Volume 9, Issue 6, June – 2024 International Journal of Innovative Science and Research Technology
ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

# Alternative II Theory Solution for a Thick Rectangular Anisotropic Plate Under in-Plane and Lateral Loads

Bertram, D. I.[1*]; Onwuka, D.O.[2]; Ibearugbulem, O.M[3]; Okere, C.E.[4]
[1,2,3,4]Department of Civil Engineering, Federal University of Technology,
P. M. B. 1526, Owerri, Imo State, Nigeria

Corresponding Author: Bertram D.I.[1*]

**Abstract:- This work investigated the application of the Alternative II refined plate theory in the analysis of an anisotropic plate subjected to in-plane and lateral loads. The kinematic equations developed from the Alternative II Refined plate theory were used together with a complete three-dimensional constitutive relation to obtain the total potential energy of an anisotropic plate under lateral and in-plane loads. General variation of the total potential energy was done, a governing equation and two compatibility equations were obtained. A polynomial displacement function was obtained by solving the governing and compatibility equations. This was used to obtain peculiar displacement functions by satisfying the boundary conditions of any plate. The stiffness coefficients were obtained using the displacement function. With the displacement functions and the stiffness coefficients, the equations for the in-plane normal and shear stresses as well as the transverse normal and shear stresses were determined for any applied lateral load when the applied in-plane load is a fraction of the buckling load. Also, the equations for the displacements of the plate were determined. Numerical values of the stresses and displacement parameters were determined for span to thickness ratios of 5, 10, 20 and 100 at angle of fiber orientations of 08 and aspect ratios of 1, 1.5 and 2.0 when the ratio of applied in-plane load to buckling load are 0, 0.25 and 0.5. Using simple percentage difference, the results from this work were compared with the works of previous researchers.**

***Keywords:-*** *Alternative II Theory, Anisotropic, In-plane and Lateral Loads, Rectangular Plate.*

## I. INTRODUCTION

Plates are flat structural members bounded by plane or curved surfaces, separated by a plane known as the thickness. They are three-dimensional elements with length and breadth usually large when compared to the thickness (Shufrin and Eisenberger, 2005). In engineering applications, plates are greatly used as structural parts to withstand loads which can result to bending or buckling of the plate. Some of the uses of plates are in the fields of civil, mechanical, marine and aeronautical engineering for the construction of roofs of structures, building floors, bridges, aircraft, vehicles, ships etc. A plate can be classified as thick plate or thin plate. This classification is based on the ratio of the span of the plate to the thickness. Plates whose span to thickness ratio are less than 20 are usually called thick plates while plates whose span to thickness ratio are higher than 50 are called thin plates. In between these two classes are the moderately thick plates (Ibearugbulem, Ezeh, Ettu and Gwarah, 2018, Ghugal and Sayyad, 2011).

Many plate theories have been developed in the past for the analysis of plates. The classical plate theory is one of the earliest theories used to solve plate problems. It has many assumptions that made the analysis appear so much simplified (Vaghefi, 2010). However, due to these assumptions, the classical plate theory, does not offer very good results when used in thick plate analysis. This is due to the development of shear stresses that occur across the thickness of the plate in thick plates (Rajesh and Meera, 2016, Liew, Hung and Lim, 1993). The assumption that a vertical cross section that is initially straight and normal to the mid-surface of the plate before deformation remains normal to the mid-surface after deformation used in the classical plate theory relaxes the consideration of these transverse shear stresses. Since significant transverse shear stresses occur in thick and moderately thick plates, it is necessary to seek for theories that would adequately treat these stresses. The first-order shear deformation theory which is developed by Reissner and Mindlin is one of the plate theories that consider the transverse shear stresses (Shimpi and Patel, 2006). In the first-order shear deformation theory, the displacement field has a linear relationship with the mid-plane displacements. It projects the shear stresses to be constant throughout the thickness of the plate. Hence, shear correction factors are needed to obtain the actual distribution of the shear stresses. These shear correction factors satisfy constitutive relations when used in the formulation of the kinematics equations and produce accepted variation of transverse shear stress across the thickness of the plate (Shufrin and Eisenberger, 2005; Sadrnejad, Daryan, and Ziaei, 2009; Ghugal and Sayyad, 2011). Higher-order shear deformation theories are other theories that consider the shear deformation of the plate. An example of the higher-order shear deformation theories is the theory developed by Reddy which uses parabolic variations in the shear stresses across the thickness of the plate (Sayyad, 2011).

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Plates are three-dimensional elements. Some scholars resort to treating plate as a two-dimensional element or treating it as a partial three-dimensional element because of the expected ease in the computation when using such assumptions. An example of this is the classical plate theory. When such assumptions are made, the results obtained are only approximate results (Sayyada and Ghugal, 2012, Rajesh and Meera, 2016). Some earlier scholars have carried out three-dimensional analysis on rectangular thick plates. Most works on three-dimensional analysis are for functionally graded (or sandwich) plates (Uymaz and Aydogdu, 2012; Srinivas and Rao, 1971; Vaghefi et al., 2010; Wuxiang and Zheng, 2009; Sburlati, 2014; Zhang, Qi, Fang and He, 2020). However, Ibearugbulem, Onwuegbuchulem and Ibearugbulem (2021) carried out analytical three-dimensional bending analysis of simply supported rectangular plate using a third order shear deformation function while Onyeka, Mama and Okeke (2022) investigated the three-dimensional stability analysis of plate using a direct variational energy method.

Anisotropic materials have gained more popularity in the field of structural engineering especially with the increase in the use of composite materials. An anisotropic material has its elastic properties vary in different directions. The elastic properties include the modulus of elasticity, Poisson ratio and the shear modulus. Composite materials, wood and most crystals are examples of anisotropic materials. A material can also be classified as orthotropic or isotropic. An orthotropic material is a special case of anisotropic materials which has the material properties vary along three mutually perpendicular axis whereas an isotropic material has the material properties the same in all directions. An example of orthotropic material is wood while an example isotropic material is steel (Caterina, 2019). Njoku, Ibearugbulem, Ettu and Anya (2023) carried out bending analysis of thick rectangular anisotropic plate using a modified first order shear deformation theory known as the Alternative I theory. In their formulation, the scholars expressed the displacement of the plate as a single unit, not having a classical component and shear deformation part. This is unlike in the Alternative II theory used in this work which treats the displacement as made up of the classical part and shear deformation part.

## II. FORMULATION OF STRESSES AND DISPLACEMENT EQUATIONS

The deformed section of a plate based on the Alternative II theory is as shown on Figure 1. The rotation of the plate is expressed as the summation of the rotation from the classical plate theory (CPT), which is given as $\phi_c$ and the rotation from shear deformation given as $\phi_s$. The displacement equations are also made up of the classical part ($u_c$ and $v_c$) and the shear deformation components ($u_s$ and $v_s$). This is as shown on Equation (1) and (2).

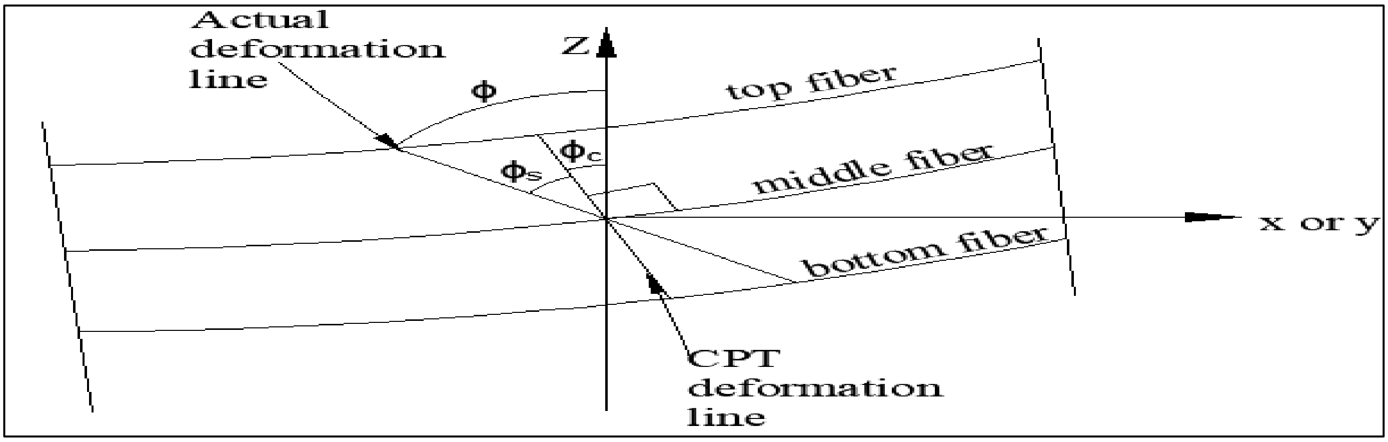


Fig 1: Deformed Section of a Plate

➢ *The Basic Assumptions Made in this Study are as Given Below:*

- The plate material is flat before loading.
- The deflection (w) of the middle in-plane surface of the plate is small when compared with the thickness of the plate. That is w/t < 0.3.
- The middle surface of the plate never stretches nor compresses before, during or after bending.
- A straight and flat x-z or y-z section, which is normal to middle x-y plane before bending shall remain straight and flat but not normal to the middle x-y surface after bending.
- The actual transverse shear stresses, that is, *x-z* and *y-z* shear stresses distributed across the thickness of the plate are the product of nominal *x-z* and *y-z* shear stresses and shear stress shape profile, g(z). That is:

✓ $\tau_{xza} = \tau_{xz}\, g(z)$
✓ $\tau_{yza} = \tau_{yz}\, g(z)$

➢ *Displacement field and Kinematics*

The displacements of the plate in the x and y directions are given in Equations 1 and 2 respectively.

$$u = u_c + u_s \tag{1}$$

$$v = v_c + v_s \tag{2}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Where $u_c$ and $v_c$ are the Classical Components of the displacement gives as:

$$u_c = z\frac{\partial w}{\partial x} \tag{3}$$

$$v_c = z\frac{\partial w}{\partial y} \tag{4}$$

Also, $u_s$ and $v_s$ are the shear deformation components of the displacement given as:

$$u_s = z\square_x \tag{5}$$

$$v_s = z\square_y \tag{6}$$

Substituting Equations (3) to (6) into Equations (1) and (2) gives the displacements u and v as.

$$u = z\frac{\partial w}{\partial x} + z\square_x \tag{7}$$

$$v = z\frac{\partial w}{\partial y} + z\square_y \tag{8}$$

The kinematic relations which are equations showing the relationship between strain and displacement are obtained using Equations (7) and (8). This is as shown below.

$$\varepsilon_x = \frac{\partial u}{\partial x} = z\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \square_x}{\partial x}\right) \tag{9}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = z\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \square_y}{\partial y}\right) \tag{10}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{11}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = z\left(2\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial \square_x}{\partial y} + \frac{\partial \square_y}{\partial x}\right) \tag{12}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2\frac{\partial w}{\partial x} + \square_x \tag{13}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 2\frac{\partial w}{\partial y} + \square_y \tag{14}$$

- *Constitutive Relations*

A complete three-dimensional constitutive relation is used to express the relationships between stresses and strains. This will consist of six stresses and six strains as shown below:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \tau_{12}\\ \tau_{13}\\ \tau_{23}\end{bmatrix} = \frac{E_0}{\Delta}\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0\\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{12}\\ \gamma_{13}\\ \gamma_{23}\end{bmatrix} \tag{15}$$

For an anisotropic material, the stresses above which are in local coordinate (1-2 coordinate) system would be transformed to the global coordinate (x-y coordinate) system as shown in Equation (16).

