ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

Modified Hourgand Graph Metric Dimensions

Denizar Kemala[1]; Nurdin[2]; Muhammad Zakir[3]
[1,2,3]Department of Mathematics, Faculty of Mathematics and Natural Sciences,
Hasanuddin University. Jl. Perintis Kemerdekaan Km. 10 Tamalanrea Makassar, Indonesia 90245

Abstract:- For example, G is a connected graph and W is a subset of the set of points V on G. Set W is called the determining set on G if every point on G has different representations towards W. A determining set with a minimum number of members is called a minimum determining set or the basis of G and the cardinality of the minimum determinant set represents the metric dimension of the graph G. And denoted by $\dim(G)$.

This paper discusses the metric dimensions of modified hourglass graphs mHg_n constructed from a complete graph K_1 with graphs C_n. Based on the results of the discussion, it was found that dim (mHg_n) with $m \geq 3$ and $3 \leq n \leq 5$ is $2m$.

Keywords:- *Hourgand Graph, Metric Dimensions, Trajectory Graph.*

I. INTRODUCTION

Graph theory is a branch of mathematics that was first introduced by Leonhard Euler in 1736. He solved the problem of the Konigsberg bridge in Russia in his work "*Solutio problematis ad geometrian situs pertinentis*". In this problem, Euler wanted to try to prove that he crossed seven bridges connecting four lands in one pass [1], He described this problem by making the four land masses into a point and the seven bridges connecting them into a side. This was the forerunner to the birth of the concept of graph theory.

Graph G is defined as a non-empty and finite set $V(G)$ with its members called points as well $E(G)$ is a (possibly empty) edge set whose members consist of unordered pairs of two distinct points that are elements $V(G)$ which are called sides/lines. In graph theory, there are various concepts, one of which is Metric Dimensions. Metric dimensions were first introduced by Slater in 1975, and then separately by Harary and Melter in 1976 [2]. For example, $V(G)$ is a set of points on a graph G, The distance between two points is denoted by $d(u, v)$ is the length of the shortest path from u to v. For ordered sets $W = \{w_1, w_2, w_3, \ldots, w_k\}$ of connected graph points G and point $v \in V(G)$, the representation of v concerning W is a k-vector (pair of k-tuples) $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$. Jika $r(v|W)$ for each point $v \in V(G)$ different, then W is called the differentiating set of $V(G)$. The set of distinctions with minimum cardinality is called the minimum distinction set (metric basis), and the cardinality of the metric basis is called the metric dimension and is denoted by $dim(G)$ [3,8,9,10,12].

Research on metric dimensions has been carried out by many previous researchers, for example, research on metric dimensions in general graphs was carried out by Klein, D.J., Yi, E (2012) who researched the comparison of the metric dimensions of a graph with new graph forms, Kousar, I. et al (2010) researched graphs that have the same metric dimensions, and Glen G Chappel is a researcher who has an important role in the development of metric dimensions specifically, especially for research on the metric dimensions of special graphs. In graph theory, there are many special types of graphs, including path graphs, cycle graphs, complete graphs, bipartite graphs, and star graphs. As in mathematics in general, in graph theory, there are also operations between two graphs. Operations on graphs use the same terms as operations on algebra, including, combination, addition, and multiplication, furthermore, there are also corona and amalgamation operations on a graph.

The hourglass graph which is denoted by (Hgn) is a new type of graph introduced by Syamsuddin in his research "Multipartite Ramsey Number Measures for Trajectory Graphs Versus Hourglass Graphs." The hourglass graph Hgn is the graph resulting from the operation of adding graph $K1$ with graph $2Cn$ $(k1 + 2Cn)$. However, here the researchers modified the hourglass graph by adding m-copies of Cn. Because this graph is new, researchers are interested in conducting research related to the metric dimensions of the modified Hourglass graph.

II. LITERATURE REVIEW

Below we will provide several supporting theories that have been implemented by several researchers in the field of graph theory.

A. Dasar-Dasar Graf

➢ *Definition*

A graph is a pair of sets (V, E), where V is a discrete set whose members are called points, while E is a pair of members of V which are called edges. [4]

Based on definition 2.1.1, the set V is called the vertex set and E is called the edge set. Sometimes some people call points as vertices and sides as points, arcs, edges, or lines. Mathematically, Definition 2.1.1 can be written as follows: Graph $G = (V(G), E(G))$ with

$V(G) = \{u\text{: } u\ disebut\ titik\}$ and $E(G) = \{(u, v)\text{:}u, v \in V(G)\}$ with (u, v) is called a side, but in this discussion, the side (u, v) will be written as uv.

