the condition for the best t_1^* is found by setting $\frac{d}{dt_1}\{TC(t_1)\}=0$ with the assistance of equation (31). After performing some simplifications, the condition is given by

From equation (11), one has

$$\theta(t_1-\mu)\left[\begin{aligned}&\frac{A}{k}+\frac{c_j}{k}\left(a\mu+\frac{1}{2}b\mu^2\right)+\frac{h}{k}\left(\frac{1}{2}a\mu^2+\frac{1}{3}b\mu^3\right)+h\mu(t_1-\mu)\\&+\frac{1}{2}\left\{(1-\gamma)c_j\theta+h(1+\theta\mu)\right\}(t_1-\mu)^2+\frac{1}{6}h\theta(t_1-\mu)^3+c_j(T-\mu)+\frac{1}{2}c_s(T-t_1)^2\end{aligned}\right]$$
$$+T\left[h\mu+\left\{h(1+\theta\mu)-\lambda c_j\theta\right\}(t_1-\mu)+\frac{1}{2}h\theta(t_1-\mu)^2-c_s\{1+\theta(t_1-\mu)\}(T-t_1)\right]=0 \quad (32)$$

Where

$$T=\frac{1}{k}\left[Q-a\mu-\frac{1}{2}b\mu^2-\frac{k\theta}{2}(t_1-\mu)^2\right]+\mu$$

B. New necessary conditions for the fuzzy model

This subsection determines the necessary conditions to obtain the best inventory plan for the company in the fuzzy environment when the best Q_{sd}^* stands at any quantity break. With the assistance of equation (16c), the company's defuzzified total cost TC_{sd} in equation (17) is transformed into the equation which involves with only one decision variable t_1. Thus, it is necessary to determine the condition for computing t_{1sd}^* under which the company's total cost $TC_{sd}(t_1)$ is minimized.

From equation (16c), one has

$$\frac{dT}{dt_1} = -\frac{k_1\theta_1+2k_2\theta_2+k_3\theta_3}{k_1+2k_2+k_3}(t_1-\mu) = -\delta(t_1-\mu) \quad (33)$$

Where

$$\delta=\frac{k_1\theta_1+2k_2\theta_2+k_3\theta_3}{k_1+2k_2+k_3} \quad (34)$$

At this situation, the condition for the best t_{1sd}^* is found by setting $\frac{d}{dt_1}\{TC_{sd}(t_1)\}=0$ with the assistance of equation (33). After performing some simplifications, the condition is given by

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$$\delta(t_1-\mu)\left[\begin{array}{l}(A_1+2A_2+A_3)+(a_1+2a_2+a_3)\mu c_j+(b_1+2b_2+b_3)\dfrac{\mu^2 c_j}{2}+(a_1h_1+2a_2h_2+a_3h_3)\dfrac{\mu^2}{2}\\ +(b_1h_1+2b_2h_2+b_3h_3)\dfrac{\mu^3}{3}+\mu(k_1h_1+2k_2h_2+k_3h_3)(t_1-\mu)\\ +[\{(1-\gamma_1)k_1\theta_1+2(1-\gamma_2)k_2\theta_2+(1-\gamma_3)k_3\theta_3)\}c_j+\{\mathrm{k}_1h_1(1+\theta_1\mu)+2\mathrm{k}_2h_2(1+\theta_2\mu)\\ +\mathrm{k}_3h_3(1+\theta_3\mu)\}]\dfrac{(t_1-\mu)^2}{2}+(k_1h_1\theta_1+2k_2h_2\theta_2+k_3h_3\theta_3)\dfrac{(t_1-\mu)^3}{6}\\ +(k_1+2k_2+k_3)c_j(T-\mu)+(k_1g_1+2k_2g_2+k_3g_3)\dfrac{(T-t_1)^2}{2}\end{array}\right]$$