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix} = \left\{[T]^{-1}\frac{E_0}{\Delta}\begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0\\ e_{21} & e_{22} & e_{23} & 0 & 0 & 0\\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & e_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & e_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & e_{66}\end{bmatrix}[T]^{-T}\right\}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix} \tag{16}$$

[T] is a transformation matrix given as shown on Equation (17).

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 2mn & 0 & 0\\ n^2 & m^2 & 0 & -2mn & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ -mn & mn & 0 & (m^2 - n^2) & 0 & 0\\ 0 & 0 & 0 & 0 & m & n\\ 0 & 0 & 0 & 0 & -n & m\end{bmatrix} \tag{17}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Where: $m$ and $n$ are $\mathrm{Cos}\ \theta$ and $\mathrm{Sin}\ \theta$ respectively while θ is the angle of orientation of the fibres.

Substituting Equation (17) into Equation (16) produces the complete stress-strain relationship as:

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{xz}\\ \tau_{yz}\end{bmatrix}=\frac{E_0}{\Delta}\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0\\ a_{12} & a_{22} & a_{23} & a_{24} & 0 & 0\\ a_{13} & a_{23} & a_{33} & a_{34} & 0 & 0\\ a_{14} & a_{24} & a_{34} & a_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & a_{55} & a_{56}\\ 0 & 0 & 0 & 0 & a_{56} & a_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{bmatrix} \tag{18}$$

➢ *Total Potential Energy Functional*

The total potential energy is the summation of the strain energy stored in the plate and the external work done on the plate. This is expressed as in Equation (19)

$$\Pi = U - V \tag{19}$$

Where U and V are the strain energy and external energy respectively.

The strain energy is given by Equation (20).

$$U = \frac{1}{2}\int_0^1\int_0^1\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right)\partial x\,\partial y\,\partial z \tag{20}$$

The external work, V, consists of the work done by the distributed lateral load, q and the in-plane compressive load $N_x$. It is as given in Equation (21).

$$V = q\int_0^1\int_0^1 w\,\partial x\,\partial y + \frac{N_x}{2}\int_0^1\int_0^1\left(\frac{dw}{dx}\right)^2\partial x\,\partial y \tag{21}$$

If the non-dimensional coordinates R= x/a, Q = y/b and S = z/t are introduced, then the stresses from Equation (18) and strains from Equations (9) to (14) are both substituted into Equation (20) to obtain the internal work which is then substituted together with Equation (21) into Equation (19) to obtain the total potential energy. The total potential energy obtained is given by Equation (22).

$$\begin{aligned}
\frac{abD_0}{2a^4}\int_0^1\int_0^1\Bigg[&a_{11}\left(\left(\frac{\partial^2 w}{\partial R^2}\right)^2 + 2a\frac{\partial^2 w}{\partial R^2}.\frac{\partial \theta_x}{\partial R} + a^2\left(\frac{\partial \theta_x}{\partial R}\right)^2\right)\\
&+\frac{2a_{12}}{\propto^2}\left(\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial Q^2} + a\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \theta_x}{\partial R} + a\propto\frac{\partial^2 w}{\partial R^2}.\frac{\partial \theta_y}{\partial Q} + a^2\propto\frac{\partial \theta_x}{\partial R}.\frac{\partial \theta_y}{\partial Q}\right)\\
&+\frac{2a_{14}}{\propto}\left(2\frac{\partial^2 w}{\partial R^2}.\frac{\partial^2 w}{\partial R\,\partial Q} + 2a\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \theta_x}{\partial R} + a\frac{\partial \theta_x}{\partial Q}.\frac{\partial^2 w}{\partial R^2} + a^2\frac{\partial \theta_x}{\partial Q}.\frac{\partial \theta_x}{\partial R} + \propto a\frac{\partial \theta_y}{\partial R}.\frac{\partial^2 w}{\partial R^2} + \propto a^2\frac{\partial \theta_y}{\partial R}.\frac{\partial \theta_x}{\partial R}\right)\\
&+\frac{a_{22}}{\propto^4}\left(\left(\frac{\partial^2 w}{\partial Q^2}\right)^2 + 2\propto a\frac{\partial^2 w}{\partial Q^2}.\frac{\partial \theta_y}{\partial Q} + \propto^2 a^2\left(\frac{\partial \theta_y}{\partial Q}\right)^2\right)\\
&+\frac{2a_{24}}{\propto^3}\Bigg(2\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial^2 w}{\partial Q^2} + 2\propto a\frac{\partial^2 w}{\partial R\,\partial Q}.\frac{\partial \theta_y}{\partial Q} + a\frac{\partial \theta_x}{\partial Q}.\frac{\partial^2 w}{\partial Q^2} + \propto a^2\frac{\partial \theta_x}{\partial Q}.\frac{\partial \theta_y}{\partial Q} +\\
&\propto a\frac{\partial \theta_y}{\partial R}.\frac{\partial^2 w}{\partial Q^2} + \propto^2 a^2\frac{\partial \theta_y}{\partial R}.\frac{\partial \theta_y}{\partial Q}\Bigg) + 12a_{33}\left(\frac{a}{t}\right)^4\left(\frac{\partial w}{\partial S}\right)^2\\
&+\frac{a_{44}}{\propto^2}\left(4\left(\frac{\partial^2 w}{\partial R\,\partial Q}\right)^2 + 4a\frac{\partial \theta_x}{\partial Q}.\frac{\partial^2 w}{\partial R\,\partial Q} + 4\propto a\frac{\partial \theta_y}{\partial R}.\frac{\partial^2 w}{\partial R\,\partial Q} + a^2\left(\frac{\partial \theta_x}{\partial Q}\right)^2 + 2\propto a^2\frac{\partial \theta_y}{\partial R}.\frac{\partial \theta_x}{\partial Q} + \propto^2 a^2\left(\frac{\partial \theta_y}{\partial R}\right)^2\right)\\
&+12a_{55}\left(\frac{a}{t}\right)^2\left(4\left(\frac{\partial w}{\partial R}\right)^2 + 4a\frac{\partial w}{\partial R}.\theta_x + a^2{\theta_x}^2\right)\\
&+\frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2\left(4\frac{\partial w}{\partial Q}.\frac{\partial w}{\partial R} + 2a\frac{\partial w}{\partial Q}.\theta_x + 2\propto a\frac{\partial w}{\partial R}.\theta_y + \propto a^2\theta_x.\theta_y\right)\\
&+\frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2\left(4\left(\frac{\partial w}{\partial Q}\right)^2 + 4\propto a\frac{\partial w}{\partial Q}.\theta_y + \propto^2 a^2{\theta_y}^2\right)\Bigg]\partial R\,\partial Q - ab\int_0^1\int_0^1\left[qw + \frac{N_x}{2a^2}\left(\frac{dw}{dR}\right)^2\right]\partial R\,\partial Q
\end{aligned} \tag{22}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Where $\propto = \frac{b}{a}$ and $D_0$ is given by Equation (23)

$$D_0 = \frac{E_0 t^3}{12\Delta} \tag{23}$$

If the total potential energy given in Equation (22) is minimized with respect to the rotations in the x-z and y-z planes, two compatibility equations are obtained. The compatibility equations obtained are shown on Equations (24) and (25).

$\frac{D\prod}{d\Phi_x} = 0.$ Therefore;

$$\begin{aligned} & a_{11}\left[2a\frac{\partial^3 w}{\partial R^3} + 2a^2\frac{\partial^2 \Phi_x}{\partial R^2}\right] + \frac{2a_{12}}{\propto^2}\left[a\frac{\partial^3 w}{\partial R\,\partial Q^2} + a^2 \propto \frac{\partial^2 \Phi_y}{\partial R\,\partial Q}\right] + \frac{2a_{14}}{\propto}\left[3a\frac{\partial^3 w}{\partial R^2\,\partial Q} + 2a^2\frac{\partial^2 \Phi_x}{\partial R\,\partial Q} + \propto a^2\frac{\partial^2 \Phi_y}{\partial R^2}\right] \\ & + \frac{2a_{24}}{\propto^3}\left[a\frac{\partial^3 w}{\partial Q^3} + \propto a^2\frac{\partial^2 \Phi_y}{\partial Q^2}\right] + \frac{a_{44}}{\propto^2}\left[4a\frac{\partial^3 w}{\partial R\,\partial Q^2} + 2a^2\frac{\partial^2 \Phi_x}{\partial Q^2} + 2\propto a^2\frac{\partial^2 \Phi_y}{\partial R\,\partial Q}\right] + 12a_{55}\left(\frac{a}{t}\right)^2\left[4a\frac{\partial w}{\partial R} + 2a^2\Phi_x\right] \\ & + \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2\left[2a\frac{\partial w}{\partial Q} + \propto a^2\Phi_y\right] = 0 \end{aligned} \tag{24}$$

Also,

$\frac{d\prod}{d\Phi_y} = 0.$ Therefore;

$$\begin{aligned} & \frac{2a_{12}}{\propto^2}\left[a \propto \frac{\partial^3 w}{\partial R^2\,\partial Q} + a^2 \propto \frac{\partial^2 \Phi_x}{\partial R\,\partial Q}\right] + \frac{2a_{14}}{\propto}\left[\propto a\frac{\partial^3 w}{\partial R^3} + \propto a^2\frac{\partial^2 \Phi_x}{\partial R^2}\right] + \frac{a_{22}}{\propto^4}\left[2\propto a\frac{\partial^3 w}{\partial Q^3} + 2\propto^2 a^2\frac{\partial^2 \Phi_y}{\partial Q^2}\right] \\ & + \frac{2a_{24}}{\propto^3}\left[3\propto a\frac{\partial^3 w}{\partial R\,\partial Q^2} + \propto a^2\frac{\partial^2 \Phi_x}{\partial Q^2} + 2\propto^2 a^2\frac{\partial^2 \Phi_y}{\partial R\,\partial Q}\right] + \frac{a_{44}}{\propto^2}\left[4\propto a\frac{\partial^3 w}{\partial R^2\,\partial Q} + 2\propto a^2\frac{\partial^2 \Phi_x}{\partial R\,\partial Q} + 2\propto^2 a^2\frac{\partial^2 \Phi_y}{\partial R^2}\right] \\ & + \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2\left[2\propto a\frac{\partial w}{\partial R} + \propto a^2\Phi_x\right] + \frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2\left[4\propto a\frac{\partial w}{\partial Q} + 2\propto^2 a^2\Phi_y\right] = 0 \end{aligned} \tag{25}$$

