A Review of Some Classical Puzzles Using Mathematical Approaches

Sadia Ferdousi[1]; Sujana Azmi Polin[2*]
[1]Department of Computer Science and Engineering, International Standard University, Dhaka-1212, Bangladesh
[2]Department of Mathematics, University of Dhaka ,Dhaka-1000, Bangladesh

Corresponding Author: Sujana Azmi Polin[2*]

Abstract:- A puzzle is a problem or game that check an individual's reasoning ability. The article explores some classical puzzles using mathematical concepts. The historical significance and the concepts of mathematics which have been applied to these puzzles have also been discussed. In this article, the famous Josephus's problem, the Tower of Hanoi problem, the Cutting Pie problem and Travelling the World Problem, have been illustrated. The explanation of Josephus's problem describes how elimination works in a circular sequence and how to maximize cuts from a circular region in the cutting-the-pie-problem. Recurrence relation is derived and the proof is established using mathematical induction. It also focuses on finding the shortest path on a dodecahedron in analyzing Travelling the World Problem which uses graph theory to find a Hamiltonian path. The existence of such a path is demonstrated. This article explores the real-life origins of these well-known problems and highlights their significance in understanding mathematical ideas. The detailed solutions to each puzzle will give readers perception into recursion, optimization, geometric properties, and the fascinating historical backgrounds of these puzzles.

Keywords:- *Puzzles, Josephus Problem, Tower of Hanoi Problem, Cutting Pie Problem, Travelling the World Problem, Hamiltonian Path.*

I. INTRODUCTION

A puzzle is something difficult to understand or explain. It can be a kind of toy, game, or test that assesses one's ability to think or know logically. In 1762, it is believed that John Spilsbury, a London cartographer and engraver, produced the first jigsaw puzzle using a marquetry saw [1]. However, Archimedes, one of the finest mathematicians, is also attributed to the creation of early forms of puzzles [2]. Archimedes is known to be the father of all puzzles. He invented the famous dissection puzzle known as Archimedes' STOMACHION OSTOMACHION, or SYNTEMACHION, composed in a Palimpsest written by an anonymous medieval scribe compiling prayers [3]. The OSTOMACHION puzzle is an influential piece, it harbours both creative and logical problems. OSTOMACHION is translated as 'bone fight' as it was originally crafted from bone and required a mental wrestle to complete. The puzzle has 14 polygons, 13 of which are different, and 2 are the same [4]. Those 14 polygons were divided into 11 triangles, 1 pentagon, and 2 quadrangles, which can be assembled into a square. At first, it was made for social gatherings, but it can be solved into 56 entirely unique squares and 17152 rotations, which can create so much mental stimulation. Many people don't realize that mental stimulation is just as vital to overall well-being as physical exercise [5]. Regular mental exercises, such as working with the OSTOMACHION or similar puzzles, have been shown to enhance memory and strengthen connections between brain cells [6]. Puzzle-solving can also boost dopamine levels in the brain, leading to improvements in problem-solving abilities, short-term memory, and visual-spatial processing [7]. The academic study of puzzles, known as Enigmatology [8], highlights the importance of puzzles and games in developing reasoning and logical thinking skills, which are crucial for mathematical proficiency [9].

His research article demonstrates the link between real-life problems that appear to have no logical solution but can be explained and resolved through mathematical logic. Its objective is to connect people more to rational approaches to problems that, on the surface, seem unsolvable. The article provides an extensive analysis of four puzzles—Josephus Problem, Tower of Hanoi, Cutting the Pie Problem, and Traveling the World Problem. The main objective of this article is to shed light on the history of each puzzle, its mathematical solution, and its real-life applications. Drawing from existing knowledge, the research compiles and synthesizes established mathematical insights to broaden readers' grasp of mathematical concepts and their logical use. These mathematical puzzles have significant potential to help understand more advanced problems and prepare readers for them.