$$+T\left[\begin{array}{l}\mu(k_1h_1+2k_2h_2+k_3h_3)+[\{(1-\gamma_1)k_1\theta_1+2(1-\gamma_2)k_2\theta_2+(1-\gamma_3)k_3\theta_3)\}c_j\\ +\{k_1h_1(1+\theta_1\mu)+2k_2h_2(1+\theta_2\mu)+k_3h_3(1+\theta_3\mu)\}](t_1-\mu)\\ +(k_1h_1\theta_1+2k_2h_2\theta_2+k_3h_3\theta_3)\dfrac{(t_1-\mu)^2}{2}-(k_1+2k_2+k_3)c_j\delta(t_1-\mu)\\ -(k_1g_1+2k_2g_2+k_3g_3)\{1+\delta(t_1-\mu)\}(T-t_1)\end{array}\right]=0 \quad (35)$$

C. Algorithm for the company's best inventory plan in crisp sense

Step 1 : Initialize $TC_{\min}(t_1,T)=\infty$ and $i=n$.

Step 2 : Exploiting the values of $a,b,A,k,\theta,\mu,h,\gamma,c_s$ and c_i, compute t_1 and T by solving equations (19) and (20). Utilizing the derived values of t_1 and T, determine the corresponding requisition amount Q from equation (11).

(i) If $Q\in[q_i,q_{i+1})$, then this solution is feasible. Determine the company's totalcost $TC_i(t_1,T)$ adopting this feasible solution.

If $TC_i(t_1,T)<TC_{\min}(t_1,T)$, then set $TC_{\min}(t_1,T)=TC_i(t_1,T)$.

Move to Step 5.

(ii) If $Q\notin[q_i,q_{i+1})$, then the solution is not feasible and move to Step 3.

Step 3 : Obtain the expression of T in terms of t_1 after setting $Q=q_i$ in equation (12), compute t_1 and T by solving equation (12) and (32) simultaneously. Exploiting t_1 and T, determine $TC_i(t_1,T)$ from equation (13). If $TC_i(t_1,T)<TC_{\min}(t_1,T)$, then set $TC_{\min}(t_1,T)=TC_i(t_1,T)$.

Move to Step 4.

Step 4 : If $i\geq 2$, move to Step 2 with $i=i-1$; else, move to Step 5.

Step 5 : The company's minimum inventory cost per unit time is $TC_{\min}(t_1,T)$ with the optimal $t_1=t_1^*$ and $T=T^*$.

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D. Algorithm for the company's best inventory plan in the fuzzy environment

Step 1 : Initialize $TC_{sd\min}(t_{1sd},T_{sd})=\infty$ and $i=n$.

Step 2 : Exploiting the values of $\tilde{a},\tilde{b},\tilde{A},\tilde{k},\tilde{\theta},\mu,\tilde{h},\tilde{\gamma},\tilde{c}_s$ and c_i, compute $t_1=t_{1sd}$ and $T=T_{sd}$ by solving equations (29) and (30). Utilizing the derived value of t_{1sd} and T_{sd}, determine the corresponding requisition amount Q_{sd} from equation (16c).

(i) If $Q_{sd}\in[q_i,q_{i+1})$, then this solution is feasible. Determine the company's total cost $TC_{isd}(t_{1sd},T_{sd})$ adopting this feasible solution. If $TC_{isd}(t_{1sd},T_{sd})<TC_{sd\min}(t_{1sd},T_{sd})$, then set $TC_{sd\min}(t_{1sd},T_{sd})=TC_{isd}(t_{1sd},T_{sd})$. Move to Step 5.

(ii) If $Q\notin[q_i,q_{i+1})$, then the solution is not feasible and move to Step 3.

Step 3 : Set $Q_{sd}=q_i$ in the equation (16c). Compute t_{1sd} and T_{sd} by solving equation (16c) and (35) simultaneously. Exploiting t_{1sd} and T_{sd}, determine $TC_{isd}(t_{1sd},T_{sd})$ from equation (17).

If $TC_{isd}(t^*_{1sd},T^*_{sd})<TC_{sd\min}(t_{1sd},T_{sd})$, then set

$TC_{sd\min}(t_{1sd},T_{sd})=TC_{isd}(t^*_{1sd},T^*_{sd})$. Move to Step 4.