From the set of compatibility equations given in Equation (24) and (25), for any of the equations to be true, the condition is that the expressions in the square brackets must be equal to zero. With this, the rotational displacements $\Phi_x$ and $\Phi_y$ can therefore be written as:

$$\Phi_x = \frac{n_R}{a}\frac{\partial w}{\partial R} \tag{26}$$

$$\Phi_y = \frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q} \tag{27}$$

Where $n_R$ and $n_Q$ are constants

Minimizing Equation (22) with respect to the deflection, w produces the governing equation. That is:

$\frac{d\prod}{dw} = 0.$ Hence,

$$\begin{aligned} & \left[a_{11}\left(2\frac{\partial^4 w}{\partial R^4} + 2a\frac{\partial^3 \Phi_x}{\partial R^3}\right) + \frac{2a_{12}}{\propto^2}\left(2\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + a\frac{\partial^3 \Phi_x}{\partial R\,\partial Q^2} + a\propto\frac{\partial^3 \Phi_y}{\partial R^2\,\partial Q}\right)\right. \\ & + \frac{2a_{14}}{\propto}\left(4\frac{\partial^4 w}{\partial R^3\,\partial Q} + 2a\frac{\partial^3 \Phi_x}{\partial R^2\,\partial Q} + a\frac{\partial^3 \Phi_x}{\partial R^2\,\partial Q} + \propto a\frac{\partial^3 \Phi_y}{\partial R^3}\right) + \frac{a_{22}}{\propto^4}\left(2\frac{\partial^4 w}{\partial Q^4} + 2\propto a\frac{\partial^3 \Phi_y}{\partial Q^3}\right) \\ & + \frac{2a_{24}}{\propto^3}\left(4\frac{\partial^4 w}{\partial R\,\partial Q^3} + 2\propto a\frac{\partial^3 \Phi_y}{\partial R\,\partial Q^2} + a\frac{\partial^3 \Phi_x}{\partial Q^3} + \propto a\frac{\partial^3 \Phi_y}{\partial R\,\partial Q^2}\right) + 24a_{33}\left(\frac{a}{t}\right)^4\frac{\partial^2 w}{\partial S^2} \\ & + \frac{a_{44}}{\propto^2}\left(8\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + 4a\frac{\partial^3 \Phi_x}{\partial R\,\partial Q^2} + 4\propto a\frac{\partial^3 \Phi_y}{\partial R^2\,\partial Q}\right) + 12a_{55}\left(\frac{a}{t}\right)^2\left(8\frac{\partial^2 w}{\partial R^2} + 4a\frac{\partial \Phi_x}{\partial R}\right) \\ & \left. + \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2\left(4\frac{\partial^2 w}{\partial R\,\partial Q} + 2a\frac{\partial \Phi_x}{\partial Q} + 2\propto a\frac{\partial \Phi_y}{\partial R}\right) + \frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2\left(8\frac{\partial^2 w}{\partial Q^2} + 4\propto a\frac{\partial \Phi_y}{\partial Q}\right)\right] \\ & - \left[\frac{2qa^4}{D_0} + \frac{2N_x a^2}{D_0}\frac{\partial^2 w}{\partial R^2}\right] = 0 \end{aligned} \tag{28}$$

If Equations (26) and (27) are substituted into Equation (28), it gives:

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

$$\Big[a_{11}\left(2\frac{\partial^4 w}{\partial R^4}+2a\frac{\partial^3}{\partial R^3}\frac{n_R}{a}\frac{\partial w}{\partial R}\right)+\frac{2a_{12}}{\propto^2}\left(2\frac{\partial^4 w}{\partial R^2\partial Q^2}+a\frac{\partial^3}{\partial R\partial Q^2}\frac{n_R}{a}\frac{\partial w}{\partial R}+a\propto\frac{\partial^3}{\partial R^2\partial Q}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)$$
$$+\frac{2a_{14}}{\propto}\left(4\frac{\partial^4 w}{\partial R^3\partial Q}+2a\frac{\partial^3}{\partial R^2\partial Q}\frac{n_R}{a}\frac{\partial w}{\partial R}+a\frac{\partial^3}{\partial R^2\partial Q}\frac{n_R}{a}\frac{\partial w}{\partial R}+\propto a\frac{\partial^3}{\partial R^3}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)$$
$$+\frac{a_{22}}{\propto^4}\left(2\frac{\partial^4 w}{\partial Q^4}+2\propto a\frac{\partial^3}{\partial Q^3}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)$$
$$+\frac{2a_{24}}{\propto^3}\left(4\frac{\partial^4 w}{\partial R\partial Q^3}+2\propto a\frac{\partial^3}{\partial R\partial Q^2}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}+a\frac{\partial^3}{\partial Q^3}\frac{n_R}{a}\frac{\partial w}{\partial R}+\propto a\frac{\partial^3}{\partial R\partial Q^2}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)+24a_{33}\left(\frac{a}{t}\right)^4\frac{\partial^2 w}{\partial S^2}$$
$$+\frac{a_{44}}{\propto^2}\left(8\frac{\partial^4 w}{\partial R^2\partial Q^2}+4a\frac{\partial^3}{\partial R\partial Q^2}\frac{n_R}{a}\frac{\partial w}{\partial R}+4\propto a\frac{\partial^3}{\partial R^2\partial Q}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)+12a_{55}\left(\frac{a}{t}\right)^2\left(8\frac{\partial^2 w}{\partial R^2}+4a\frac{\partial}{\partial R}\frac{n_R}{a}\frac{\partial w}{\partial R}\right)$$
$$+\frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2\left(4\frac{\partial^2 w}{\partial R\partial Q}+2a\frac{\partial}{\partial Q}\frac{n_R}{a}\frac{\partial w}{\partial R}+2\propto a\frac{\partial}{\partial R}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)+\frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2\left(8\frac{\partial^2 w}{\partial Q^2}+4\propto a\frac{\partial}{\partial Q}\frac{n_Q}{a\propto}.\frac{\partial w}{\partial Q}\right)\Big]$$
$$-\left[\frac{2qa^4}{D_0}+\frac{2N_x a^2}{D_0}\frac{\partial^2 w}{\partial R^2}\right]=0 \qquad (29)$$

Simplifying Equation (29) gives:

$$a_{11}(1+n_R)\frac{\partial^4 w}{\partial R^4}+\left(\frac{(a_{12}+2a_{44})(2+n_R+n_Q)}{\propto^2}\right)\frac{\partial^4 w}{\partial R^2\partial Q^2}+\frac{a_{14}}{\propto}(4+3n_R+n_Q)\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{a_{22}}{\propto^4}(1+n_Q)\frac{\partial^4 w}{\partial Q^4}$$
$$+\frac{a_{24}}{\propto^3}(4+3n_Q+n_R)\frac{\partial^4 w}{\partial R\partial Q^3}+12a_{33}\left(\frac{a}{t}\right)^4\frac{\partial^2 w}{\partial S^2}+24a_{55}\left(\frac{a}{t}\right)^2(2+n_R)\frac{\partial^2 w}{\partial R^2}$$
$$+\frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2(2+n_R+n_Q)\frac{\partial^2 w}{\partial R\partial Q}+\frac{24a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(2+n_Q)\frac{\partial^2 w}{\partial Q^2}-\frac{qa^4}{D_0}-\frac{N_x a^2}{D_0}\frac{\partial^2 w}{\partial R^2}=0 \qquad (30)$$

Equation (30) can be separated for cases of pure bending and for buckling. For pure bending, when there is no in-plane load, $N_x = 0$ and Equation (30) becomes:

$$a_{11}(1+n_R)\frac{\partial^4 w}{\partial R^4}+\left(\frac{(a_{12}+2a_{44})(2+n_R+n_Q)}{\propto^2}\right)\frac{\partial^4 w}{\partial R^2\partial Q^2}+\frac{a_{14}}{\propto}(4+3n_R+n_Q)\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{a_{22}}{\propto^4}(1+n_Q)\frac{\partial^4 w}{\partial Q^4}$$
$$+\frac{a_{24}}{\propto^3}(4+3n_Q+n_R)\frac{\partial^4 w}{\partial R\partial Q^3}+12a_{33}\left(\frac{a}{t}\right)^4\frac{\partial^2 w}{\partial S^2}+24a_{55}\left(\frac{a}{t}\right)^2(2+n_R)\frac{\partial^2 w}{\partial R^2}$$
$$+\frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2(2+n_R+n_Q)\frac{\partial^2 w}{\partial R\partial Q}+\frac{24a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(2+n_Q)\frac{\partial^2 w}{\partial Q^2}-\frac{qa^4}{D_0}$$
$$=0 \qquad (31)$$

For the case of buckling, when q = 0, Equation (30) becomes:

$$a_{11}(1+n_R)\frac{\partial^4 w}{\partial R^4}+\left(\frac{(a_{12}+2a_{44})(2+n_R+n_Q)}{\propto^2}\right)\frac{\partial^4 w}{\partial R^2\partial Q^2}+\frac{a_{14}}{\propto}(4+3n_R+n_Q)\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{a_{22}}{\propto^4}(1+n_Q)\frac{\partial^4 w}{\partial Q^4}$$
$$+\frac{a_{24}}{\propto^3}(4+3n_Q+n_R)\frac{\partial^4 w}{\partial R\partial Q^3}+12a_{33}\left(\frac{a}{t}\right)^4\frac{\partial^2 w}{\partial S^2}+24a_{55}\left(\frac{a}{t}\right)^2(2+n_R)\frac{\partial^2 w}{\partial R^2}$$
$$+\frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2(2+n_R+n_Q)\frac{\partial^2 w}{\partial R\partial Q}+\frac{24a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(2+n_Q)\frac{\partial^2 w}{\partial Q^2}-\frac{N_x a^2}{D_0}\frac{\partial^2 w}{\partial R^2}=0 \qquad (32)$$

Equations (31) and (32) are identical and each can be solved to obtain w.