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

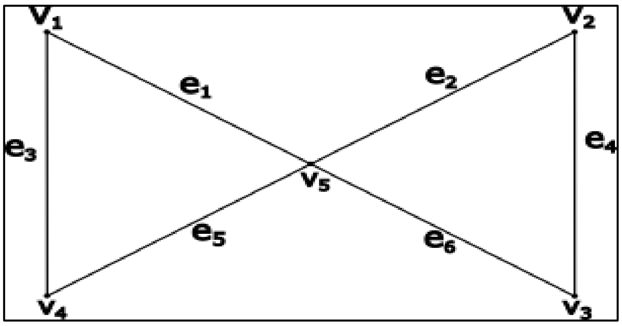
➢ *Definition*

The cardinality of a set is the number of elements in that set, cardinality is usually denoted by "| |".

The order of graph G is the number of elements of $V(G)$ and is denoted by $p(G)$ while the size is the number of edges in graph G and is denoted by $q(G)$. So if $p(G)$ is the order of graph G and $q(G)$ is its size, then $p(G) = |V(G)|$ and $q(G) = |E(G)|$.

Definition 2.1.3 The degree of a point vi in a graph G, denoted by "$d(v_i)$", *is the number of edges associated with point vi or* $d(v_i) = |N_G(v_i)|$.

- Example: Suppose a graph G with $V(G) = \{v1, v2, v3, v4, v5\}$ and $E(G) = \{e1, e2, e3, e4, e5\}$ with $e1 = v1v5$, $e2 = v2v5$, $e3 = v1v4$, $e4 = v2v3$, $e5 = v4v5$.



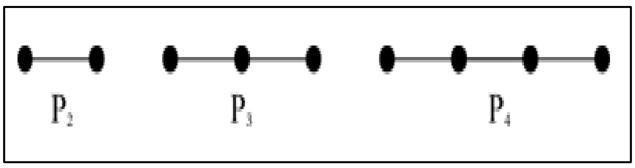
Picture 1: Graph G

From Figure 2.1.1 above, it can be seen that there are four points of degree two, namely $v1$, $v2$, $v3$, $v4$, and one point of degree four, namely $v5$, and the graph above has order $p(G) = 5$ and size $q(G) = 6$.

➢ *Definition*

A path graph is a graph consisting of a sequence of vertices and edges $v1, e1, v2, e2, \dots, en{-}1, vn$ with $ei = vivi{+}1$, $i = 1,2, \dots, n - 1$ and denoted by P_n.

The following will show the trajectory graph in Figure 2 below:

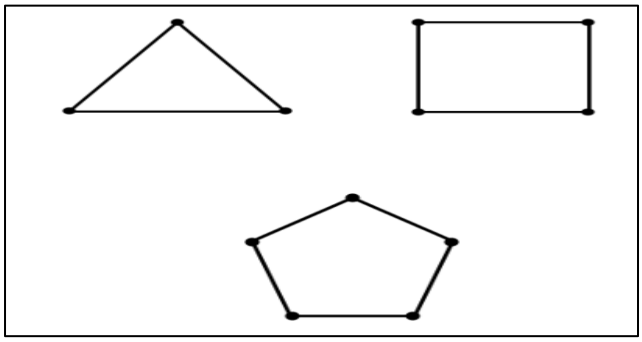


Picture 2: Trajectory Graph

Picture 2 above is an example of a trajectory graph P_2, P_3, P_4.

➢ *Definition*

Cycle graph with n vertices and n edges where $n \geq 3$ and denoted by C_n is a graph with a set of points $V(C_n) = V(P_n)$ and set of edges $E(C_n) = E(P_n) \cup \{v_n v_1\}$. Next, we will show several examples of cycle graphs below:


Picture 3: Cycle Graph

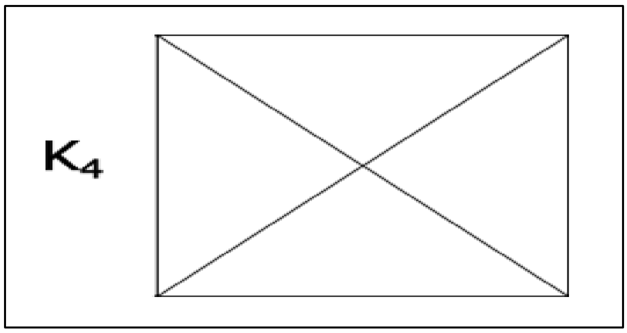
From the three images above you can see several cycle graphs C_3, C_4, C_5.

A graph G is said to be a connected graph if for every two points u and v there is always a path containing points u and v. Based on this understanding, a cycle graph is a connected graph.

➢ *Definition*

Graph G is said to be a complete graph if every two vertices on graph G are adjacent. The complete graph with n points is denoted Kn.