Step 4 : If $i\geq 2$, move to Step 2 with $i=i-1$; else, move to Step 5.

Step 5 : The company's minimum inventory cost per unit time in fuzzy environment is $TC_{sd\min}(t_{1sd},T_{sd})$ with the optimal $t_{1sd}=t^*_{1sd}$ and $T_{sd}=T^*_{sd}$.

IX. NUMERICAL EXAMPLES

Two numerical examples are illustrated below with a view to examine the working performance of the algorithms 6.6.3 and 6.6.4.

Example 1. Inventory procedure in crisp sense

Suppose the following rebate scheme on the unit acquisition price based upon the purchased amount is presented by the supplier.

Table 1: Unit acquisition price based on purchased amount in Crisp model

Amount (units)	$Q\in[q_1,q_2)=[0,400)$	$Q\in[q_2,q_3)=[400,460)$	$Q\in[q_3,q_4)=[460,\infty)$
Acquisition cost ($/unit)	$c_1=4.75$	$c_2=4.5$	$c_3=4$

The values of the other known parameters are:

$$A=200,\ a=100,\ b=1.5,\ \mu=2,\ \theta=.05,\ h=.03,\ \gamma=.04,\ c_s=3.$$

The best inventory planning for the company under this rebate scheme to minimize company's cost is achieved by utilizing the algorithm 6.6.3 as follows:

Initially set $TC_{\min}(t_1,T)=\infty$ and $i=3$.

Iteration 1: $c_3=4$ *for* $Q\in[q_3,q_4)=[460,\infty)$

Exploiting the values of $A,a,b,k,\theta,\mu,h,\gamma,c_s$ and c_3, compute $t_1=4.362$ and $T=4.561$ by solving equations (19) and(20) .Utilizing the derived values of t_1 and T, from equation (11), determine the corresponding requisition amount $Q=481$ units. Since $Q\in[q_3,q_4)=[460,\infty)$,this solution is feasible. Adopting this feasible solution in equation (13), determine the company's total cost $TC_3(t_1,T)=473.35$.

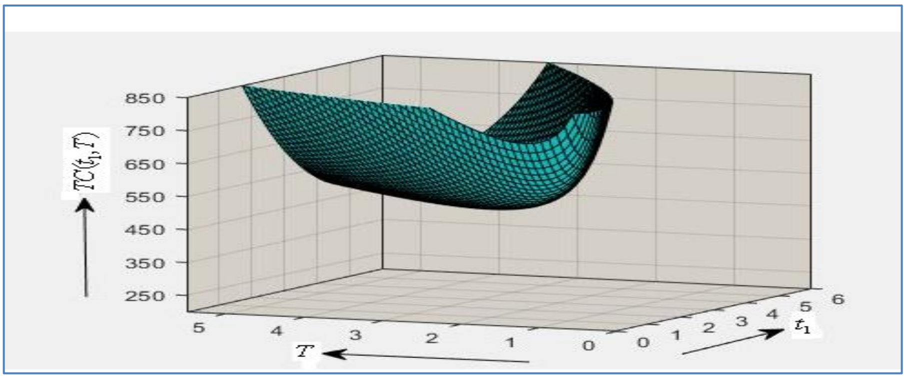

Fig. 4: Total cost function $TC(t_1,T)$ Vs. t_1 and T .

Furthermore, as $TC_3(t_1,T) = 473.35 < \infty = TC_{min}(t_1,T)$, set $TC_{min}(t_1,T) = 473.35$. Therefore, the company's minimum inventory cost per unit time is $TC_{\min}(t_1,T) = 473.35$ with the optimal $t_1^* = 4.362$, $T^* = 4.561$ and $Q^* = 481$. From this purchased quantity amount, the best backordering quantity amount is $R^* = 20$ units, determined from equation (10) while the stock amount at starting moment of each cycle after backordering is $S^* = 461$ units from equation (8).