Equation (31) can be rearranged as follows:

$$\left(\frac{\partial^4 w}{\partial R^4}+\frac{\xi_1}{\propto^2}\frac{\partial^4 w}{\partial R^2\partial Q^2}+\frac{\xi_2}{\propto^4}\frac{\partial^4 w}{\partial Q^4}-\frac{qa^4}{D_0 a_{11}(1+n_R)}\right)+\left(\frac{\xi_3}{\propto}\frac{\partial^4 w}{\partial R^3\partial Q}+\frac{\xi_4}{\propto^3}\frac{\partial^4 w}{\partial R\partial Q^3}\right)+\left(\xi_5\frac{\partial^2 w}{\partial R^2}+\frac{\xi_6}{\propto}\frac{\partial^2 w}{\partial R\partial Q}+\frac{\xi_7}{\propto^2}\frac{\partial^2 w}{\partial Q^2}\right)+\left(\xi_8\frac{\partial^2 w}{\partial S^2}\right)=0 \qquad (33)$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Where:

$$\xi_1 = \frac{(\mathrm{a}_{12} + 2\mathrm{a}_{44})(2 + \mathrm{n}_R + \mathrm{n}_Q)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34}$$

$$\xi_2 = \frac{\mathrm{a}_{22}(1 + \mathrm{n}_Q)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34a}$$

$$\xi_3 = \frac{\mathrm{a}_{14}(4 + 3\mathrm{n}_R + \mathrm{n}_Q)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34b}$$

$$\xi_4 = \frac{\mathrm{a}_{24}(4 + 3\mathrm{n}_Q + \mathrm{n}_R)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34c}$$

$$\xi_5 = \frac{24\mathrm{a}_{55}\left(\frac{a}{t}\right)^2 (2 + \mathrm{n}_R)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34d}$$

$$\xi_6 = \frac{24\mathrm{a}_{56}\left(\frac{a}{t}\right)^2 (2 + \mathrm{n}_R + \mathrm{n}_Q)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34e}$$

$$\xi_7 = \frac{24\mathrm{a}_{66}\left(\frac{a}{t}\right)^2 (2 + \mathrm{n}_Q)}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34f}$$

$$\xi_8 = \frac{12\mathrm{a}_{33}\left(\frac{a}{t}\right)^4}{\mathrm{a}_{11}(1 + \mathrm{n}_R)} \tag{34g}$$

For Equation (33) to hold, then the expressions in each bracket must be equal to zero. Therefore:

$$\frac{\partial^4 w}{\partial R^4} + \frac{\xi_1}{\propto^2}\frac{\partial^4 w}{\partial R^2\,\partial Q^2} + \frac{\xi_2}{\propto^4}\frac{\partial^4 w}{\partial Q^4} - \frac{\mathrm{q}a^4}{\mathrm{D}_0\mathrm{a}_{11}(1 + \mathrm{n}_R)} = 0 \tag{35}$$

$$\frac{\xi_3}{\propto}\frac{\partial^4 w}{\partial R^3\,\partial \mathrm{Q}} + \frac{\xi_4}{\propto^3}\frac{\partial^4 w}{\partial \mathrm{R}\,\partial Q^3} = 0 \tag{36}$$

$$\xi_5\frac{\partial^2 w}{\partial R^2} + \frac{\xi_6}{\propto}\frac{\partial^2 w}{\partial \mathrm{R}\,\partial \mathrm{Q}} + \frac{\xi_7}{\propto^2}\frac{\partial^2 w}{\partial Q^2} = 0 \tag{37}$$

$$\xi_8\frac{\partial^2 w}{\partial S^2} = 0 \tag{38}$$

The deflection of the plate, w can be expressed as a product of the deflections in the orthogonal directions x, y and z. That is:

$$\mathrm{w} = \mathrm{w_x}.\mathrm{w_y}.\mathrm{w_s} \tag{39}$$

Equation (38) can therefore be written as:

$$\mathrm{w_x}.\mathrm{w_y}.\xi_8\frac{\partial^2 \mathrm{w_s}}{\partial S^2} = 0 \tag{40}$$

For non-trivial solution of Equation (40), it follows that:

$$\frac{\partial^2 \mathrm{w_s}}{\partial S^2} = 0 \tag{41}$$

If Equation (41) is integrated once, it gives a constant. This is as shown on Equation (42).

$$\frac{\partial \mathrm{w_s}}{\partial S} = \Omega_1 \tag{42}$$

The implication of Equation (42) is that even though strain is zero at the mid surface, strain can be measured at outer fibers no matter how small. Integrating Equation (42) with respect to s gives:

$$\mathrm{w}_S = \Omega_0 + \Omega_1 \mathrm{S} \tag{43}$$

As strain is zero at middle surface, it follows that from Equation (42), $\Omega_1 = 0$. Hence, Equation (43) gives:

$$\mathrm{w}_S = \Omega_0 \tag{44}$$

The implication of Equation (44) is that the z component of deflection, $\mathrm{w_s}$ of the plate is a constant. It is not a variable.

Therefore, it cannot be differentiated with respect to any of the three cardinal coordinates (R, Q and S). Hence, the deflection, w is a function of only x and y (R and Q) only.

Solving Equation (35) gives the deflection of the plate, w in the form written in Equation (45).

$$\mathrm{w} = \left[\mathrm{a}_0 + \mathrm{a}_1\mathrm{R} + \mathrm{a}_2\frac{\mathrm{R}^2}{2} + \mathrm{a}_3\frac{\mathrm{R}^3}{6} + \mathrm{a}_4\frac{\mathrm{R}^4}{24}\right] \times \left[\mathrm{f}_0 + \mathrm{f}_1\mathrm{Q} + \mathrm{f}_2\frac{\mathrm{Q}^2}{2} + \mathrm{f}_3\frac{\mathrm{Q}^3}{6} + \mathrm{f}_4\frac{\mathrm{Q}^4}{24}\right] \tag{45}$$

Equation (45) can be written as:

$$\mathrm{w} = [\mathrm{a_i}][\mathrm{h_x}] \times [\mathrm{b_i}]\,[\mathrm{h_y}] \tag{46}$$

In a more concise form, the deflection of the plate is expressed as a product of a coefficient, A and a shape function, h as:

$$\mathrm{w} = \mathrm{A\,h} \tag{47}$$

Substituting Equation (47) into Equations (26) and (27) would give:

$$\square_x = \frac{\mathrm{n}_R}{a}\frac{\mathrm{d[A\,h]}}{\mathrm{dR}} = \frac{\mathrm{An}_R}{a}\frac{\mathrm{d}h}{\mathrm{dR}} = \frac{B_R}{a}\frac{\mathrm{d}h}{\mathrm{dR}} \tag{48}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

$$\square_y = \frac{n_Q}{a \propto}\frac{d[A\,h]}{dQ} = \frac{An_Q}{a \propto}.\frac{dh}{dQ} = \frac{B_Q}{a \propto}.\frac{dh}{dQ} \qquad (48a)$$

If Equations (47), (48) and (48a) are substituted into Equation (22), removing terms containing dS since the deflection is not a function of S and simplifying gives:

$$\Pi = \frac{abD_0}{2a^4}\int_0^1\int_0^1 \Big[a_{11}(A^2 + 2AB_R + B_R{}^2)K_{RRRR} + \frac{2a_{12}}{\propto^2}(A^2 + AB_R + AB_Q + B_RB_Q)K_{RRQQ}$$
$$+ \frac{2a_{14}}{\propto}(2A^2 + 3AB_R + B_R{}^2 + AB_Q + B_RB_Q)K_{RRRQ} + \frac{a_{22}}{\propto^4}(A^2 + 2AB_Q + B_Q{}^2)K_{QQQQ}$$
$$+ \frac{2a_{24}}{\propto^3}(2A^2 + 3AB_Q + AB_R + B_RB_Q + B_Q{}^2)K_{RQQQ}$$
$$+ \frac{a_{44}}{\propto^2}(4A^2 + 4AB_R + 4AB_Q + B_R{}^2 + 2B_RB_Q + B_Q{}^2)K_{RRQQ} + 12a_{55}\left(\frac{a}{t}\right)^2(4A^2 + 4AB_R + B_R{}^2)K_{RR}$$
$$+ \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2(4A^2 + 2AB_R + 2AB_Q + B_RB_Q)K_{RQ} + \frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(4A^2 + 4AB_Q + B_Q{}^2)K_{QQ} - \frac{2qa^4A}{D_0}k_q$$
$$- \frac{N_x a^2 A^2}{D_0}K_{RR}\Big]\partial R\,\partial Q \qquad (49)$$

Where:

$$K_{RRRR} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R^2}\right)^2 dR\,dQ, \quad K_{RRQQ} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R\,\partial Q}\right)^2 dR\,dQ, \quad K_{RQ} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R\,\partial Q}\right) dR\,dQ,$$

$$K_{RRRQ} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\,\partial Q}\right) dR\,dQ, \quad K_{QQQQ} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dR\,dQ, \quad K_{QQ} = \int_0^1\int_0^1\left(\frac{\partial h}{\partial Q}\right)^2 dR\,dQ,$$

$$K_{RQQQ} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R\,\partial Q}.\frac{\partial^2 h}{\partial Q^2}\right) dR\,dQ, \quad K_{RR} = \int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dR\,dQ, \quad K_q = \int_0^1\int_0^1 h\,dR\,dQ$$

If Equation (49) is minimized with respect to $B_R$ and simplified, Equations (50) is obtained.

$$\left(A_{11}K_{RRRR} + \frac{2a_{14}}{\propto}K_{RRRQ} + \frac{a_{44}}{\propto^2}K_{RRQQ} + 12a_{55}\left(\frac{a}{t}\right)^2 K_{RR}\right)B_R$$
$$+ \left(\frac{a_{12}}{\propto^2}K_{RRQQ} + \frac{a_{14}}{\propto}K_{RRRQ} + \frac{a_{24}}{\propto^3}K_{RQQQ} + \frac{a_{44}}{\propto^2}K_{RRQQ} + \frac{12a_{56}}{\propto}\left(\frac{a}{t}\right)^2 K_{RQ}\right)B_Q$$
$$+ \Big(a_{11}K_{RRRR} + \frac{a_{12}}{\propto^2}K_{RRQQ} + \frac{3a_{14}}{\propto}K_{RRRQ} + \frac{a_{24}}{\propto^3}K_{RQQQ} + \frac{2a_{44}}{\propto^2}K_{RRQQ} + 24a_{55}\left(\frac{a}{t}\right)^2 K_{RR}$$
$$+ \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2 K_{RQ}\Big)A = 0 \qquad (50)$$

In a similar manner, if Equation (49) is minimized with respect to $B_Q$, it gives Equations (51).