II. LITERATURE REVIEW

The Josephus Problem, a classical theoretical puzzle, has been a topic of interest in mathematical literature for many years, with numerous articles published in the *Mathematical Gazette* and other esteemed journals**.** W.J. Robinson (1960) was among the first to explore the origins of the Josephus Problem in this journal, providing foundational insights into its history [10]. More recently, in 2018, Peter Schumer, a Professor of Mathematics at Middlebury College, contributed an article titled *"The Josephus Problem: Once More Around,"* in which he delved into the historical context and various aspects of the problem [11]. Additionally, Osvaldo Marrero and Paul C. Pasles from Villanova University published a significant article in 2023 titled *"The Multivariate Probabilistic Josephus Problem."* This expository work introduced new results and potential observations, further expanding the understanding of the problem [12]. The *Mathematical Gazette* also featured an article by Derek Holton in November 2014 titled *"More Problem Solving– The Creative Side of Mathematics,"* where the Josephus Problem was discussed alongside the Tower of Hanoi, highlighting creative problem-solving approaches in mathematics [13]. The Tower of Hanoi, another well-known mathematical puzzle, has also been extensively studied. In April 1985, K. Kotovsky**,** J.R. Hayes, and H.A. Simon published a paper analyzing the varying difficulty levels of different isomorphic versions of the Tower of Hanoi problem [14]. This work contributed to understanding why some versions of the problem are more challenging than others. Later, in 2001, J.R. Anderson and **S.** Dougla**ss** explored a version of the Tower of Hanoi that allowed for a separate analysis of the effects of goal retention on storage time and retrieval time [15]. In 1997, Sandi Klavžar and Uroš Milutinović examined a particular variant of the Tower of Hanoi problem in the *Czechoslovak Mathematical Journal*. Their study focused on graphs that are isomorphic to the Tower of Hanoi graphs. They proved that there are at most two shortest paths between any two vertices and provided a formula for computing the distance between two vertices in time. Their results also showed that the graphs are Hamiltonian [16]. Further, in January 2013, an online article titled *"The Generalized Towers of Hanoi Problem"* presented a simple recursive algorithm for solving the generalized version of this problem [17]. Additionally, in 1996, Colin Gerety and Paul Cull published a paper examining the relationship between the Tower of Hanoi problem and time, adding another dimension to the study of this classic puzzle [18]. The Traveling the World Problem has also garnered significant attention in mathematical research. In October 2000, Luke Desforges published an article in *Annals of Tourism Research* titled *"Traveling the World: Identity and Travel Biography,"* where he introduced the concepts of identity, subjectivity, and self in the context of tourism studies, relating them to the TSP (Traveling Salesman Problem) [19]. In January 2019, James Cooper published a research article explaining algorithms and providing an example application of the TSP algorithm to a given graph and its digraph variant [20]. In August 2012, Javad Salimi Sartakhti**,** Saeed Jalili, and Ali Gholami Rudi published a paper on a new light-based solution to the Hamiltonian Path Problem, a variation of the TSP. Their solution involved designing filters to remove invalid Hamiltonian paths [21]. Furthermore, the book *"Traveling Salesman Problem: Theory and Applications,"* edited by Donald Davendra, discusses various approaches to solving the TSP across different chapters, providing a comprehensive overview of the problem's theoretical and practical aspects [22]. Fair division problems, particularly those involving pie cutting, have also been a subject of mathematical inquiry. In 2018, an article titled *"Cutting a Pie Is Not a Piece of Cake"* was published in the *American Mathematical Monthly* by Julius B. Barbanel**,** Stevens J. Brams, and Walter Stromquist. The authors discussed the general problem of fair division, introducing mathematical formalism by representing a pie as a circle and exploring the possibilities of dividing it equally [23]. Another article, *"Pie Cutting,"* was written and published by Jonathan Ratner, contributing further to this field [24]. In 2017, Julius B. Barbanel and Stevens J. Brams published another article focusing on a two-person pie-cutting procedure. They explored whether a two-person moving-knife procedure could yield an envy-free, undominated, and equitable allocation of a pie. The authors presented two procedures: one that yields an envy-free, almost undominated, and nearly equitable allocation and another that removes the "almost" conditions by broadening the definition of a "procedure." However, they noted that this latter approach raises philosophical rather than mathematical issues.

III. MATHEMATICAL ANALYSIS AND RESOLUTION

A. *Josephus Problem*

➢ *Problem Discussion*

In the year 67 CE, the city of Jotapata in Galilee fell under attack by the Roman Army under the command of Vespasian. The Roman army sieged Jotapa brutally for 47 days. Josephus, the Jewish general of Jotapa. While the Romans were bathing the inhabitants of Jotapata in blood, Josephus somehow hid himself in a cave with forty others 'persons of distinction.' One of the others who were hiding in that cavern with Josephus was captured while he was wondering about it and revealed the location of Josephus to the Roman army. However, the Romans wanted Josephus alive. Josephus was ready to surrender himself to the Roman army. But others like him preferred mass suicide to slavery. They wanted to murder Josephus themselves. To satisfy this angry mob, Josephus orchestrated a scheme in which all the caverneans will kill themselves in order, with the second person killing the first person, similarly the third person killing the second person, and so on. Josephus succeeded in his plan to be one of the last two alive individuals. He then convinced the second person to surrender with him. The question of how Josephus did it must arise. This historical enigma brought about a famous mathematical puzzle for us to solve [26].

➢ *Mathematical Elucidation of the Problem*

We consider, a group of m individuals numbered from $1 - 10$ standing around a circle. The technique begins from murdering the second of every two remaining individuals until one of the two individuals remains. We consider, the value of the number m has been determined for example, when m= 20, Then the first person to get killed will be on position 2. The elimination will continue in following order If there are 20 persons standing in a circle, then we can take $m = 20$. Then the first person to get killed will be on position 2. The elimination will continue in following order,

Table 1: Sequence of Eliminations for m=20 on the First Round in the Josephus Problem

Person that kills	Person that gets killed
1	2
3	4
5	6
7	8
9	10
11	12
13	14
15	16
17	18
19	20

The remaining persons will be 1,3,5,7,9,11,13,15,17,19. So another elimination round will start.