Figure 6.3 exhibits the behavior of the company's total cost $TC(t_1,T)$ for crisp inventory model against variables t_1 and T . As seen from Figure 6.3, the company's total cost $TC(t_1,T)$ is jointly convex in t_1 and T , and therefore, $TC(t_1,T)$ holds the global minimum value at the unique point (t_1^*, T^*) . To examine the behavior of the company's total cost $TC(t_1,T)$ more explicitly, two additional graphs (Figures 6.4 and 6.5) are provided against each variable separately.

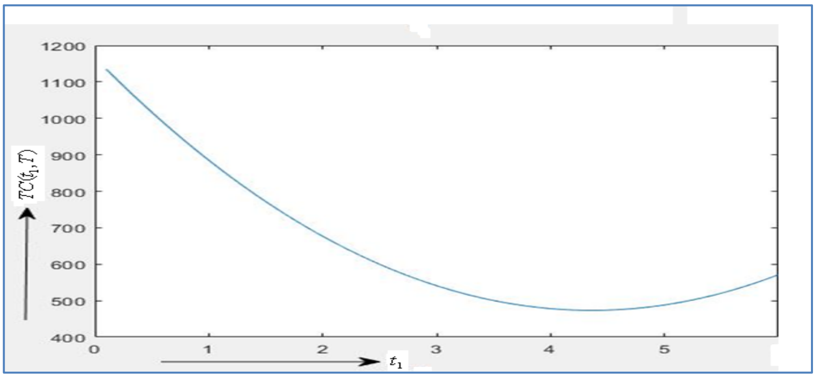

Fig. 5: Total cost function $TC(t_1,T)$ Vs. t_1

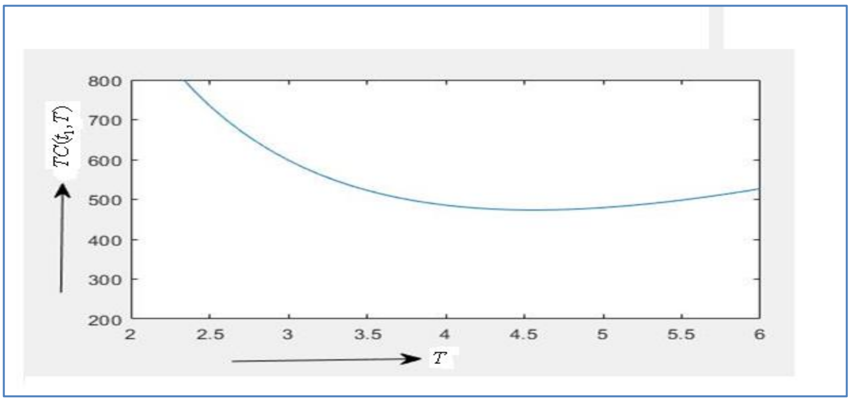

Fig. 6: Total cost function $TC(t_1,T)$ Vs. T .

Example 2. Inventory procedure in fuzzy sense.

This example adopts the same rebate scheme on unit acquisition price and other data in Example 1 along with the following triangular fuzzy numbers to justify the proposed fuzzy model.

$$\tilde{A}=(150,200,250),\ \tilde{a}=(80,100,120),\ \tilde{b}=(1,1.5,2),\ \mu=2,\ \tilde{\theta}=(.04,.05,.06),$$
$$\tilde{h}=(.01,.03,.05),\ \tilde{\gamma}=(.03,.04,.05),\ \tilde{k}=(82,103,124),\ \tilde{c}_s=(2,3,4)\ .$$

The best inventory planning for the company under the rebate scheme to minimize company's total cost in fuzzy environment is achieved by utilizing the algorithm 6.6.4 as follows:

Initially set $TC_{sd\min}(t_{1sd},T_{sd})=\infty$ and i = 3.