$$\left(\frac{A_{12}}{\propto^2}K_{RRQQ} + \frac{a_{14}}{\propto}K_{RRRQ} + \frac{a_{24}}{\propto^3}K_{RQQQ} + \frac{a_{44}}{\propto^2}K_{RRQQ} + \frac{12a_{56}}{\propto}\left(\frac{a}{t}\right)^2 K_{RQ}\right)B_R$$
$$+ \left(\frac{a_{22}}{\propto^4}K_{QQQQ} + \frac{2a_{24}}{\propto^3}K_{RQQQ} + \frac{a_{44}}{\propto^2}K_{RRQQ} + \frac{12a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2 k_{QQ}\right)B_Q$$
$$+ \Big(\frac{a_{12}}{\propto^2}K_{RRQQ} + \frac{a_{14}}{\propto}K_{RRRQ} + \frac{a_{22}}{\propto^4}K_{QQQQ} + \frac{3a_{24}}{\propto^3}K_{RQQQ} + \frac{2a_{44}}{\propto^2}K_{RRQQ} + \frac{24a_{56}}{\propto}\left(\frac{a}{t}\right)^2 K_{RQ}$$
$$+ \frac{24a_{66}}{\propto^2}\left(\frac{a}{t}\right)^2 K_{QQ}\Big)A = 0 \qquad (51)$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

If Equations (50) and (51) are solved simultaneously, it would

$$B_R = \mathrm{T_R}.A \qquad (52)$$

$$B_Q = \mathrm{T_Q}.A \qquad (52a)$$

Where:

$$T_R = \frac{(j_{12}j_{23} - j_{13}j_{22})}{(j_{11}j_{22} - j_{12}j_{21})} \qquad (53)$$

$$T_Q = \frac{(j_{21}j_{13} - j_{11}j_{23})}{(j_{11}j_{22} - j_{12}j_{21})} \qquad (53a)$$

$$j_{11} = \mathrm{a}_{11}K_{RRRR} + \frac{2\mathrm{a}_{14}}{\propto}K_{RRRQ} + \frac{\mathrm{a}_{44}}{\propto^2}K_{RRQQ} + 12\mathrm{a}_{55}\left(\frac{a}{t}\right)^2 \mathrm{K_{RR}} \qquad (54)$$

$$j_{12} = j_{21} = \frac{\mathrm{a}_{12}}{\propto^2}K_{RRQQ} + \frac{\mathrm{a}_{14}}{\propto}\mathrm{K}_{RRRQ} + \frac{\mathrm{a}_{24}}{\propto^3}K_{RQQQ} + \frac{\mathrm{a}_{44}}{\propto^2}\mathrm{K}_{RRQQ} + \frac{12\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2 \mathrm{K_{RQ}} \qquad (54a)$$

$$j_{13} = \mathrm{a}_{11}K_{RRRR} + \frac{\mathrm{a}_{12}}{\propto^2}K_{RRQQ} + \frac{3\mathrm{a}_{14}}{\propto}K_{RRRQ} + \frac{\mathrm{a}_{24}}{\propto^3}\mathrm{K}_{RQQQ} + \frac{2\mathrm{a}_{44}}{\propto^2}\mathrm{K}_{RRQQ} + 24\mathrm{a}_{55}\left(\frac{a}{t}\right)^2 \mathrm{K_{RR}} + \frac{24\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2 \mathrm{K_{RQ}} \qquad (54b)$$

$$j_{22} = \frac{\mathrm{a}_{22}}{\propto^4}K_{QQQQ} + \frac{2\mathrm{a}_{24}}{\propto^3}\mathrm{K}_{RQQQ} + \frac{\mathrm{a}_{44}}{\propto^2}\mathrm{K}_{RRQQ} + \frac{12\mathrm{a}_{66}}{\propto^2}\left(\frac{a}{t}\right)^2 k_{QQ} \qquad (54c)$$

$$j_{23} = \frac{\mathrm{a}_{12}}{\propto^2}K_{RRQQ} + \frac{\mathrm{a}_{14}}{\propto}\mathrm{K}_{RRRQ} + \frac{\mathrm{a}_{22}}{\propto^4}K_{QQQQ} + \frac{3\mathrm{a}_{24}}{\propto^3}K_{RQQQ} + \frac{2\mathrm{a}_{44}}{\propto^2}\mathrm{K}_{RRQQ} + \frac{24\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2 K_{RQ} + \frac{24\mathrm{a}_{66}}{\propto^2}\left(\frac{a}{t}\right)^2 \mathrm{K_{QQ}} \qquad (54d)$$

Minimizing Equation (49) with respect to A gives:

$$\begin{aligned}\mathrm{a}_{11}(2A + 2B_R)\mathrm{K}_{RRRR} &+ \frac{2\mathrm{a}_{12}}{\propto^2}(2A + B_R + B_Q)\mathrm{K}_{RRQQ} + \frac{2\mathrm{a}_{14}}{\propto}(4A + 3B_R + B_Q)\mathrm{K}_{RRRQ} + \frac{\mathrm{a}_{22}}{\propto^4}(2A + 2B_Q)\mathrm{K}_{QQQQ}\\ &+ \frac{2\mathrm{a}_{24}}{\propto^3}(4A + 3B_Q + B_R)\mathrm{K}_{RQQQ} + \frac{\mathrm{a}_{44}}{\propto^2}(8A + 4B_R + 4B_Q)\mathrm{K}_{RRQQ} + 12\mathrm{a}_{55}\left(\frac{a}{t}\right)^2(8A + 4B_R)\mathrm{K_{RR}}\\ &+ \frac{24\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2(8A + 2B_R + 2B_Q)\mathrm{K_{RQ}} + \frac{12\mathrm{a}_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(8A + 4B_Q)\mathrm{K_{QQ}} - \frac{2\mathrm{q}a^4}{\mathrm{D_0}}K_q - \frac{2A\mathrm{N_x}a^2}{\mathrm{D_0}}K_{RR} = 0 \qquad (55)\end{aligned}$$

Substituting Equations (52) and (52a) into Equation (55) and simplifying gives:

$$\begin{aligned}2A\mathrm{a}_{11}(1 + \mathrm{T_R})\mathrm{K}_{RRRR} &+ \frac{2A\mathrm{a}_{12}}{\propto^2}(2 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRQQ} + \frac{2A\mathrm{a}_{14}}{\propto}(4 + 3\mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRRQ} + \frac{2A\mathrm{a}_{22}}{\propto^4}(1 + \mathrm{T_Q})\mathrm{K}_{QQQQ}\\ &+ \frac{2A\mathrm{a}_{24}}{\propto^3}(4 + 3\mathrm{T_Q} + \mathrm{T_R})\mathrm{K}_{RQQQ} + \frac{4A\mathrm{a}_{44}}{\propto^2}(2 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRQQ} + 48A\mathrm{a}_{55}\left(\frac{a}{t}\right)^2(2 + \mathrm{T_R})\mathrm{K}_{RR}\\ &+ \frac{48A\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2(4 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RQ} + \frac{48A\mathrm{a}_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(2 + \mathrm{T_Q})\mathrm{K_{QQ}} - \frac{2A\mathrm{N_x}a^2}{\mathrm{D_0}}\mathrm{K}_{RR} = \frac{2\mathrm{q}a^4}{\mathrm{D_0}}\mathrm{K}_q \qquad (56)\end{aligned}$$

Further simplification of Equation (56) gives:

$$\mathrm{A}\left(\mathrm{K_T} - \frac{\mathrm{N_x}a^2\mathrm{K}_{RR}}{\mathrm{D_0}}\right) = \frac{\mathrm{q}a^4\mathrm{K}_q}{\mathrm{D_0}} \qquad (57)$$

Where:

$$\begin{aligned}\mathrm{K_T} = \mathrm{a}_{11}(1 + \mathrm{T_R})\mathrm{K}_{RRRR} &+ \frac{\mathrm{a}_{12}}{\propto^2}(2 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRQQ} + \frac{\mathrm{a}_{14}}{\propto}(4 + 3\mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRRQ} + \frac{\mathrm{a}_{22}}{\propto^4}(1 + \mathrm{T_Q})\mathrm{K}_{QQQQ}\\ &+ \frac{\mathrm{a}_{24}}{\propto^3}(4 + 3\mathrm{T_Q} + \mathrm{T_R})\mathrm{K}_{RQQQ} + \frac{2\mathrm{a}_{44}}{\propto^2}(2 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RRQQ} + 24\mathrm{a}_{55}\left(\frac{a}{t}\right)^2(2 + \mathrm{T_R})\mathrm{K}_{RR}\\ &+ \frac{24\mathrm{a}_{56}}{\propto}\left(\frac{a}{t}\right)^2(4 + \mathrm{T_R} + \mathrm{T_Q})\mathrm{K}_{RQ} + \frac{24\mathrm{a}_{66}}{\propto^2}\left(\frac{a}{t}\right)^2(2 + \mathrm{T_Q})\mathrm{K_{QQ}} \qquad (58)\end{aligned}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Equation (57) can be written as shown in Equation (59)

$$A = \frac{\frac{qa^4K_q}{D_0}}{K_T - \frac{N_x a^2 K_{RR}}{D_0}} \tag{59}$$

For pure buckling, the denominator of Equation (59) becomes zero at buckling and $N_x$ will be equal to the buckling load, $N_c$. That is:

$$K_T - \frac{N_c a^2 K_{RR}}{D_0} = 0 \tag{60}$$

That is:

$$K_T = \frac{N_c a^2 K_{RR}}{D_0} \tag{61}$$

Hence,

$$N_c = \frac{K_T D_0}{K_{RR}.\,a^2} \tag{62}$$

Let the ratio of applied in-plane load to critical buckling load be $N_r$, that is,

$$N_x = N_r N_c \tag{63}$$

Where: $N_r$ ranges from zero (case of no in-plane load) to one (when in-plane load is equal to the critical buckling load). That is:

$$0 \le N_r \le 1 \tag{64}$$

$N_x$ is the applied in-plane compression load.

$N_c$ is the critical buckling load.