Table 2: Sequence of Eliminations for m=20 on the second round in the Josephus Problem

Person that kills	Person that gets killed
1	3
5	7
9	11
13	15
17	19

The remaining persons will be 1,5,9,13,17. Again 1 will eliminate 3 and the process will continue as

Table 3: Sequence of Eliminations for m=20 on the third round in the Josephus Problem

Person that kills	Person that gets killed
1	3
5	7
9	11
13	15
17	19

Table 4: Sequence of Eliminations for m=20 on the Fourth Round in the Josephus Problem

Person that kills	Person that gets killed
1	5
9	13
17	1
9	17

Lastly 9 remains. So, for $m = 20$ the survivor will be the person standing on position 9.

➢ *Analysis*

Let us consider that, m =total number of individuals standing in the primary circle p =each step counts; i.e., $p - 1$ =number of individuals being skipped and p^{th} term is executed. Individuals in the circle are numbered from 1 to m. The problem has an explicit solution when every second individual will be murdered, i. e. $p = 2$(A conceptual solution is given below when $p \neq 2$.) Expressing the solution recursively. We consider that, $f(m)$ designate the position of the survivor when there are primarily m individuals and ($p = 2$). On the first pass around the circle, all the even-numbered individuals are eliminated [28]. During the second pass around the circle, the new second person is eliminated, followed by the new fourth person, as if the first round never occurred. When the beginning number of individuals is even, the individual in position q during the second time around the circle was originally in position $2q - 1$ (for every choice of q). Let m= $2j$. The person at $f(j)$ who will now survive was originally in position $2f(j) - 1$. Thus, we get the following recurrence relation:

$$f(2j) = 2f(j) - 1 \tag{3.1}$$

If the starting number of people was odd, Person 1 would be regarded as having died at the conclusion of the first pass around the circle. Once again, during the second pass around the circle, the new second person is eliminated, followed by the new fourth person, and so on. In this case, the person in position q was originally in position $2q + 1$. This gives us the recurrence.

$$f(2j + 1) = 2f(j) + 1 \tag{3.2}$$

When the values are being in tabular form of m and $f(m)$ it is shown that there is a pattern:

Table 5: Position of the Survivor $f(m)$ for Various Values of m in the Josephus Problem with $p = 2p$

m		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(\boldsymbol{m})$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

This suggests that $f(m)$ is an increasing odd sequence that restarts with $f(m) = 1$ whenever the index m is a power of 2. Therefore, if choosing b and l such that $m = 2^b + l$ and $0 < l < 2^b$, then $f(m) = 2l + 1$. Clearly, values in the table satisfy this equation. Or it could be assumed that after l people are dead there are only 2^b people and we go to the $2l + 1$. He must be the survivor. Below a proof by induction is given [29].

- *Theorem 3.1* If $m = 2^b + l$ and $0 < l < 2b$, then $f(m) = 2l + 1$

- *Proof:* Using strong induction on m. The base case $m = 1$ is true. Considering the cases separately when m is even and m is odd. If m is even, then let us choose l^1 and b_1, such that $m/2 = 2^{b_1} + l^1$ and $0 < 1 < 2^{b_1}$. Noting that $l^1 = l/2$. Let us have $f(m) = 2f(m/2) - 1 =$

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$2((2l^1)+1)-1 = 2l+1$, where the second equality follows from the induction hypothesis. If m is odd, then let us choose l_1, and b_1 such that: $\frac{m-1}{2} = 2^{b_1} + l_1$, $and\ 0 < l < 2^{b_1}$ Noting that $l_1 = (l-1)/2$. Now for this case again let us have $f(m) = 2f((m—1)/2) + 1 = 2((2l_1) + 1) + 1 = 2l + 1$, where the second equality follows from the induction hypothesis. This completes the proof [30].

B. *Tower of Hanoi*

➢ *Problem Discussion*

As claimed in the legends of the tower of Hanoi, its origin is the tower of Brahma, which is a temple in the Indian city of Banaras. The temple priests had to move a tower consisting of 64 damaged gold disks from one part of the temple to another and one disk at a time. In that Hindu temple, the puzzle was apparently used to raise the mental discipline of young priests [31].

➢ *Mathematical Elucidation of the Problem*

The Tower of Hanoi problem can be solved in some number of moves using the recursion relation. There is a sequence of moves that transfers the entire stack of disks from one pole to another, following the three essential rules of the puzzle [32]. They are given below:

- The disks must be moved one at a time.
- No disks can ever be in the air or on the ground while another one is being moved.
- There can never be a larger disk on top of a smaller disk.