Iteration 1: $c_3=4$ *for* $Q\in[q_3,q_4)=[460,\infty)$

Exploiting the values of $\tilde{A}$, ã, b̃, μ, $\tilde{\theta}$, h̃, $\tilde{\gamma}$, $\tilde{k}$, $\tilde{c}_s$ and c_3, compute $t_{1sd}=4.312$ and $T_{sd}=4.506$ by solving equations (29) and (30). Utilizing the derived value of t_{1sd} and T_{sd}, from equation (16c), we determine the corresponding requisition amount $Q_{sd}=475$ units. Since $Q_{sd}\in[q_3,q_4)=[460,\infty)$,the solution is feasible. Adopting this feasible solution in equation (17), determine the company's total cost $TC_{sd3}(t_{1sd},T_{sd})=474.088$ from the fuzzy model.

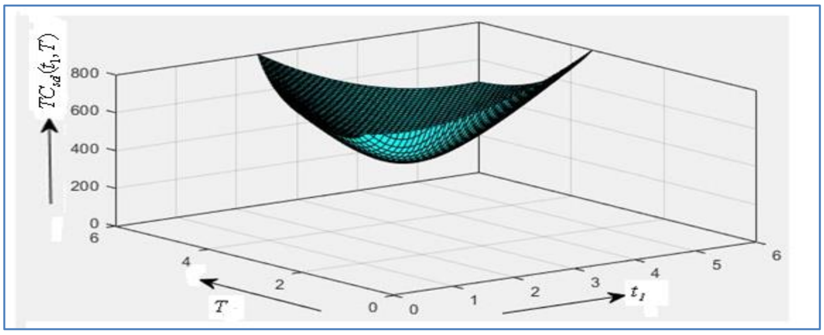

Fig. 7: Defuzzified total cost function $TC_{sd}(t_1,T)$ Vs. t_1 and T .

Again $TC_{sd3}(t_{1sd},T_{sd})=474.088<\infty=TC_{sd\,min}(t_{1sd},T_{sd})$, set $TC_{sd\,min}(t_{1sd},T_{sd})=474.088$. Therefore, using the fuzzy model the company's minimum cost per unit time is $TC_{sd\,min}(t_{1sd},T_{sd})=474.088$ with the optimal $t_{1sd}^*=4.321$, $T_{sd}^*=4.506$ and $Q_{sd}^*=475$ units. With this purchased quantity amount, the best backordering quantity amount is $R_{sd}^*=20$ units, determined from equation (16b) while the stock amount at starting moment of each cycle after backordering is $S_{sd}^*=455$ units from equation (16a).

Figure 6.6 displays the behavior of the company's defuzzified total cost $TC_{sd}(t_1,T)$ for fuzzy inventory model against variables t_1 and T . As seen from Figure 6.6, the company's defuzzified total cost function $TC_{sd}(t_1,T)$ is jointly convex in t_1 and T , and therefore, $TC_{sd}(t_1,T)$ holds the global minimum value at the unique point (t_{1sd}^*,T_{sd}^*). To examine the behavior of the company's defuzzified total cost function $TC_{sd}(t_1,T)$ more clearly, two additional graphs (Figures 6.7 and 6.8) are provided against each variable separately.