A complete graph has a special characteristic, namely that it has the same degree. A graph where every vertex has the same degree is called a regular graph. If the degree of the regular graph is r, denoted $r - reguler$, then the complete graph $Kn = (n - 1) - reguler$, because every vertex in the graph Kn is sequenced jat $n - 1$. [4]. Below is a complete graph image:



Picture 4: Complete Graph

➢ *Definition*

Distance from point u to point v on graph G is denoted by $d(u, v)$ is the length of the shortest path from u to v, $d(u, v) \geq 0$ for all pairs of vertices u, v in the graph G and $d(u, v) = 0$ if and only if $u = v$. If there is no path from u to v, then $d(u, v) = \infty$. [5, 8,9,10]

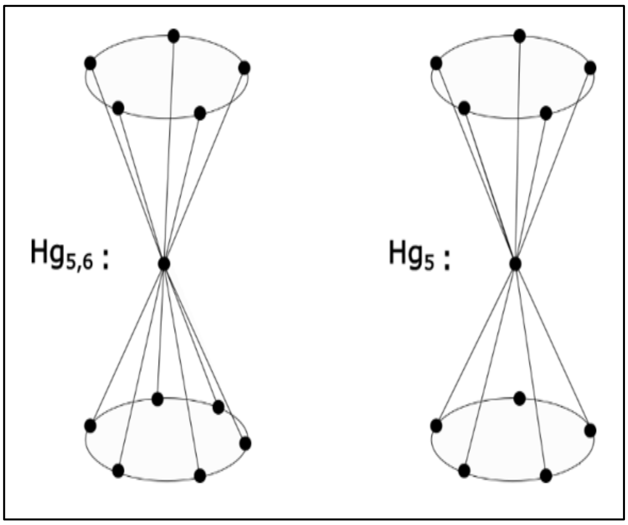
➢ *Definition*

Hourglass graph ($Hg_{n,r}$) is the sum graph between the complete grapequation the combination of two cycle graphs with n vertices on one cycle and r vertices on the other cycle. If $n = r$ then it is an hourglass graph $Hg_{n,r}$ can be said to be a balanced hourglass graph, and is denoted by Hg_n. [6]

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

Below we will show a representation of two examples of hourglass graphs, namely $Hg_{5,6}$ and Hg_5:

The following figure 2.1.5 is an illustration of the hourglass graph and the complete graph.



Picture 5: Graph $Hg_{5,6}$ and Graph Hg_5

From Figure 5 above the graph, points are complete K_1 on an hourglass graph $Hg_{n,r}$ as the center point. Centre point $Hg_{n,r}$ degree $n + r$, n on point C_n degree 3 and r on C_r also has degree 3. If $n = r$ the hourglass graph is called balanced and is denoted by Hg_n.

- Next, we will define the graph Hg_n in mathematical form as follows:

✓ $V(Hg_n) = \{x_{ij}, y|1 \leq i \leq 2, 0 \leq j \leq n-1\}$,
✓ $E(Hgn) = \{xijy, xijxi(j{+}1)\ mod\ n|1 \leq i \leq 2{,}0 \leq j \leq n-1\}$

B. Metric Bases and Dimensions

The following will explain the definition of basis and metric dimensions in graphs:

➢ *Definition*

Suppose G is a connected graph and there is an ordered set $W = \{w_1, w_2, w_3, \dots, w_n\} \subseteq V(G)$, **representation of points $\boldsymbol{v} \in \boldsymbol{V(G)}$** concerning W is an ordered pair of k-tuples, i.e $r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_n))$.

➢ *Definition*

A set W is called a determining set if for every two points $x, y \in V(G)$ different ones fulfill $r(x|W) \neq r(y|W)$. [7, 11, 12]

➢ *Definition*

The set of determinants that has the minimum cardinality is called the minimum separated set.. [7]

➢ *Definition*

The basis of a graph G is the minimum determinant set of G.[7,8,9,10, 11,12].

➢ *Definition*

The metric dimension of the graph G is the number of cardinalities of the minimum determinant set (basis) and is denoted **dim** $(\boldsymbol{G})$. [7,11]

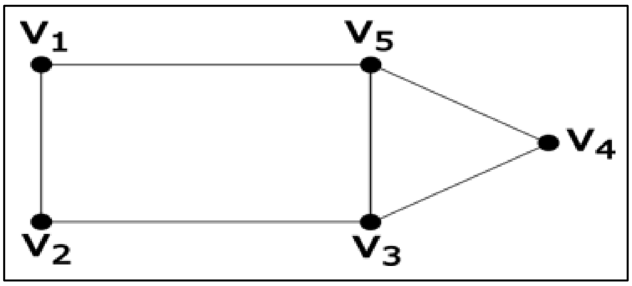
Suppose selected $W_2 = \{v_1, v_4\}$ the representation of each point on the graph G concerning W_2 is

Table 1: Table of Representation of Each Point on the Graph G Against the Set W with $|W| < |W_2|$