➢ *Analysis*

The most straightforward way to solve this problem is to use a recursive algorithm. Recursive relation. To move n disks first, there has to move first $(n-1)$ disks out of the way. Later, we move the bottom disk. And again, it needed to move $(n-1)$ disks, this time on to the final pole. It is better to make a claim about the number of moves that this algorithm requires given a problem of n disks, and the motive is to prove that the claim is correct. It can be done by making a chart for each disk to move [33]. That is given below:

Table 6: Number of Moves Required for the Tower of Hanoi Problem with n Disks

Number **of disks (n)**	**Number of moves**
1	1
2	$1+1+1=3$
3	$3+1+3=7$
4	$7+1+7=15$
… … … …	… … … … …
$\boldsymbol{n-1}$	$M(n-1)+1+M(n-1)$
n	2^n-1

Looking at the table it can be shown that to move one disk it requires one move. To move two disks first it was needed to move smaller disk and then it was needed to move bigger one and again needed to move the smaller one. So, it requires total three moves. To move three disks first it is needed to move two disks which requires three moves and then is needed to move the bigger disk which requires one move and then again moving two disks requires three moves so total seven moves are required. Now suppose it is needed to move n disks total number of moves required to move $(n-1)$ disks are M. Then we can describe the number of moves for n disks in terms of number M. This can be denoted by $M(n)$.

It must be proved that $M(n) = 2^n - 1$

Proving our claim:

$$M(n) = M(n-1) + 1 + M(n-1) = 2M(n-1) + 1$$

$$M(1) = 1$$

$$M_n = 2M_{n-1} + 0$$

$$M_n - 2M_{n-1} = 0$$

$$M_n = r^n C_1$$

$$r - 2 = 0$$

$$h^r = 2$$

$$M_n = 2^n + C$$

Since C is an arbitrary constant so let the value of $C = -1$

Then for any n number of people the solution must have the form,

$$M_n = 2^n - 1$$

For basic case: $n = 1: M(n) = 2^1 - 1 = 1$

It must be proved by us that if the number of moves required to move n disks is $2^n - 1$ then the number of moves required to move $(n+1)$ disks is $2^{n+1} - 1$.

From the above solution, $M(n) = 2^n - 1$

$$M(n+1) = 2M(n) + 1 = 2(2^n - 1) + 1 = 2.2^n - 2 + 1 = 2^{n+1} - 1$$

Since the assumption for $n+1$ disk is satisfied by the above recursive relation so it can be said that the assumption is generally true for n number of disks. Hence the assumption is correct.

C. *Traveling the world Problem*

➢ *Problem Discussion*

In 1859, Sir William Rowan Hamilton, an Irish mathematician, demonstrated a mathematical game known as the icosian game [34]. The game took place on the surface of a wooden dodecahedron, a shape made up of 20 vertices (corners). Each corner of the dodecahedron was assigned the name of a city. The goal of the game was to trace a route that visits every vertex exactly once and returns to the starting point. The solution to this challenge was demonstrated using a Hamiltonian path [35]. The idea of traveling the world hinges on two key factors. Firstly, the traveler must visit each vertex exactly once and must take the shortest possible route without revisiting any vertex. This problem should be framed as finding a Hamiltonian path. If a Hamiltonian path can be demonstrated for these vertices, the shortest route for the traveller can be determined. This chapter explores Hamiltonian paths and addresses whether a dodecahedron possesses such a path.

➢ *Hamiltonian Path*

Here, it can be shown in figure 1 that a path is given by ABCFED, which has 6 vertices and 9 edges. It is needed to find out whether this path is a hamilton path or not. Now, if it starts from point A and goes to point E, then at point D and passes through as shown in figure 2, from this figure, it is seen that it could draw a path through each vertex, and it touches each vertex only once. Hence, the given figure has a Hamiltonian path [36].