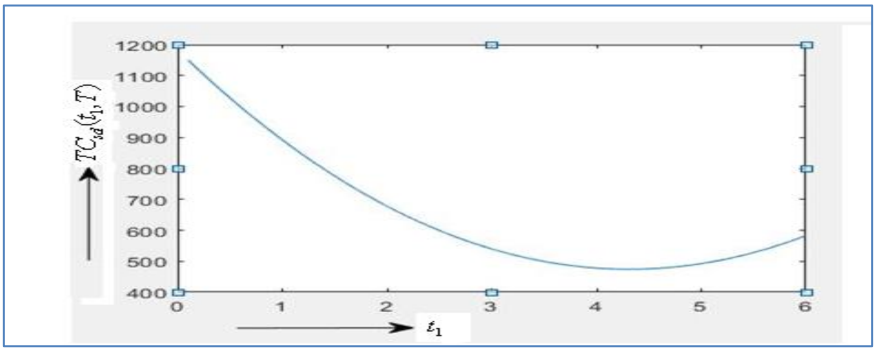

Fig. 8: Defuzzified total cost function $TC_{sd}(t_1,T)$ Vs. t_1

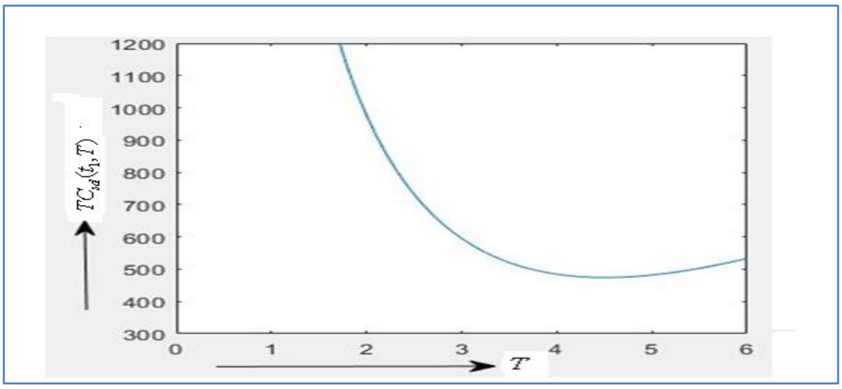

Fig. 9: Defuzzified total cost function $TC_{sd}(t_1,T)$ Vs. T

A. Comparison between the Two Optimal Results:

A comparison between the crisp and the fuzzy inventory model on the basis of optimal results is displayed in the following table:

Table 2: Comparison between the best outcomes in Crisp and Fuzzy inventory models:

Model	**Optimal value of t_1/t_{1sd}**	**Optimal value of T/T_{sd}**	**Optimal value of S/S_{sd}**	**Optimal value of R/R_{sd}**	**Optimal value of Q/Q_{sd}**	**Optimal value of total cost TC/TC_{sd}**
Crisp	4.362	4.561	461	20	481	473.351
Fuzzy	4.312	4.506	455	20	475	474.088

X. SENSITIVITY ANALYSIS

For the fuzzy model, we have analyzed how the optimal inventory strategy of the company evolves as we vary each system parameter. We highlight several significant observations and extract guidelines for minimizing the company's costs in this section.Thereby, computational results by altering each fuzzy parameter from -20% to +20% are delivered in Table 2. In addition, Table 2 is constructed adopting the numerical data used in Example 2. It is worth noting that Table 2 exposes the company's best inventory strategy for additional 36 new inventory examples.

Table 3: Consequences of system parameters on the optimal solution for the inventory procedure in fuzzy environment:

Parameter	**Original value**	**% of changes**	**New values**	t^*_{1sd}	T^*_{sd}	S^*_{sd}	R^*_{sd}	Q^*_{sd}	TC^*_{sd}
$\tilde{A}$	(150, 200,250)	-20	(120,160,200)	4.190	4.373	441	19	460	464.987
		-10	(135,180,225)	4.189	4.373	441	19	460	469.560
		+10	(165,220,275)	4.491	4.699	476	21	497	478.433
		+20	(180,240,300)	4.663	4.884	496	23	519	482.607
$\tilde{a}$	(80, 100, 120)	-20	(64,80,96)	5.086	5.335	439	21	460	390.045
		-10	(72,90,108)	4.592	4.807	440	20	460	432.048
		+10	(88,110,132)	4.157	4.339	480	21	501	516.00