$\boldsymbol{r(v_1|W_2) = (0,2)}$	$\boldsymbol{r(v_2|W_2) = (1,2)}$	$\boldsymbol{r(v_3|W_2) = (2,1)}$
$r(v_4|W_2) = (2,0)$	$r(v_5|W_2) = (1,1)$ $(v_3|W2) = (2,1)$	

Because there is no equal representation, it can be concluded that W_2 is the set of determinants in the graph G. Next, we will investigate whether W_2 is the basis of the graph G, by calculating the representation of any element point $V(G)$ that has a cardinality less than W_2. As stated in the following table:

- Example: For example, G is a graph with a set of points $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(G) = \{v_1v_2, v_1v_5, v_5v_4, v_5v_3, v_3v_4\}$. The shape of the graph G can be seen in the following image:



Picture 6: Graph G

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

Suppose selected $W_1 = \{v_1, v_3\}$, representation of each point on the graph G against W_1 is :

$r(v_1\|W_1) = (0,2)$	$r(v_2\|W_1) = (1,1)$	$r(v_3\|W_3) = (2,0)$
$r(v_4\|W_1) = (2,1)$	$r(v_5\|W1) = (1,1)$ $(v_3\|W3) = (2,0)$	

Because there is the same representation, namely $r(v_2|W_1) = r\{v_5|W_1) = (1,1)$. so W_1 is not a determining set in the graph G.

	$r(v_1\|W)$	$r(v_2\|W)$	$r(v_3\|W)$	$r(v_4\|W)$	$r(v_5\|W)$
$W = v_1$	(0)	(1)	(2)	(2)	(1)
$W = v_2$	(1)	(2)	(2)	(1)	(0)
$W = v_3$	(2)	(1)	(1)	(0)	(1)
$W = v_4$	(2)	(1)	(0)	(1)	(2)
$W = v_5$	(1)	(0)	(1)	(1)	(2)

From Table 2.2.1, it can be seen that all subsets of $V(G)$ which has a number of members less than $|W_2|$ is not a determining set because every point in the graph G has the same representation of W. So W_2 is the minimum determinant set. So that $W_2 = \{v_1, v_4\}$ is the basis of the graph G, and dim $G = 2$.

- **Theorem 2.2.1** Let G be a connected graph with order $n \geq 2$, so $\dim(G) = 1$ if and only if $G = P_n$.
- **Theorem 2.2.2** If Cn is a cycle graph with n vertices and $n \geq 3$, so $\dim(C_n) = 2$
- **Proof:** For example, $v_1, v_2, v_3, \ldots, v_n$ are the points on the cycle with $n \geq 3$ on the graph G.
  For cycles with odd n. For example $W = \{u_{n-1}, u_n\}$ we will prove that W is the determining set. Representation of each point on a graph C_n concerning S is

$$r(v_1|W) = (2,1)$$
$$r(v_2|W) = (3,2)$$
$$r(v_3|W) = (4,3)$$
$$\vdots$$
$$r\left(v_{\frac{n-1}{2}}|W\right) = \left(\frac{n-1}{2}, \frac{n-3}{2}\right)$$
$$r\left(v_{\frac{n-1}{2}}|W\right) = \left(\frac{n-1}{2}, \frac{n-1}{2}\right)$$
$$r\left(v_{\frac{n+1}{2}}|W\right) = \left(\frac{n-3}{2}, \frac{n-1}{2}\right)$$
$$r\left(v_{\frac{n+3}{2}}|W\right) = \left(\frac{n-5}{2}, \frac{n-3}{2}\right)$$
$$\vdots$$
$$r(v_{n-2}|W) = (1,2)$$
$$r(v_{n-1}|W) = (0,1)$$
$$r(v_n|W) = (1,0)$$

because for every $u, v \in V(C_n)$, $u \neq v$ applies $r(u|W) \neq r(v|W)$, so $W = \{v_{n-1}, v_n)$ is the determining set. Next it will be proven that $W = \{v_{n-1}, v_n)$ is the set of determinants with minimum cardinality. Because graph C_n is a cycle graph, by Theorem 2.2.1, then $dim(C_n) \neq 1$, so that there is no set of determinants with cardinality less than 2. Therefore, $|W| = 2$ is a set of determinants with minimum cardinality, so that $dim(C_n) = 2$ for odd n.