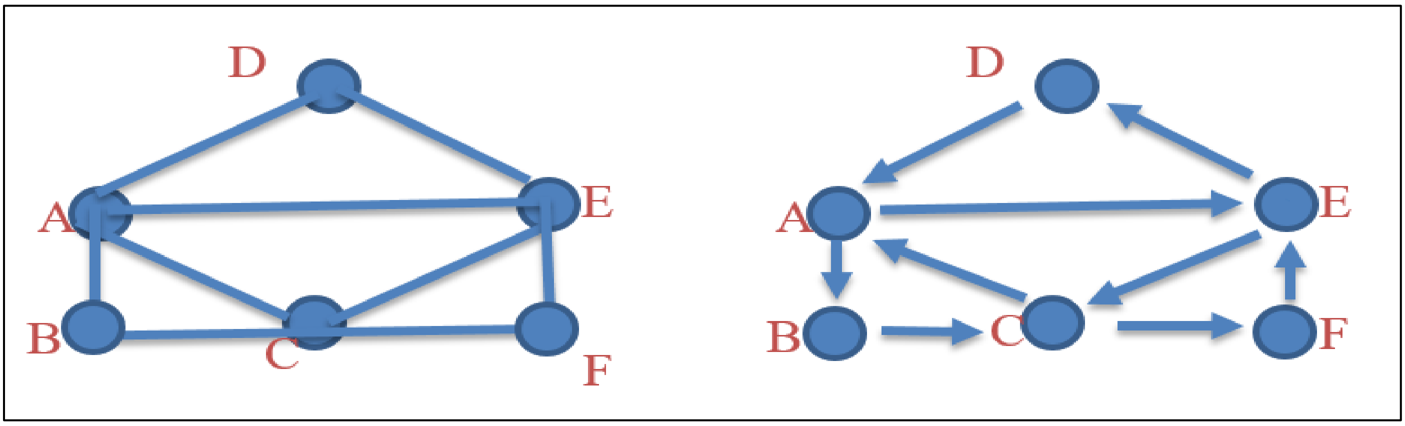

Fig 1: Hamiltonian Path

➢ *Mathematical Elucidation of the Problem*

At first glance, it might appear that there should be a straightforward method to determine whether a graph contains a Hamiltonian path. There are no known straightforward criteria that are both necessary and sufficient to determine the existence of Hamiltonian circuits. Nevertheless, several theorems provide sufficient conditions for the existence of Hamiltonian circuits. Additionally, certain properties can be employed to demonstrate that a graph lacks a Hamiltonian circuit. For example, a graph with a vertex of degree one cannot contain a Hamiltonian circuit, as such a circuit requires each vertex to be connected by exactly two edges within the circuit. Additionally, if a vertex in the graph has a degree of two, both edges incident to this vertex must be included in any Hamiltonian path. Additionally, it is important to note that once a Hamiltonian circuit has passed through a vertex, any remaining edges incident to that vertex, excluding the two used in the circuit, can be disregarded. Moreover, a Hamiltonian circuit cannot include any smaller circuits within it. These two conditions are among the most crucial for determining the existence of a Hamiltonian circuit [37]. They were established by Gabriel A. Dirac in 1952 and by Ore in 1960.

- *Theorem (Ore's Theorem)* If G is a simple graph with n vertices $(n > 2)$ such that $deg(u) + deg(v) \geq \frac{n}{2}$ for each nonadjacent pair of vertices u and v ,then G has a Hamilton path.
- *Proof:* This theorem will be proved by contradiction. Let G satisfies the above condition but is not have a Hamiltonian path. First, let us form G* from G by adding edges until we get a non-Hamiltonian maximal graph. This means that adding one edge should make it Hamiltonian. Then a Hamiltonian path exists. Let us consider that the Hamiltonian path exists like this u is the starting vertex, then let us have intermediate vertices such that $x_1, x_2, \dots \dots . x_{n-3}, x_{n-2}, \dots$ and v be the final vertex. Now by adding the vertex, say u v, make it Hamiltonian so this means this path exists due to v through x_1. If x_i is adjacent to u $then$ x_i cannot be adjacent to u. Now $x_1, x_2, \dots \dots . x_{n-3}, x_{n-2}, x_{i-1} \dots x_i$ this makes a Hamiltonian path, now by getting

$$Deg(v) \leq (n-1) - \deg(u)$$

$$Deg(v) + \deg(u) \leq (n-1)$$

But this is a contradiction hence Ore's theorem is proved [38].

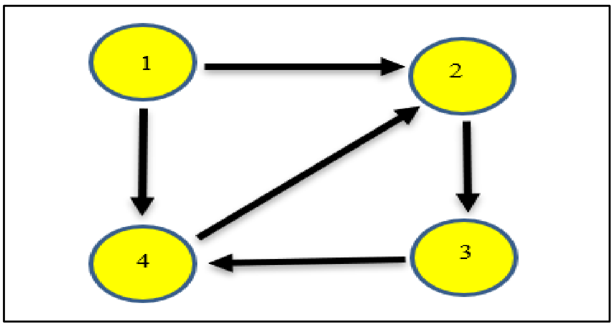

Fig 2: Hamiltonian Path Satisfying the Ore's Theorem

Here number of vertices 4 and the only pair of nonadjacent vertices are 1 and 3. Then the degree of nonadjacent vertices are given below,

$$Deg(A) + Deg(E) = 4 \geq 4$$

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Since every pair of nonadjacent vertices satisfies the Ore's theorem. Hence the path given in figure 2 has a Hamiltonian path. Any path which satisfies Ore's theorem has a Hamiltonian path but the converse is not always true. It can be shown by given examples. In figure 3,