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		+20	(96,120,144)	4.022	4.194	505	21	526	557.752
$\tilde{b}$	(1, 1.5, 2)	-20	(.8,1.2,1.6)	4.344	4.541	457	20	477	472.099
		-10	(.9,1.35,1.8)	4.328	4.524	456	20	476	473.094
		+10	(1.1,1.65,2.2)	4.296	4.489	454	20	474	475.080
		+20	(1.2,1.8,2.4)	4.280	4.472	453	20	473	476.070
μ	2	-20	1.6	4.211	4.423	447	22	469	477.028
		-10	1.8	4.259	4.462	451	21	472	475.512
		+10	2.2	4.369	4.555	460	19	479	472.753
		+20	2.4	4.431	4.609	466	18	484	471.507
$\tilde{h}$	(.01, .03, .05)	-20	(.008,.024,.04)	4.384	4.573	464	19	483	472.657
		-10	(.009,.027,.045)	4.347	4.539	459	20	479	473.376
		+10	(.011,.033,.055)	4.278	4.474	451	20	471	474.793
		+20	(.012,.036,.06)	4.244	4.443	447	21	468	475.491
$\tilde{\theta}$	(.04, .05, .06)	-20	(.032,.04,.048)	4.654	4.84	491	19	510	471.377
		-10	(.036,.045,.054)	4.470	4.66	472	19	491	472.802
		+10	(.044,.055,.066)	4.174	4.372	441	20	461	475.256
		+20	(.048,.06,.072)	4.146	4.345	439	21	460	476.352
$\tilde{\gamma}$	(.03, .04, .05)	-20	(.024,.032,.04)	4.3	4.494	454	20	474	474.192
		-10	(.027,.036,.045)	4.306	4.5	454	20	474	474.140
		+10	(.033,.044,.055)	4.318	4.512	456	20	476	474.035
		+20	(.036,.048,.06)	4.325	4.519	457	20	477	473.983
$\tilde{c}_s$	(2, 3, 4)	-20	(1.6,2.4,3.2)	4.298	4.540	453	25	478	473.756
		-10	(1.8,2.7,3.6)	4.306	4.521	454	22	476	473.940
		+10	(2.2,3.3,4.4)	4.317	4.494	456	18	474	474.210
		+20	(2.4,3.6,4.8)	4.321	4.484	456	17	473	474.312
c_j	(4.75, 4.5, 4)	-20	(3.8,3.6,3.2)	4.667	4.854	496	19	515	389.548
		-10	(4.275,4.05,3.6)	4.477	4.668	474	20	494	431.892
		+10	(5.225,4.95,4.4)	4.177	4.374	440	20	460	516.155
		+20	(5.7,5.4,4.8)	4.166	4.176	438	22	460	558.166

The following observations are accumulated from Table 2:

- The company's optimal purchased quantity (Q_{sd}^*) remains fixed if the values of $\tilde{A}$, $\tilde{a}$ and $\tilde{\gamma}$ increased from -20% to -10% and it increases when the value of the said parameters increases from +10% to +20%. Also (Q_{sd}^*) changes in a positive way in accordance with the changes in μ.On the other hand, the optimal (Q_{sd}^*) declines for the increasing values of the remaining parameters $\tilde{b},\tilde{h},\tilde{\theta},c_j\,(j=1,2,3)$ and $\tilde{c}_s$. When the cost of placing each order rises, the company aims to increase the optimal purchase quantity to reduce the average ordering cost per unit. Likewise, if the initial market demand increases, the overall demand also escalates, prompting the company to procure a larger order size.

- Maximum backordering level (R_{sd}^*) rises when $\tilde{A},\tilde{a},\tilde{h},\tilde{\theta}$ and $c_j\,(j=1,2,3)$ increase,while (R_{sd}^*) shrinks concerning the positive variations of the parameters μ and $\tilde{c}_s$. It is worth mentioning that the parameters $\tilde{b}$ and $\tilde{\gamma}$ has no influence on the optimal value of backordering level.

- The company's initial best positive stock level (S_{sd}^*) rises for the variation of the values of $\tilde{A},\tilde{a},\mu,\tilde{\gamma}$ and $\tilde{c}_s$ in positive way, while (S_{sd}^*) reduces concerning the change in the parameters $\tilde{b},\tilde{h},\tilde{\theta}$ and $c_j\,(j=1,2,3)$ in a positive way.

- The optimal period for stock available inventory (t_{1sd}^*) decreases initially and then increases for the positive changes in ordering cost $\tilde{A}$. In addition, it is noticed that the best t_{1sd}^* increases when the parameters $\mu,\tilde{\gamma}$ and $\tilde{c}_s$ increase, while (t_{1sd}^*) decreases with respect to $\tilde{a},\tilde{b},\tilde{h},\tilde{\theta}$ and c_j.