That is, from Equation (63),

$$N_r = \frac{N_x}{N_c} \tag{65}$$

Substituting Equation (62) into Equation (63) gives:

$$N_x = \frac{N_r K_T D_0}{K_{RR} a^2} \tag{66}$$

Substituting Equation (66) into Equation (59) would give:

$$A = \frac{\frac{qa^4K_q}{D_0}}{K_T - \frac{N_r K_T D_0}{K_{RR} a^2}.\frac{a^2 K_{RR}}{D_0}} \tag{67}$$

Simplifying Equation (67) gives:

$$A = \frac{\frac{qa^4K_q}{D_0}}{K_T - N_r K_T} \tag{68}$$

From Equation (68),

$$A = \frac{\frac{qa^4K_q}{D_0}}{K_T\,(1 - N_r)} \tag{69}$$

Equation (69) can be written as shown on Equation (70)

$$A = \frac{K_q}{K_T}\frac{qa^4}{D_0}\beta \tag{70}$$

Where:

$$\beta = \frac{1}{(1 - N_r)} \tag{71}$$

Substituting Equation (70) into Equations (52) and (52a) respectively gives:

$$B_R = T_R.\frac{K_q}{K_T}\frac{qa^4}{D_0}\beta \tag{72}$$

$$B_Q = T_Q.\frac{K_q}{K_T}\frac{qa^4}{D_0}\beta \tag{73}$$

If Equations (70), (72) and (73) are substituted into Equations (47), (48) and (48a) respectively, the following equation are obtained:

$$w = \frac{K_q}{K_T}\frac{qa^4}{D_0}\beta\,h \tag{74}$$

$$\square_x = T_R.\frac{K_q}{K_T}\frac{qa^3}{D_0}\beta\frac{dh}{dR} \tag{75}$$

$$\square_y = T_Q.\frac{K_q}{\propto K_T}\frac{qa^3}{D_0}\beta\frac{dh}{dQ} \tag{76}$$

- *Expressions for Stresses and Displacements*

Substituting $x = aR$, $y = bQ$, $z = tS$ and $\propto = b/a$ in Equations (8) to (13) and substituting the outcome into Equations (18) gives the six stresses as shown in Equations (77) to (77e).

$$\sigma_x = \frac{E_0}{\Delta}\left[a_{11}\frac{tS}{a}\left(\frac{\partial^2 w}{a\,\partial R^2} + \frac{\partial\square_x}{\partial R}\right) + a_{12}\frac{tS}{\propto a}\left(\frac{\partial^2 w}{\propto a\,\partial Q^2} + \frac{\partial\square_y}{\partial Q}\right) + a_{13}\frac{\partial w}{t\,\partial S} + a_{14}\frac{tS}{a}\left(2\frac{\partial^2 w}{\propto a\,\partial R\,\partial Q} + \frac{\partial\square_x}{\propto\partial Q} + \frac{\partial\square_y}{\partial R}\right)\right] \tag{77}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

$$\sigma_y = \frac{E_0}{\Delta}\left[a_{12}\frac{tS}{a}\left(\frac{\partial^2 w}{a\,\partial R^2}+\frac{\partial \square_x}{\partial R}\right)+a_{22}\frac{tS}{\propto a}\left(\frac{\partial^2 w}{\propto a\,\partial Q^2}+\frac{\partial \square_y}{\partial Q}\right)+a_{23}\frac{\partial w}{t\,\partial S}+a_{24}\frac{tS}{a}\left(2\frac{\partial^2 w}{\propto a\,\partial R\,\partial Q}+\frac{\partial \square_x}{\propto \partial Q}+\frac{\partial \square_y}{\partial R}\right)\right] \quad (77a)$$

$$\sigma_z = \frac{E_0}{\Delta}\left[a_{13}\frac{tS}{a}\left(\frac{\partial^2 w}{a\,\partial R^2}+\frac{\partial \square_x}{\partial R}\right)+a_{23}\frac{tS}{\propto a}\left(\frac{\partial^2 w}{\propto a\,\partial Q^2}+\frac{\partial \square_y}{\partial Q}\right)+a_{33}\frac{\partial w}{t\,\partial S}+a_{34}\frac{tS}{a}\left(2\frac{\partial^2 w}{\propto a\,\partial R\,\partial Q}+\frac{\partial \square_x}{\propto \partial Q}+\frac{\partial \square_y}{\partial R}\right)\right] \quad (77b)$$

$$\tau_{xy} = \frac{E_0}{\Delta}\left[a_{14}\frac{tS}{a}\left(\frac{\partial^2 w}{a\,\partial R^2}+\frac{\partial \square_x}{\partial R}\right)+a_{24}\frac{tS}{\propto a}\left(\frac{\partial^2 w}{\propto a\,\partial Q^2}+\frac{\partial \square_y}{\partial Q}\right)+a_{34}\frac{\partial w}{t\,\partial S}+a_{44}\frac{tS}{a}\left(2\frac{\partial^2 w}{\propto a\,\partial R\,\partial Q}+\frac{\partial \square_x}{\propto \partial Q}+\frac{\partial \square_y}{\partial R}\right)\right] \quad (77c)$$

$$\tau_{xz} = \frac{E_0}{\Delta}\left[a_{55}\left(2\frac{\partial w}{a\,\partial R}+\square_x\right)+a_{56}\left(2\frac{\partial w}{\propto a\,\partial Q}+\square_y\right)\right] \quad (77d)$$

$$\tau_{yz} = \frac{E_0}{\Delta}\left[a_{56}\left(2\frac{\partial w}{a\,\partial R}+\square_x\right)+a_{66}\left(2\frac{\partial w}{\propto a\,\partial Q}+\square_y\right)\right] \quad (77e)$$

If Equations (74), (75) and (76) are substituted into Equations (77) to (77e) bearing in mind that the derivative of w with respect to S is zero, and subsequently, substituting Equation (23) into the resultant equations. Then the equations for the stress parameters can be expressed as shown in Equations (78) to (78e).

$$\bar{\sigma}_x = \sigma_x.\frac{t^2}{qa^2} = 12S\left[a_{11}(1+T_R)\frac{\partial^2 h}{\partial R^2}+\frac{a_{12}}{\propto^2}(1+T_Q)\frac{\partial^2 h}{\partial Q^2}+\frac{a_{14}}{\propto}(2+T_R+T_Q)\frac{\partial^2 h}{\partial R\,\partial Q}\right]\beta\frac{K_q}{K_T} \quad (78)$$

$$\bar{\sigma}_y = \sigma_y.\frac{t^2}{qa^2} = 12S\left[a_{12}(1+T_R)\frac{\partial^2 h}{\partial R^2}+\frac{a_{22}}{\propto^2}(1+T_Q)\frac{\partial^2 h}{\partial Q^2}+\frac{a_{24}}{\propto}(2+T_R+T_Q)\frac{\partial^2 h}{\partial R\,\partial Q}\right]\beta\frac{K_q}{K_T} \quad (78a)$$

$$\bar{\sigma}_z = \sigma_z.\frac{t^2}{qa^2} = 12S\left[a_{13}(1+T_R)\frac{\partial^2 h}{\partial R^2}+\frac{a_{23}}{\propto^2}(1+T_Q)\frac{\partial^2 h}{\partial Q^2}+\frac{a_{34}}{\propto}(2+T_R+T_Q)\frac{\partial^2 h}{\partial R\,\partial Q}\right]\beta\frac{K_q}{K_T} \quad (78b)$$

$$\bar{\tau}_{xy} = \tau_{xy}.\frac{t^2}{qa^2} = 12S\left[a_{14}(1+T_R)\frac{\partial^2 h}{\partial R^2}+\frac{a_{24}}{\propto^2}(1+T_Q)\frac{\partial^2 h}{\partial Q^2}+\frac{a_{44}}{\propto}(2+T_R+T_Q)\frac{\partial^2 h}{\partial R\,\partial Q}\right]\beta\frac{K_q}{K_T} \quad (78c)$$

$$\bar{\tau}_{xz} = \frac{\tau_{xz}}{q}\left(\frac{t}{a}\right) = 12\left[a_{55}(2+T_R)\frac{\partial h}{\partial R}+\frac{a_{56}}{\propto}(2+T_Q)\frac{\partial h}{\partial Q}\right]\beta\frac{K_q}{K_T}\left(\frac{a}{t}\right)^2 \quad (78d)$$

$$\bar{\tau}_{yz} = \frac{\tau_{yz}}{q}\left(\frac{t}{a}\right) = 12\left[a_{56}(2+T_R)\frac{\partial h}{\partial R}+\frac{a_{66}}{\propto}(2+T_Q)\frac{\partial h}{\partial Q}\right]\beta\frac{K_q}{K_T}\left(\frac{a}{t}\right)^2 \quad (78e)$$

If Equation (23) is substituted into Equation (74) and simplified, it gives the out-of-plane displacement parameter, $\bar{w}$ as:

$$\bar{w} = w\frac{E_0 t^3}{qa^4} = 12\Delta\left(\frac{K_q}{K_T}\right)h \quad (79)$$

By substituting relevant parameters into Equation (7), the expression for the in-plane displacement parameter, $\bar{u}$ is obtained as:

$$\bar{u} = u\frac{E_0 t^2}{qa^3} = 12\Delta(1+T_R)S\frac{K_q}{K_T}\frac{dh}{dR} \quad (80)$$

Similarly, by substituting relevant parameters into Equation (8), the expression for the in-plane displacement parameter, $\bar{v}$ is obtained as:

$$\bar{v} = v\frac{E_0 t^2}{qa^3} = 12\Delta(1+T_Q)\frac{S}{\propto}\frac{K_q}{K_T}\frac{\partial h}{\partial Q} \quad (81)$$

## III. NUMERICAL EXAMPLE

An anisotropic three-dimensional all round simply supported (SSSS) rectangular plate with the properties similar to the properties of a unidirectional graphite/epoxy material given by Sarvestani, Naghashpour and Heidari-Rarani (2015), would be analyzed. The material properties are $E_I$ = 132 Gpa, $E_2 = E_3$=10.8 Gpa, $G_{12} = G_{13}$ = 5.65 Gpa, $G_{23}$ = 3.38 Gpa, $\mu_{12} = \mu_{13}$ = 0.24, $\mu_{23}$ = 0.59. The in-plane normal stresses ($\sigma_x$) and ($\sigma_y$) and the out-of-plane normal stress, $\sigma_{z,}$ are obtained at coordinate (0.5, 0.5, 0.5). The in-plane shear stress ($\tau_{xy}$) is obtained at (0, 0, 0.5). The out-plane shear stress ($\tau_{xz}$) is obtained (0, 0.5, 0), while the out-plane shear stress ($\tau_{yz}$) is obtained at (0.5, 0, 0). The transverse displacement (w) is obtained at (0.5, 0.5, 0). The in-plane displacements, u is obtained at (0, 0.5, 0.5) while the in-plane displacements, v is obtained at (0.5, 0, 0.5). The shape function for the given plate is h = $(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$.

Volume 9, Issue 6, June – 2024 International Journal of Innovative Science and Research Technology
ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

## IV. RESULTS AND DISCUSSION

The numerical values of the stresses and displacement parameters determined for span to thickness ratios of 5, 10, 20 and 100 at angle of fiber orientations, θ of 08 and aspect ratios of 1, 1.5 and 2.0 for ratios of applied in-plane load to buckling load, Nr of 0, 0.25 and 0.5 are presented in Table 1, 2 and 3. It is observed that the application of in-plane loads increases the stresses and displacements of the plate at any given value of applied lateral load. When the applied in-plane load is half of the buckling load, the stresses and displacements get doubled. It is observed that the values of normal stress in the x-direction ($\sigma_x$), transverse shear stress $\tau_{xz}$ and in-plane displacement u, increase with an increase in both the span to thickness $\left(\frac{a}{t}\right)$ value and the aspect ratio. Also, the values normal stress in the y-direction ($\sigma_y$), normal stress in the z-direction ($\sigma_z$), transverse shear stress $\tau_{yz}$ and in-plane displacement v, decrease with an increase in both the span to thickness $\left(\frac{a}{t}\right)$ value and the aspect ratio. For the lateral displacement, w the values decrease with an increase in the span to thickness $\left(\frac{a}{t}\right)$ value but increase with an increase in the aspect ratio.