For a cycle with even n, for example $W = \{v_{(n-1)}, v_n\}$. It will be proven that W is a determining set. Representation of each point on the graph C_n with respect to W is

$$r(v_1|W) = (2,1)$$
$$r(v_2|W) = (3,2)$$
$$r(v_3|W) = (4,3)$$
$$\vdots$$
$$r\left(v_{\frac{n-1}{2}}|W\right) = \left(\frac{n}{2}, \frac{n-2}{2}\right)$$
$$r\left(v_{\frac{n}{2}}|W\right) = \left(\frac{n-2}{2}, \frac{n}{2}\right)$$
$$r\left(v_{\frac{n+2}{2}}|W\right) = \left(\frac{n-4}{2} \frac{n-2}{2}\right)$$
$$r\left(v_{\frac{n+3}{2}}|W\right) = \left(\frac{n-5}{2}, \frac{n-3}{2}\right)$$
$$\vdots$$
$$r(v_{n-2}|W) = (1,2)$$
$$r(v_{n-1}|W) = (0,1)$$
$$r(v_n|W) = (1,0)$$

because for every $u, v \in V(C_n)\ u \neq v$, applies $r(u|W) \neq r(v|W)$,

so $W = \{v_{n-1}, v_n)$ is the determining set.

Next it will be proven that $W = \{v_{n-1}, v_n)$ is the set of determinants with minimum cardinality. Because graph C_n is a cycle graph, by Theorem 2.2.1, then $dim(C_n) \neq 1$, so that there is no set of determinants with cardinality less than 2. Therefore, $|W| = 2$ is a set of determinants with minimum cardinality, so that $dim(C_n) = 2$ for n is even.

Based on I and II, it is proven that the graph is cyclical (C_n) with n odd and n even then $\dim(C_n) = 2$.

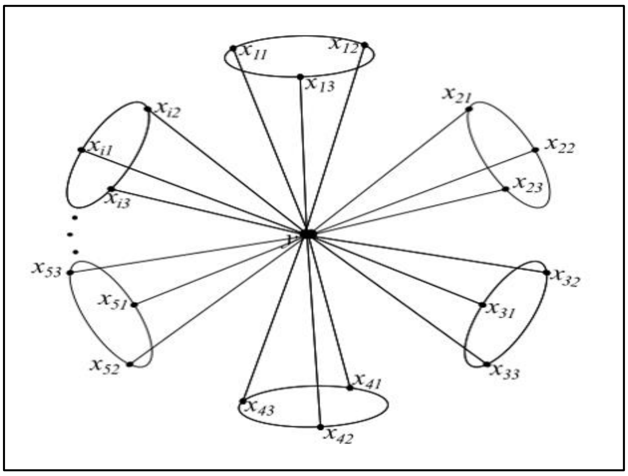
III. RESULT

In this section, the research results and evidence will be discussed. Previously we would define a modified hourglass graph as follows.

A. Modified Hourglass Graph

Hourglass graph Hg_n is the graph resulting from the graph add operation k_1 with $2C_n$, so $(k_1 + 2C_n)$. The graph is a development of the hourglass graph Hg_n what is meant is increase C_n as much as m-coffee, that is $(k_1 + mC_n)$, For $m \geq 3$ *dan* $n \geq 3$. Next, graph $(k_1 + mC_n)$ denoted by mHg_n. Formally, the set of vertices and the set of edges of an hourglass graph are as follows. $V(mHg_n) = \{x_{i,j}, y|i = 1,2,3, \ldots, m; j = 1,2,3, \ldots, n\}$.

$E(mHg_n) = \{x_{i,j}y|i = 1,2,3, \ldots, m; j = 1,2,3, \ldots, n\} \cup \{x_{i,1}x_{i,n}, x_{i,j}x_{i,j+1}|i = 1,2,3, \ldots, m; j = 1,2,3, \ldots, n-1\}$. Here's an example



Picture 7: mHg_3

Based on the definition of a set of points and a set of graph edges mHg_n This results in several properties related to the distance of each point on the graph mHg_n for i, k = 1,2,3, … , m, and j, l = 1,2,3, … , n, that is:

- $d(x_{i,j}, y) = 1$
- $d\left(x_{i,j}, y_{k,i}\right) = \begin{cases} 0 & i = k, j = l \\ 1 & i = k, l - j + 1 \\ 2 & i \neq k \end{cases}$

B. Modified Hourglass Graph Metric Dimensions

In this section, several properties related to the gram meter dimension of the hourglass are given.

- *Lemma 1.*

If $W \subseteq V(mHg_3)$ is a set of determinants, then $|W| > 2m - 1$.

- *Proof:*

Suppose W is a set of determinants with $|W| \leq 2m - 1$. Then there is a bar that has two different points, for example, x and y, such that $x, y \notin W$. Because the distance between points on the same blade is one, and the distance between points on one blade and another blade is two, then points x and y have the same representation of W. Thus it is a contradiction that W is a set of determinants. So that $|W| > 2m - 1$.

Next will be shown for the case $|W| \leq 2m - 1$ as follows:

I. Only contains one point on one bar contained in the set W

For example $W = \{x_{13}, x_{22}, x_{23}, \dots , x_{m2}, x_{m3}\}$. It will be proven that W is a determining set.

$$r(x_{11}|W) = (1,2,2,\dots ,2,2),$$
$$r(x_{12}|W) = (1,2,2,\dots ,2,2).$$

It is clear that if one bar contains only one point at W then there will be two points that have the same representation.