- The company's best cycle length (T_{sd}^*) for each business cycle declines noticeably when $\tilde{a},\tilde{\theta},\tilde{b},\tilde{h},\tilde{c}_s$ and $c_j\,(j=1,2,3)$ increase, while the optimal (T_{sd}^*) for the company increase when the parameters $\tilde{A},\mu$ and $\tilde{\gamma}$ increase.

- The company's minimum cost (TC_{sd}^{*}) rises for increasing the value of all parameters except the parameters μ and $\tilde{\gamma}$. The company's minimum cost (TC_{sd}^{*}) decreases as the parameters μ and $\tilde{\gamma}$ increase. Parameters $\tilde{a}$ and c_j have remarkably high effects on (TC_{sd}^{*}) . The minimum cost (TC_{sd}^{*}) increases significantly for a small increment in the values of $\tilde{A}$, $\tilde{a}$ and c_j .

XI. MANAGERIAL IMPLICATIONS

- If the cost of placing orders is high, the company should consider implementing measures to reduce this expense. One approach is to increase the order size, thereby decreasing the number of orders and the associated cost per unit.
- A higher value of the parameter μ signifies a longer lifespan for the product, indicating better product quality. The analysis indicates that optimal costs decrease with higher values of μ. Therefore, the company should prioritize receiving high-quality items from the supplier during replenishment to minimize overall costs.
- To address product deterioration, the company must stock more products than customer demand, resulting in increased purchasing and holding costs. To mitigate this, the company manager should focus on minimizing item deterioration by providing suitable storage conditions and ensuring a supply of high-quality items from the supplier.
- The salvage value significantly impacts overall inventory costs. A higher salvage value means the company will recoup more money when disposing of a deteriorated item, reducing the net cost. Thus, the company manager should enhance the storage and maintenance processes to preserve item condition and potentially increase salvage value.
- The unit procurement cost has the most significant impact on the company's total cost per unit of time. Even a slight increase in the unit procurement cost results in a considerable escalation of total costs. Therefore, the company should exercise caution in minimizing the unit procurement cost, such as opting for a supplier offering the lowest unit procurement price or capitalizing on order-based discount programs to obtain a lower procurement price through larger purchase quantities.

XII. CONCLUSION

This paper investigates an inventory model for a non-instantaneous deteriorating item with fully backlogged shortages, considering a lot size-based price discount policy. The demand rate is assumed to be time-dependent, and a salvage value is assigned to deteriorated products. The proposed model is developed in both crisp and fuzzy environments, where all related parameters are expressed as triangular fuzzy numbers. Defuzzification is carried out using the signed distance method.

The analytical examination of the curvature of the company's total cost is conducted in both crisp and fuzzy scenarios. To determine the optimal inventory plan under the offered discount scheme on the unit acquisition price, two algorithms are presented for both situations. In the fuzzy setting, a sensitivity analysis is conducted to investigate how the company's optimal inventory strategy evolves in reaction to modifications in inventory parameters. For instance, when the cost of placing each order is elevated, it is recommended for the company to boost the purchase quantity to lower the average ordering cost per unit.Additionally, recommendations are made for the company to decrease the rate of deterioration and enhance the salvage value of products by ensuring a favorable storage environment and obtaining high-quality items from the supplier.

A comparison of the optimal results presented in Table 1 indicates that the total minimum inventory cost derived in the crisp model is slightly lower than that of the fuzzy model. However, the fuzzy model is considered more realistic, as it addresses the uncertainty of parameters commonly encountered in real-world situations.

Similar to any research endeavor, this study has certain limitations. For example, the model assumes a zerolead time, implying that the supplier has the item readily available for immediate shipment. In reality, lead time may not always be zero, as the supplier might need to manufacture or collect the item from another source, potentially increasing inventory costs. Therefore, enhancing the proposed model's realism could involve incorporating lead time for replenishment. The model could further be extended by introducing stock-dependent demand, time-dependent holding costs, and deterioration rates while considering other forms of fuzzy numbers.

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