Table 1: Numerical Values of Non-Dimensional Stresses and Displacements at Nr = 0

| | b/a | $\overline{\sigma_x}$ | $\overline{\sigma_y}$ | $\overline{\sigma_z}$ | $\overline{\tau_{xy}}$ | $\overline{\tau_{xz}}$ | $\overline{\tau_{yz}}$ | $\bar{w}$ | $\bar{u}$ | $\bar{v}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{a}{t} = 5$ | 1.0 | 0.6751 | 0.1469 | 0.0999 | -0.0698 | 0.4065 | 0.1165 | 0.0202 | -0.0168 | -0.0249 |
| | 1.5 | 0.8295 | 0.1070 | 0.0794 | -0.0606 | 0.4762 | 0.0703 | 0.0248 | -0.0214 | -0.0219 |
| | 2.0 | 0.8853 | 0.0839 | 0.0669 | -0.0495 | 0.4988 | 0.0485 | 0.0264 | -0.0232 | -0.0180 |
| $\frac{a}{t} = 10$ | 1.0 | 0.7063 | 0.1269 | 0.0888 | -0.0638 | 0.4206 | 0.1025 | 0.0137 | -0.0179 | -0.0202 |
| | 1.5 | 0.8460 | 0.0920 | 0.0709 | -0.0526 | 0.4825 | 0.0607 | 0.0166 | -0.0220 | -0.0168 |
| | 2.0 | 0.8946 | 0.0739 | 0.0612 | -0.0423 | 0.5020 | 0.0420 | 0.0177 | -0.0235 | -0.0135 |
| $\frac{a}{t} = 20$ | 1.0 | 0.7160 | 0.1208 | 0.0853 | -0.0619 | 0.4249 | 0.0981 | 0.0120 | -0.0182 | -0.0188 |
| | 1.5 | 0.8507 | 0.0878 | 0.0685 | -0.0504 | 0.4843 | 0.0580 | 0.0146 | -0.0222 | -0.0153 |
| | 2.0 | 0.8972 | 0.0711 | 0.0596 | -0.0403 | 0.5029 | 0.0402 | 0.0155 | -0.0236 | -0.0122 |
| $\frac{a}{t} = 100$ | 1.0 | 0.7192 | 0.1187 | 0.0842 | -0.0613 | 0.4264 | 0.0967 | 0.0115 | -0.0183 | -0.0183 |
| | 1.5 | 0.8523 | 0.0864 | 0.0677 | -0.0497 | 0.4849 | 0.0571 | 0.0139 | -0.0222 | -0.0148 |
| | 2.0 | 0.8980 | 0.0703 | 0.0591 | -0.0396 | 0.5032 | 0.0396 | 0.0148 | -0.0236 | -0.0118 |

Table 2: Numerical Values of Non-Dimensional Stresses and Displacements at Nr = 0.25

| | b/a | $\overline{\sigma_x}$ | $\overline{\sigma_y}$ | $\overline{\sigma_z}$ | $\overline{\tau_{xy}}$ | $\overline{\tau_{xz}}$ | $\overline{\tau_{yz}}$ | $\bar{w}$ | $\bar{u}$ | $\bar{v}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{a}{t} = 5$ | 1.0 | 0.9002 | 0.1959 | 0.1332 | -0.0931 | 0.5420 | 0.1553 | 0.0269 | -0.0224 | -0.0332 |
| | 1.5 | 1.1060 | 0.1426 | 0.1059 | -0.0808 | 0.6349 | 0.0937 | 0.0330 | -0.0285 | -0.0292 |
| | 2.0 | 1.1804 | 0.1118 | 0.0891 | -0.0661 | 0.6650 | 0.0646 | 0.0352 | -0.0309 | -0.0240 |
| $\frac{a}{t} = 10$ | 1.0 | 0.9418 | 0.1693 | 0.1184 | -0.0850 | 0.5607 | 0.1366 | 0.0182 | -0.0238 | -0.0270 |
| | 1.5 | 1.1281 | 0.1227 | 0.0945 | -0.0702 | 0.6434 | 0.0810 | 0.0222 | -0.0293 | -0.0224 |
| | 2.0 | 1.1928 | 0.0985 | 0.0815 | -0.0563 | 0.6694 | 0.0560 | 0.0236 | -0.0314 | -0.0180 |
| $\frac{a}{t} = 20$ | 1.0 | 0.9546 | 0.1611 | 0.1138 | -0.0825 | 0.5665 | 0.1308 | 0.0160 | -0.0242 | -0.0251 |
| | 1.5 | 1.1343 | 0.1170 | 0.0913 | -0.0672 | 0.6458 | 0.0774 | 0.0194 | -0.0296 | -0.0204 |
| | 2.0 | 1.1962 | 0.0949 | 0.0795 | -0.0537 | 0.6705 | 0.0536 | 0.0207 | -0.0315 | -0.0163 |
| $\frac{a}{t} = 100$ | 1.0 | 0.9590 | 0.1583 | 0.1122 | -0.0817 | 0.5685 | 0.1289 | 0.0153 | -0.0244 | -0.0244 |
| | 1.5 | 1.1364 | 0.1152 | 0.0903 | -0.0662 | 0.6466 | 0.0762 | 0.0186 | -0.0296 | -0.0198 |
| | 2.0 | 1.1973 | 0.0937 | 0.0788 | -0.0528 | 0.6709 | 0.0529 | 0.0197 | -0.0315 | -0.0158 |

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Table 3: Numerical values of non-dimensional stresses and displacements at Nr = 0.5

| | b/a | $\overline{\sigma_x}$ | $\overline{\sigma_y}$ | $\overline{\sigma_z}$ | $\overline{\tau_{xy}}$ | $\overline{\tau_{xz}}$ | $\overline{\tau_{yz}}$ | $\overline{w}$ | $\overline{u}$ | $\overline{v}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{a}{t} = 5$ | 1.0 | 1.3502 | 0.2938 | 0.1998 | -0.1396 | 0.8130 | 0.2330 | 0.0404 | -0.0336 | -0.0498 |
| | 1.5 | 1.6590 | 0.2139 | 0.1588 | -0.1211 | 0.9523 | 0.1405 | 0.0495 | -0.0428 | -0.0438 |
| | 2.0 | 1.7706 | 0.1677 | 0.1337 | -0.0991 | 0.9975 | 0.0970 | 0.0528 | -0.0463 | -0.0360 |
| $\frac{a}{t} = 10$ | 1.0 | 1.4127 | 0.2539 | 0.1775 | -0.1275 | 0.8411 | 0.2049 | 0.0273 | -0.0357 | -0.0405 |
| | 1.5 | 1.6921 | 0.1840 | 0.1418 | -0.1053 | 0.9651 | 0.1214 | 0.0333 | -0.0440 | -0.0336 |
| | 2.0 | 1.7892 | 0.1478 | 0.1223 | -0.0845 | 1.0040 | 0.0840 | 0.0354 | -0.0470 | -0.0270 |
| $\frac{a}{t} = 20$ | 1.0 | 1.4319 | 0.2416 | 0.1707 | -0.1238 | 0.8498 | 0.1963 | 0.0240 | -0.0364 | -0.0376 |
| | 1.5 | 1.7014 | 0.1755 | 0.1370 | -0.1008 | 0.9687 | 0.1160 | 0.0292 | -0.0444 | -0.0307 |
| | 2.0 | 1.7943 | 0.1423 | 0.1192 | -0.0805 | 1.0058 | 0.0804 | 0.0310 | -0.0472 | -0.0245 |
| $\frac{a}{t} = 100$ | 1.0 | 1.4385 | 0.2374 | 0.1683 | -0.1225 | 0.8527 | 0.1933 | 0.0229 | -0.0366 | -0.0366 |
| | 1.5 | 1.7045 | 0.1727 | 0.1354 | -0.0993 | 0.9699 | 0.1143 | 0.0279 | -0.0445 | -0.0297 |
| | 2.0 | 1.7960 | 0.1405 | 0.1182 | -0.0792 | 1.0064 | 0.0793 | 0.0296 | -0.0473 | -0.0237 |

*A. Comparison of Some Results of This Research with Results from Existing Literature.*

Results of the transverse shear stress, $\tau_{xz}$ and the deflection w, obtained for an orthotropic rectangular plate by Shimpi and Patel (2006) and Reddy (1984) are compared with the results obtained from this study using simple percentage difference. In the works of Shimpi and Patel (2006) and Reddy (1984), the authors used parameters for aspect ratio in the form of a/b and h/a for thickness to span ratio, while in this work, the parameters for aspect ratio and span to thickness ratios are expressed in the form of b/a and a/t respectively. Hence, a/b values of 0.5, 1.0 and 2.0 used in their study, corresponds to b/a values of 2.0, 1.0 and 0.5 respectively in the present study. Also, h/a values of 0.05, 0.1 and 0.14 used in their study, correspond to a/t values of 20, 10 and 7.14286 respectively used in this study.

- *Comparison of Non-Dimensional Shear Stress, $\overline{\tau_{xz}}$ of this Study with Previous Study on Simply Supported Orthotropic Rectangular Plate (Under Uniformly Distributed Transverse Load).*

The results of the non-dimensional transverse shear stress, $\overline{\tau_{xz}}$ parameters for simply supported orthotropic rectangular plate with uniformly distributed lateral load obtained in this study as compared with that of Shimpi and Patel (2006) and Reddy (1984) are presented on Table (4). It is observed from Table (4) that the percentage difference is least at $\left(\frac{a}{t}\right)$ value of 7.14286 for all aspect ratios. This implies that the percentage difference becomes smaller with decrease in the span to thickness value. This shows that the different theories agree more in thick plate analysis than thin plate analysis. With the works of Shimpi and Patel (2006), the variation of $\overline{\tau_{xz}}$ has a maximum percentage difference of 19.49% at $\left(\frac{a}{t}\right)$ value of 20 and $\left(\frac{b}{a}\right)$ value of 1.0 while the least variation is observed as a percentage difference of 2.89% when the $\left(\frac{a}{t}\right)$ value is 7.14286 and $\left(\frac{b}{a}\right)$ value is 2.0. When compared with the works of Reddy (1984), the variation shows a maximum percentage difference of 19.24% at a span to thickness $\left(\frac{a}{t}\right)$ value of 20 and aspect ratio $\left(\frac{b}{a}\right)$ of 1.0 while the least variation is obtained as a percentage difference of 4.28% when the span to thickness $\left(\frac{a}{t}\right)$ value is 7.14286 and aspect ratio $\left(\frac{b}{a}\right)$ is 2.0. This indicates that the Alternative II theory converges better with the works of Shimpi and Patel (2006) and the works of Reddy (1984) at higher values of $\left(\frac{b}{a}\right)$ ratio. The comparison of the results of the transverse shear stress, $\overline{\tau_{xz}}$ obtained from this research with that from previous scholars is reasonable as the maximum percentage difference of 19.49%. The variations are simply due to the different theories used by the different scholars. While Shimpi and Patel (2002) used a refined plate theory having just two variables with a third order shear deformation profile, Reddy (1984) used a higher order shear deformation theory.