There are no points contained on one bar in the set W. For example $W = \{x_{21}, x_{22}, x_{23}, \dots , x_{m2}, x_{m3}\}$. It will be proven that W is a determining set.

$$r(x_{11}|W) = (2,2,2, \dots ,2,2),$$
$$r(x_{12}|W) = (2,2,2,\dots ,2,2),$$
$$r(x_{13}|W) = (2,2,2,\dots ,2,2).$$

It can be seen that there are 3 points that have the same representation, namely points contained in the same bar. So it can be concluded from cases I and II that a bar must contain at least 2 points in the differentiator set W.

- *Lemma 2.*

If $w_i, w_j \in W$, $W \subseteq V(mHg_n)$ and $i \neq j$ with $1 \leq i, j \leq 5$ so $r(w_i|W) \neq r(w_j|W)$.

- *Proof:*

For example $W = \{w_1, w_2, w_3, w_4, w_5\}$ and $w_i, w_j \in W$ So representation w_i and w_j with respect to W i.e. $r(w_i \,|\, W) = (\dots, \underbrace{x}_{i}, \dots, \underbrace{0}_{j}, \dots)$ will be 0 for $(d(wj, wj))$ and not 0 for $(d(w_i, w_j))$, $r(w_j \,|\, W) = (\dots, \underbrace{x}_{i}, \dots, \underbrace{0}_{j}, \dots)$ will be 0 for $(d(w_i, w_i))$ and not 0 for $(d(w_i, w_j))$.

So it's proven $r(w_i|W) \neq r(w_j|W)$ for $i \neq j$.

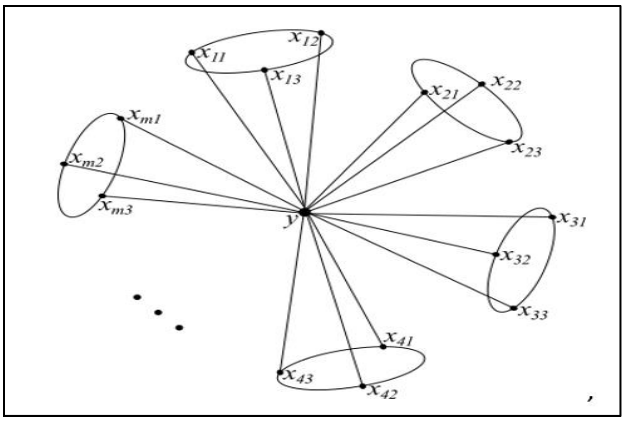
- *Theorem 1.*

Modified hourglass graph metric dimensions (mHg_n), *for* $m \geq 3$ *and* $3 \leq n \leq 5$ is $2m$.

- *Proof*

In this proof the following three cases will be reviewed:

- *Case 1: $n = 3$*



Picture 8: mHg_3

Based on Lemma 1 we obtain $\dim(mHg_3) > 2m - 1$ or $\dim(mHg_3) \geq 2m$. To prove $\dim(mHg_3) \leq 2m$ then we will look for the differentiating set W with cardinality $|W| = 2m$.

For example $W = \{x_{12}, x_{13}, x_{22}, x_{23}, \dots , x_{m2}, x_{m3}\}$. It will be proven that W is a determining set.

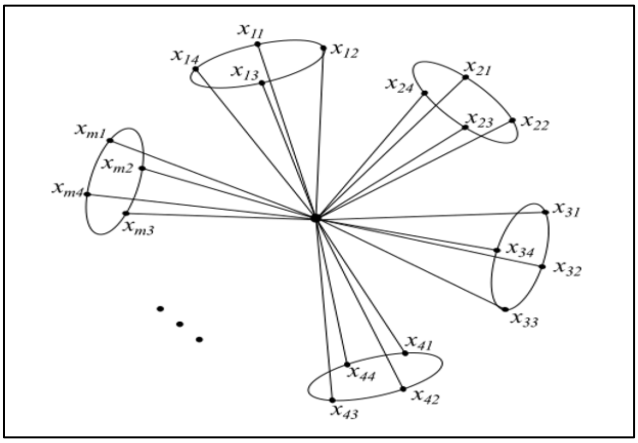
ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

$$r(x_{11}|W) = (1,1,2,2,\ldots,2,2),$$
$$r(x_{12}|W) = (0,1,2,2,\ldots,2,2),$$
$$r(x_{13}|W) = (1,0,2,2,\ldots,2,2),$$
$$r(x_{21}|W) = (2,2,1,1,\ldots,2,2),$$
$$r(x_{22}|W) = (2,2,0,1,\ \ldots,2,2),$$
$$r(x_{23}|W) = (2,2,1,0,\ \ldots,2,2),$$
$$\vdots$$
$$r(x_{m1}|W) = (2,2,2,2,\ldots,1,1),$$
$$r(x_{m3}|W) = (2,2,2,2,\ldots,1,0),$$
$$r(y|W) = (1,1,1,1,\ \ldots\ ,1,1).$$