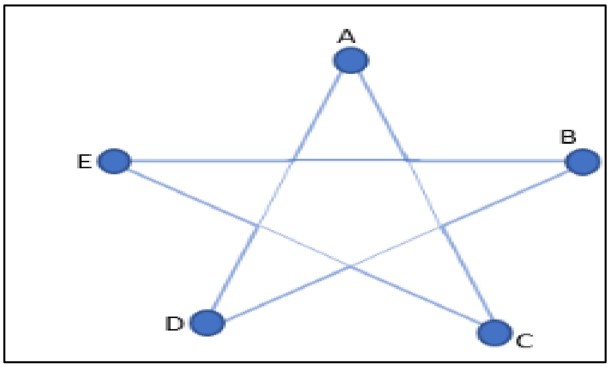

Fig 3: A Hamiltonian Path that does not Satisfy Ore's Theorem

Here, the number of vertices is 5, and the nonadjacent vertices are AE, AB, DE, DC, CB, and CD. Then the degree of nonadjacent vertices is given below,

$$Deg(A) + Deg(E) = 4 < 5$$

$$Deg(A) + Deg(B) = 4 < 5$$

$$Deg(D) + Deg(E) = 4 < 5$$

$$Deg(D) + Deg(C) = 4 < 5$$

$$Deg(C) + Deg(B) = 4 < 5$$

$$Deg(C) + Deg(D) = 4 < 5$$

Ore's theorem is not satisfied for this path, but there are several Hamiltonian paths in that figure. It can be shown as ADBEC or ECADB etc. Despite not satisfying Ore's theorem, Hamiltonian paths are found in figure 3. Hence, Ore's theorem provides sufficient conditions for an existing Hamiltonian path, but it does not provide a necessary condition for an existing Hamiltonian path.

- *Finding a Hamiltonian Path in Dodecahedron*

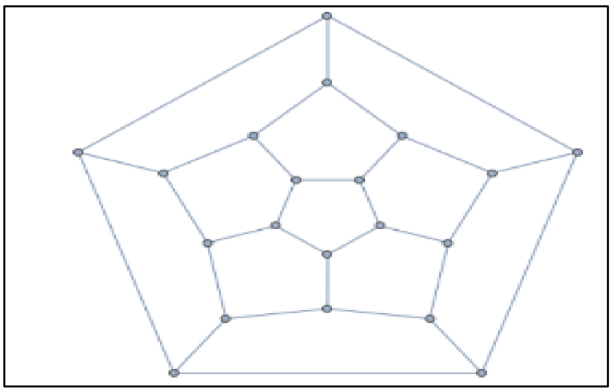
Fig 4: Dodecahedron

To find a Hamiltonian path in a dodecahedron, shown in figure 4, a 3-dimensional form of a dodecahedron is given below,

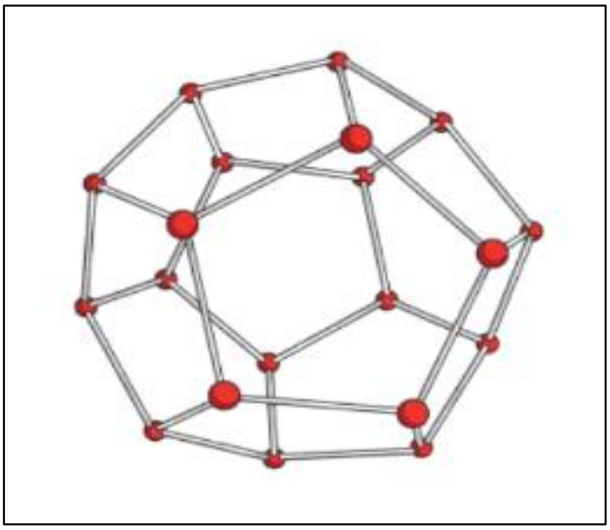
Fig 5: *3D* form of a dodecahedron

A regular dodecahedron or pentagonal dodecahedron is a regular dodecahedron composed of twelve regular pentagonal faces, three meeting at each vertex. It is one of the five Platonic solids. It has 12 faces, 20 vertices, 30 edges, and 160 diagonals. Here in Figure 6, let us consider any pair of nonadjacent vertices 9 and 11, then

$$Deg(9) + Deg(11) = 2 + 2 = 4 < 20$$

Hence Ore's theorem is not satisfied. But it is shown in figure 5.5 that there exists a Hamiltonian path in dodecahedron. Since it is proved that Ore's theorem does not provide necessary condition for finding Hamiltonian path hence without satisfying the theorem Hamiltonian path could be found. In figure 6 one Hamiltonian path has been found. From 1 to vertex 20 it is shown in the figure.