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN1201

Table 4: Comparison of Present Study Non-Dimensional Transverse Shear Stresses ($\overline{\tau_{xz}}$) of Simply Supported Orthotropic Rectangular Plate Under Uniformly Distributed Transverse Load With Those of Previous Scholars

| **Plate dimensional parameters** | | $\overline{\tau_{xz}} = \frac{\tau_{xz}}{q}$ at x = 0, y = b/2, z = 0 | | | | |
|---|---|---|---|---|---|---|
| $\frac{b}{a}$ | $\frac{a}{t}$ | **Present Study** | **Shimpi and Patel (2006), (SP)** | **Reddy (1984), (R)** | **Percentage difference between Present study and previous works. (%)** | |
| | | | | | **SP** | **R** |
| **2.0** | **20** | 13.320 | 14.048 | 13.98 | 5.47 | 4.95 |
| | **10** | 6.651 | 6.9266 | 6.958 | 4.14 | 4.62 |
| | **7.14286** | 4.741 | 4.8782 | 4.944 | 2.89 | 4.28 |
| **1.0** | **20** | 9.099 | 10.873 | 10.85 | 19.49 | 19.24 |
| | **10** | 4.523 | 5.3411 | 5.382 | 18.09 | 18.99 |
| | **7.14286** | 3.207 | 3.7313 | 3.805 | 16.35 | 18.65 |

**Note**: $\left(\frac{E_2}{E_1} = 0.52500, \frac{G_{12}}{E_1} = 0.26293, \frac{G_{13}}{E_1} = 0.15991, \frac{G_{23}}{E_1} = 0.26681, \mu_{12} = 0.44046, \mu_{21} = 0.23124\right)$

- *Comparison of Non-Dimensional Displacement, $\overline{W}$ of this Study with Previous Study for Simply Supported Orthotropic Rectangular Plate (Under Uniformly Distributed Transverse Load).*

The non-dimensional displacement, $\overline{w}$ of simply supported orthotropic rectangular plate under uniformly distributed lateral load as compared with the works of Shimpi and Patel (2006) and Reddy (1984) are presented on Table (5). From Table (5), it is seen that the results from this study vary only slightly with the works of the previous scholars. The values follow the same trend as the displacement parameters decrease with a decrease in the span to thickness $\left(\frac{a}{t}\right)$ value. The non-dimensional displacement, $\overline{w}$ obtained in this study has a maximum percentage difference of 7.52% at span to thickness $\left(\frac{a}{t}\right)$ value of 20 and aspect ratio $\left(\frac{b}{a}\right)$ of 2.0 when compared with the works of Shimpi and Patel (2006), while its minimum percentage difference is 5.33% at $\left(\frac{a}{t}\right)$ value of 7.14286 and $\left(\frac{b}{a}\right)$ value of 1.0. When compared with the works of Reddy (1984), the variation in the values obtained from this study has a maximum percentage difference of 7.52% at $\left(\frac{a}{t}\right)$ value of 20 and $\left(\frac{b}{a}\right)$ value of 2.0 while its minimum percentage difference is 5.07% when the span to thickness $\left(\frac{a}{t}\right)$ value is 7.14286 and aspect ratio $\left(\frac{b}{a}\right)$ is 1.0. Hence, the values obtained for the non-dimensional displacement, $\overline{w}$ for an orthotropic rectangular plate are in much agreement with previous studies as the maximum percentage difference is obtained as 7.82%. The little variations are due to the different theories used by the different scholars. Therefore, the Alternative II theory is adequate for thick plate analysis.

Table 5: Comparison of Present Study Non-Dimensional Displacement, ($\overline{w}$) of Simply Supported Orthotropic Rectangular Plate Under Uniformly Distributed Transverse Load With Those of Previous Scholars

| Plate dimensional parameters | | $\overline{w} = \frac{wE}{\Delta t q}$ at x = a/2, y = b/2 | | | | |
|---|---|---|---|---|---|---|
| $\frac{b}{a}$ | $\frac{a}{t}$ | Present Study | Shimpi and Patel (2006), (**SP**) | Reddy (1984), (**R**) | Percentage difference between Present study and previous works. (%) | |
| | | | | | **SP** | **R** |
| **2.0** | **20** | 23294.1 | 21542 | 21542 | 7.52 | 7.52 |
| | **10** | 1522.02 | 1408.5 | 1408.5 | 7.46 | 7.46 |
| | **7.14286** | 418.192 | 387.23 | 387.5 | 7.40 | 7.34 |
| **1.0** | **20** | 11037.8 | 10443 | 10450 | 5.39 | 5.33 |
| | **10** | 727.495 | 688.57 | 689.5 | 5.35 | 5.22 |
| | **7.14286** | 201.836 | 191.07 | 191.6 | 5.33 | 5.07 |
| **0.5** | **20** | 2180.48 | 2048.7 | 2051.0 | 6.04 | 5.94 |
| | **10** | 148.862 | 139.08 | 139.8 | 6.57 | 6.09 |
| | **7.14286** | 42.865 | 39.79 | 40.21 | 7.17 | 6.19 |

**Note**: $\left(\frac{E_2}{E_1} = 0.52500, \frac{G_{12}}{E_1} = 0.26293, \frac{G_{13}}{E_1} = 0.15991, \frac{G_{23}}{E_1} = 0.26681, \mu_{12} = 0.44046, \mu_{21} = 0.23124\right)$

## V. CONCLUSION

Based on the results obtained from this study, it can be concluded that using the complete three-dimensional constitutive relations produces reasonable values for the normal stresses in the thickness direction of a plate. Also, from the comparison done with the works of previous scholars, it is seen that the Alternative II refined plate theory produced reasonable results, hence can be used in thick plate analysis.

## REFERENCES

[1]. Caterina L. (2019). Elastic constants determination of anisotropic materials by depth-sensing indentation. SN Applied Sciences (2019) 1:1263 | https://doi.org/10.1007/s42452-019-1301-y

[2]. Ghugal, Y. M., & Sayyad, A. S. (2011). Free vibration of thick orthotropic plates using trigonometric shear deformation theory. Latin American Journal of Solids and Structures, 8(3), 229–243. doi:10.1590/S1679-78252011000300002.

[3]. Ibearugbulem O.M, Onwuegbuchulem U.C and Ibearugbulem C. N (2021). Analytical Three-Dimensional Bending Analyses of Simply Supported Thick Rectangular Plate. *International Journal of Engineering Advanced Research (IJEAR) eISSN: 2710-7167 [Vol. 3 No. 1 March 2021] Journal website: http://myjms.mohe.gov.my/index.php/ijear*.

[4]. Ibearugbulem, O.M., Ezeh, J.C., Ettu, L.O, Gwarah, L.S. (2018). Bending Analysis of Rectangular Thick Plate Using Polynomial Shear Deformation Theory. IOSR Journal of Engineering (IOSRJEN) Vol. 08, Issue 9, ||V (III) || pp. 53-61

[5]. Liew, K.M, Hung K.C and Lim, M.K, (1993). A continuum three-dimensional vibration analysis of thick rectangular plates. Int. J. Solids Structures Vol. 30, No. 24, PP. 3357-3379.

[6]. Njoku, K.O, Ibearugbulem, O.M, Ettu, L.O, Anya U.C. (2023). Bending Analysis of Thick Anisotropic Rectangular Plate using Modified First Shear Deformation Theory. International Journal of Innovative Science and Research Technology ISSN No:-2456-2165.

[7]. Onyeka F.C, Mama B.O, and Okeke T.E (2022). Exact Three-Dimensional Stability Analysis of Plate Using a Direct Variational Energy Method. *Civil Engineering Journal, Available online at www.CivileJournal.org*

[8]. Rajesh, K. and Meera, S. K. (2016). Linear free vibration analysis of rectangular Mindlin plates using coupled displacement field method. Mathematical Models in Engineering. 1(1), 41-47.

[9]. Sadrnejad, S. A., Daryan, A. S., & Ziaei, M. (2009). Vibration equations of thick rectangular plates using mindlin plate theory. Journal of Computer Science, 5(11), 838–842. doi:10.3844/jcssp.2009.838.842.

[10]. Sarvestani, H.Y, Naghashpour, A. and Heidari-Rarani, M., (2015). Bending analysis of a general cross-ply laminate using 3D elasticity solution and layer wise theory. Int J Adv Struct Eng, 7:329–340b DOI 10.1007/s40091-014-0073-2.

[11]. Sayyad, A. S and Ghugal, Y. M. (2012). Bending and free vibration analysis of thick isotropic plates by using exponential shear deformation theory. Applied and Computational Mechanics 6, pp. 65–82

[12]. Sayyad, A. S. (2011). Comparison of various shear deformation theories for the free vibration of thick isotropic beams. International Journal of Civil and Structural Engineering Volume 2, No 1, Pp. 85-97.

[13]. Sburlati, R. (2014). Three-Dimensional Analyses of Functionally Graded Multi-Layered Systems. Procedia Engineering 88, pp. 235 – 241.

[14]. Shimpi R.P & Patel H.G. (2006). A two variable refined plate theory for orthotropic plate analysis. International Journal of Solids and Structures 43 (2006) 6783–6799.

[15]. Shufrin, I., & Eisenberger, M. (2005). Stability and vibration of shear deformable plates - First order and higher order analyses. International Journal of Solids and Structures, 42(3–4), 1225–1251. doi:10.1016/j.ijsolstr.2004.06.067.

[16]. Srinivas and Rao, A.K, (1971). A Three-dimensional Solution for Plates and Laminates. Journal of the Franklin Institute, Vol. 291, No. 6, pp. 469 – 481.

[17]. Uymaz, B, and Aydogdu, M, (2012). Three-dimensional shear buckling of fg plates with various boundary conditions. International Conference on Mechanics of Nano, Micro and Macro Composite Structures.

[18]. Vaghefi, R, Baradaran, G.H, Koohkan, H (2010). Three-dimensional static analysis of thick functionally graded plates by using meshless local Petrov–Galerkin (MLPG) method. Engineering Analysis with Boundary Elements 34, pp. 564 – 573.

[19]. Wuxiang, L., Zheng Z. (2009). Three-Dimensional Analysis of Simply Supported Functionally Graded Plate with Arbitrary Distributed Elastic Modulus. Tsinghua Science and Technology, Vol. 14, No. S2, pp. 58 - 63, December 2009.

[20]. Zhang, P., Qi, C., Fang, H., He, W. (2020), Three dimensional mechanical behaviors of in-plane functionally graded plates, Composite Structures (2020), doi: https://doi.org/10.1016/j.compstruct.2020.112124.