Because based on Lemma 2 the representation of each point is different, then W is a set of determinants at mHg_3. Because $|W| = 2m$ so $\dim(mHG_3) \leq 2m$. (1)

Based on Lemma 1 and equation (1) we obtain $\dim(mHg_3) = 2m$.

- *Case 2, for $n = 4$*



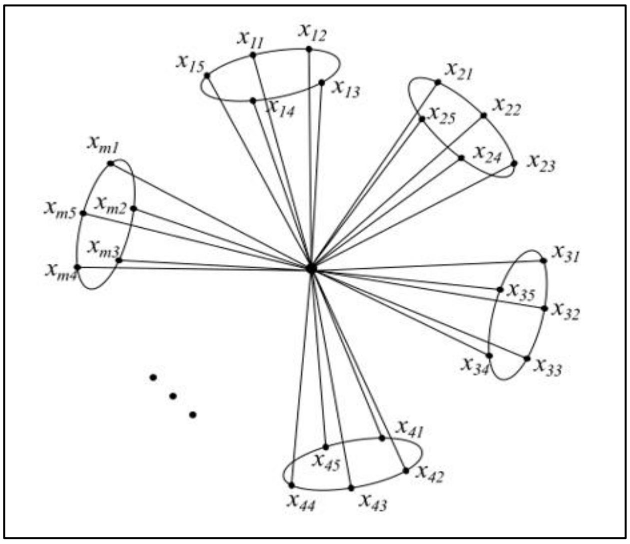
Picture 9: $mHg4$

For example $W = \{x_{12}, x_{13}, x_{22}, x_{23},\ \ldots\ , x_{m2}, x_{m3}\}$ It will be shown that W is the determinant set. Representation of each point on mHg_4 with respect to W is

$$r(x_{11}|W) = (1,2,2,2,\ \ldots\ ,2,2),$$
$$r(x_{12}|W) = (0,1,2,2,\ \ldots\ ,2,2),$$
$$r(x_{13}|W) = (1,0,2,2,\ \ldots\ ,2,2),$$
$$r(x_{14}|W) = (2,1,2,2,\ \ldots\ ,2,2),$$
$$r(x_{21}|W) = (2,2,1,2,\ \ldots\ ,2,2),$$
$$r(x_{22}|W) = (2,2,0,1,\ \ldots\ ,2,2),$$
$$r(x_{23}|W) = (2,2,1,0,\ \ldots\ ,2,2),$$
$$r(x_{24}|W) = (2,2,2,1,\ \ldots\ ,2,2),$$
$$\vdots$$
$$r(x_{m1}|W) = (2,2,2,2,\ \ldots\ ,1,2),$$
$$r(x_{m2}|W) = (2,2,2,2,\ \ldots\ ,0,1),$$
$$r(x_{m3}|W) = (2,2,2,2,\ldots,1,0),$$
$$r(x_{m4}|W) = (2,2,2,2,\ \ldots\ ,2,1),$$
$$r(y|W) = (1,1,1,1,\ \ldots\ ,1,1).$$

Because for every point $u, v \in V(mHg_4)$ for $u \neq v$ or based on Lemma 2, then $W = \{x_{12}, x_{13}, x_{22}, x_{23},\ \ldots\ , x_{m2}, x_{m3}\}$ is the determining set. Next we will show that W is the minimum determinant set. Example $W = X$ with $X = \{x_{i2}, x_{i3} | 1 \leq i \leq m\}$ so $|X| = 2m$. Note that the members of X are m pairs of points each drawn from $V(C_4^i)$. Based on the condition of distance between points, it is known that the distance between each point located on one blade and the point on the other blade is the same. Thus, because $\dim(C_4^i) = 2$ (Theorem 2.1) then if you only take one point (arbitrarily) member in $V(C_4^i)$ to load in W call $x_{ij} \in V(C_4^i)$ for each i and j with $1 \leq j \leq 4$, then there must be at least two other points, namely $x_{i(j+1 \bmod 4)}$ dan $x_{i(j+3 \bmod 4)}$ has the same representation as W. Therefore members $X \subset W$ cannot be reduced or $|X| \leq 2m$. Then it can be concluded that W with cardinality $|W| = |X| = 2m$ is the minimum determinant set then for the case $n = 4$ is obtained $\dim(mHg_4) = 2m$.