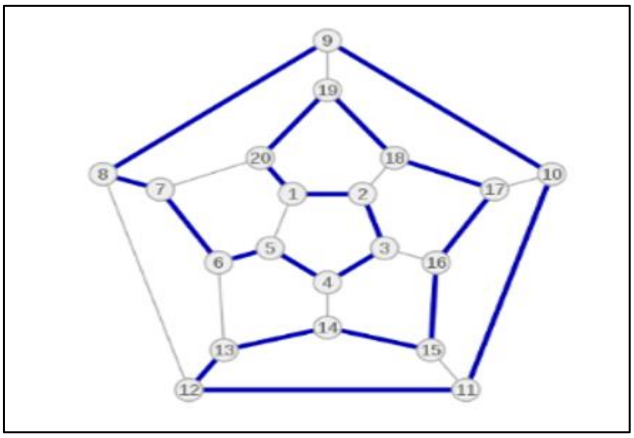

Fig 6: Hamiltonian Path

Using this concept, anyone can travel through the vertices of the world, touching it only once, and could travel the world by covering the shortest distance in the shortest time [39].

D. Cutting the Pie Problem

➢ *Problem Discussion*

In 1993, David Gale inquired whether it is always possible to divide a pie among n claimants in a manner that is both envy-free and undominated. This question addresses whether such a division can always be achieved. The pie is divided by n radii, with each claimant's preferences represented by individual measures. The measures assign positive values to pieces with positive areas. For n=3n, the answer to Gale's question is negative, demonstrated by presenting three measures where pie division cannot be both envy-free and undominated. The measures are absolutely continuous, with respect to each other and with respect to the area [40].

➢ *Mathematical Elucidation of the Problem*

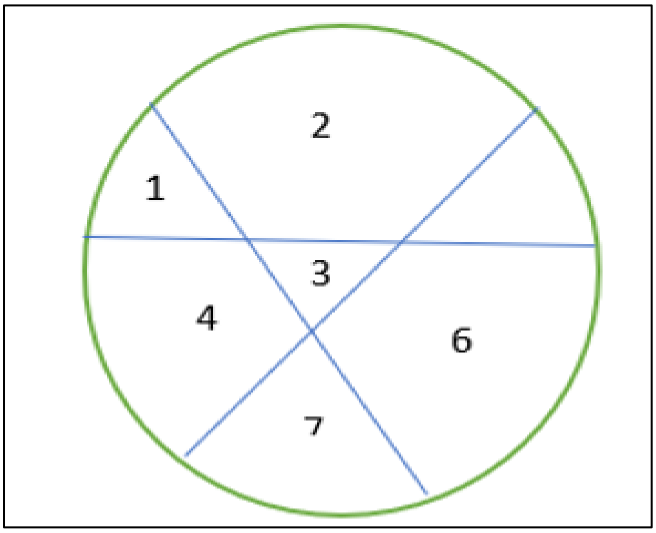

Fig 7: Mathematical Elucidation of the Problem

To address this problem, the approach began with small circles. The experiments were carried out with one, two, and three cuts. After several unsuccessful attempts with four cuts, it was realized that enlarging the circles was crucial for determining the remaining solutions. At this stage, a pattern was observed in the maximum number of pieces: the number of pieces increased with each additional cut, building upon the previous total [41]. The table below reflects this pattern to the current findings:

Table 7: Number of Regions Created by Cuts in a Circle

*Number **of cuts***	***Number of Pieces***
1	2
2	$4(2+2)$
3	$7(4+3)$
4	$11(7+4)$

This was how the answers for future circles had been double-checked, like the one with 5 cuts. Originally, the number of pieces was 15, but the pattern suggested 16 (11 +5). It had been tried a little more and then 16 pieces were obtained. The same process happened with the rest of the circles as well. During this process, something had been learned about how the intersection of lines worked and that every line had to cross every other line in a unique place. So, no crossing through existing intersections. That was how it ended up maximizing the total piece. Because of this, it had also begun to count the intersections. In the end, circles were drawn for one through six cuts and the pattern was used for the rest. Here is the table that ended up, including intersections, and the pattern which had been found for intersections.

Table 8: Number of Regions and Intersections Created by Cuts in a Circle

Number **of cuts**	**Number of pieces**	**Number of intersections**
1	$2(1+1)$	$0(0+0)$
2	$4(2+2)$	$1(0+1)$
3	$7(4+3)$	$3(1+2)$
4	$11(7+4)$	$6(3+3)$
5	$16(11+5)$	$10(6+4)$
6	$22(16+6)$	$15(10+5)$
7	$29(22+7)$	$21(15+6)$
8	$37(29+8)$	$28(21+7)$
9	$46(37+9)$	$36(28+8)$
10	$56(46+10)$	$45(36+9)$

Having this table, one also searches for an equation. The focus was on the total of the pieces, but it also looked at the intersection totals. The equations have been found through trial and error and by experimenting with numbers. The variables used here are c, which is the number of cuts, p, which was the number of pieces, and i, which was the number of intersections. Here is the equation for the total number of pieces:

$$p = \left(\sum_{s=0}^{c} s\right) + 1 \qquad (3.3)$$

And it can also be shown from the above table that an equation can be generated for the number of intersections.

$$i = (\sum_{s=0}^{c} s) - c \quad (3.4)$$

Hence, this is the general formula for a circular pie to be cut into the maximum number of pieces with a fair division. But this solution is not general for any plane.

➢ *Analysis*

It is shown in this solution that how to get the maximum cut with fair division in any plane. This problem was first solved by the Swiss mathematician Jacob Steiner in 1826. To solve this problem, it has to start by looking at small cases. Remembering to begin with the smallest of all. The plane with no lines has one region with one line it has two regions. And with two lines it has four regions. Now it can be easily imagined that the solution must be $L(n) = 2^n$ where n,is the number of cuts. But this is not the solution for cutting a plane into maximum pieces. Again, a table is generated by the trial-and-error method and the table is given below:

Table 9: Maximum Number of Regions and Intersections Created by Cuts in a Plane

Number of cuts	Number of pieces	Number of intersections
1	2	0
2	4	1
3	7	3
4	11	6
5	16	10
6	22	15
7	29	21
8	37	28
9	46	36
10	56	45

From the above table a pattern is seen. Where $L(n)$ is the maximum number of pieces. And $L(n-1)$ is the number of maximum pieces of previous number of cuts and n is the total number of cuts. The recurrence is therefore,

$$n = 0, L(n) = 1;$$

$$n > 0, L(n) = L(n-1) + n$$

A recurrence can often understood by unfolding it all the way to the end as follows,

$$L(n) = L(n-1) + n$$

$$= L(n-2) + (n-1) + n$$

$$= L(n-3) + (n-2) + (n-1) + n.........$$

$$= L(0) + 1 + 2 + 3 .... + (n-2) + (n-1) + n$$

$$= 1 + S(n)$$

Where $S(n) = 1 + 2 + 3 + 4 + 5 + \cdots .. + (n-1) + n$

In other words, $L(n)$ is one more than the sum $S(n)$ of the first n positive integers. To evaluate $S(n)$, a trick can be used that Gauss reportedly used in 1786,

$$S(n) = 1 + 2 + 3 + 4 \ldots \ldots \ldots .. + (n-1) + n \quad (3.5)$$

$$S(n) = n + (n-1) + (n-2) + (n-3) + \cdots 2 + 1 \quad (3.6)$$

Adding these two equations actually adding $S(n)$ to its reversal,

$$2S(n) = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

$$S(n) = \frac{n(n+1)}{2},$$

$$for\ n \geq 0$$

Now the actual solution can be generated since evaluating $S(n)$ has been done.

$$L(n) = \frac{n(n+1)}{2} + 1, for\ n \geq 0$$

Now it is good to construct a rigorous proof by induction. The key induction step is

$$L(n) = L(n-1) + n$$

$$= \frac{(n-1)(n+1)}{2} + n$$

$$= \frac{n(n+1)}{2} + 1$$

Hence the formula generated for maximum pieces from cutting a plane is generalized and the general solution is given by, [41].

IV. CONCLUSION

Puzzles seem just for recreation, but solving puzzles has a deeper meaning than we can imagine. Mathematical concepts are everywhere, and puzzles are one of the most significant proofs. Josephus didn't want to die, so he subconsciously managed to survive his death by using a mathematical concept. Josephus's problem is a well-known puzzle that is observed regularly in popular mathematics. In this problem, 'One soldier survived by using the induction method.' In the Josephus problem, recursion and how positional patterns emerge in elimination sequences are demonstrated, getting insights into algorithmic problem-solving. In the Tower of Hanoi problem, to move 64 disks from the first peg to the third, the monks would need over 590 billion years, assuming they can move one disk per second. The function $2^n - 1$ was found by recognizing the geometric progressions in the recursive formula and using it in an explicit pattern. This function can be used to find the most optimal number of moves it would take to move any number of disks to the third peg. Another problem that has been studied, the problem of traveling the world using a dodecahedron, is the concept of finding a Hamiltonian path. In the last problem, the word 'pie' represented the circular region and how to find out a generalized formula for any number of cuts to get the divided maximum number of pieces. Further, the formula has been established for any plane by using the trial-and-error method—a generalized formula for any number of cuts to get the divided maximum number of pieces. These problems demonstrate how mathematical analysis can uncover patterns, establish algorithms, and contribute to optimization, decision-making, and computational efficiency. Future work on these problems could include investigating more complex variations, such as introducing probabilistic elements in the Josephus problem, optimizing moves for multi-peg versions of the Tower of Hanoi, exploring non-Euclidean Hamiltonian paths in higher-dimensional spaces, or examining envy-free divisions for irregularly shaped pies in the Cutting Pie problem. These extensions could lead to discoveries and practical applications in computer science, economics, and logistics, further highlighting the broad impact and relevance of this research.

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