Kasus 3, untuk $n = 5$



Picture 10: mHg_5

For example $W = \{x_{12}, x_{14}, x_{21}, x_{24},\ \ldots\ , x_{m2}, x_{m4}\}$, It will be shown that W is the determinant set. Representation of each point on mHg_5 with respect to W is

$$r(x_{11}|W) = (1,2,2,2,\ \ldots\ ,2,2),$$
$$r(x_{12}|W) = (0,2,2,2,\ \ldots\ ,2,2),$$
$$r(x_{13}|W) = (1,1,2,2,\ \ldots\ ,2,2),$$
$$r(x_{14}|W) = (2,0,2,2,\ \ldots\ ,2,2),$$
$$r(x_{15}|W) = (2,1,2,2,\ \ldots\ ,2,2),$$
$$r(x_{21}|W) = (2,2,1,2,\ \ldots\ ,2,2),$$
$$r(x_{22}|W) = (2,2,0,2,\ \ldots\ ,2,2),$$
$$r(x_{23}|W) = (2,2,1,1,\ \ldots\ ,2,2),$$
$$r(x_{24}|W) = (2,2,2,0,\ \ldots\ ,2,2),$$
$$r(x_{21}|W) = (2,2,2,1,\ \ldots\ ,2,2),$$
$$\vdots$$
$$r(x_{m1}|W) = (2,2,2,2,\ \ldots\ ,1,2),$$
$$r(x_{m2}|W) = (2,2,2,2,\ \ldots\ ,0,2),$$
$$r(x_{m3}|W) = (2,2,2,2,\ \ldots\ ,1,1),$$
$$r(x_{m4}|W) = (2,2,2,2,\ \ldots\ ,2,0),$$
$$r(x_{m5}|W) = (2,2,2,2,\ \ldots\ ,2,1),$$
$$r(y|W) = (1,1,1,1,\ \ldots\ ,1,1).$$

Because based on lemma 2 W is a determining set. Next, it will be proven that $W = \{x_{12}, x_{14}, x_{21}, x_{24},\ \ldots\ , x_{m2}, x_{m4}\}$ is the minimum determinant set. For example $W = X$, with $X = \{x_{i2}, x_{i4} | 1 \leq i \leq m\}$. So $|X| = 2m$.

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/IJISRT24JUN485

Note that the members of X are m pairs of points each drawn from $V(C_5{}^i)$. Based on the condition of distance between points, it is known that the distance between each point located on one blade and the point on the other blade is the same. Thus, because dim($C_5{}^i$) = 2. (Theorem 2.1) then if you only take one point (arbitrarily) member $V(C_5{}^i)$ to load di W, call $x_{ij} \in V(C_5{}^i)$ for a i and j with $1 \leq j \leq 5$ then there must be at least two other points that have the same representation of W. Therefore, $X \subset W$ cannot be reduced or $|X| \geq 2m$. So for the case $n = 5$, we obtain dim(mHg_5) = $2m$.

Based on the description of the three previous cases, it is proven that dim(mHg_n) = $2m$ for $3 \leq n \leq 5$.

IV. CONCLUSION

From the description above, this research can be concluded that. Based on cases 1 to case 3, it is known that the metric dimensions of the hourglass are modified (mHg_n) = $2m$.

REFERENCES

[1]. F. Harary, Graf Theory, USA: Addision Wesley Publishing Company, 1994.
[2]. F. M. R. A. Harary, ""On the metric dimension of a graph,,",," *Ars Combinatoria,* pp. p. 2: 191 -195,, 1976.
[3]. G. Chartrand and L. Lesniak, graphs and digraphs third edition, United State of America: Chapman & Hall, 2000.
[4]. Hasmawati, Pengantar dan jenis-jenis Graf, Makassar: Unhas Press, 2020.
[5]. R. Wijaya, "Dimensi Metrik Hasil Identifikasi," *Jurnal matematika UNAD,* vol. 11, pp. 199-209, 2022.
[6]. Syamsuddin, "Bilangan Ramsey Multipartit Ukuran untuk Graf Lintasan versus Graf Jam Pasir," 2023.
[7]. S. M. Saputro, "Th metric Dimension of Comb Products Graphs," in *Graphs theory Conference in Honor of Egawa's 60th Birthday*, september 10 to 14.
[8]. G. Chartrand, Introductory Graph Theory, Dover, 1985.
[9]. Clark and D. A. Hoiton, A First Look at Graph Theory, World Scientific Publishing, 1991.
[10]. F. Harary, R. Z. Norman and D. Cartwright, Structural Models, Wiley, 1965.
[11]. O. Ore, Graphs and their Uses, 2nd edn, New Mathematical Library 10, Mathematical Association of America, 1990.
[12]. R. J. Wilson and J. J. Watkins, Graphs; An Introductory Approach, Wiley, 1990.