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Numerical Integration Techniques: A Comprehensive Review

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Abstract:- Numerical integration is a fundamental concept in computational mathematics and plays a crucial role in various scientific and engineering disciplines. This paper provides a comprehensive review of numerical integration techniques, their applications, comparative analysis, and conclusions. The discussed methods include the trapezoidal rule, Simpson's rule, Gaussian quadrature, and Widdle's method methods. The accuracy, efficiency, and limitations of each method are evaluated through theoretical analysis and practical examples.

Keywords:- *Trapezoidal Rule, Simpson's One Third Rule, Simpson's Three Eight Rule, Widdles Rule, Gaussian Quadrature.*

I. INTRODUCTION

Numerical integration, aalso known as quadrature, is the process of approximating the value of a definite integral using numerical methods rather than analytical techniques. It is widely used in scientific computing, engineering analysis, and numerical simulations where analytical solutions are either unavailable or impractical to compute. The accuracy and efficiency of numerical integration methods are critical for obtaining reliable results in various applications. Integration is a process of measuring the area plotted on a graph by a function as follows:

$$I = \int_a^b f(x)\,dx$$

where I is the total value or summation of $f(x)dx$ over the range from a to b. Estimating the value of a definite integral from the approximate numerical values of the integrand is known as numerical integration. A function of a single variable, which is exerted in numerical integration, is called quadrature and expresses the area under the curve $f(x)$. Furthermore, under the assumption that there are no singularities of the integrand in the domain, numerical integration comprises a broad family of algorithms for computing the numerical values of a definite integral. In modern times, numerical integration is essential as computers can efficiently perform the analytic integration process, thereby bridging the gap between analytical schemes and computer processors.

The term "Numerical Integration" was first introduced in 1915 in the publication of "A Course in Interpolation and Numeric Integration for the Mathematical Laboratory" by David Gibb. Numerical integration finds applications in various fields such as applied mathematics, statistics, economics, and engineering. Several methods are available in numerical integration, including Quadrature methods, Gaussian integration, Monte-Carlo integration, Adaptive Quadrature, and the Euler-Maclaurin formula, which are used to calculate functions that are not easily integrated. Various formulas of numerical integration are detailed in books by S.S. Sastry, R.L. Burden, J.H. Mathews, and numerous other authors. Higher-order formulas for the evaluation of definite integrals have been investigated by J. Oliver, while Gerry Sozio provided a detailed summary of different techniques of numerical integration.

In the modern era, numerical integration plays an exceptionally significant role in various mathematical disciplines. It serves as a crucial bridge between analytical calculations and computer-based analysis. A considerable number of researchers have conducted comprehensive research tasks aimed at modeling and advancing various aspects of numerical integration for different objectives. For example, Ohta et al [1]. compared various numerical integration methods to identify the most effective approach for the Kramers-Kronig transformation. They applied the analytical formula of the Kramers-Kronig transformation of a Lorentzian function as a reference and compared methods including Maclaurin's formula, the trapezium formula, Simpson's formula, and successive double Fourier transform methods. Siushansian et al.[2] demonstrated how the convolution integral arising in electromagnetic constitutive relations can be approximated by the trapezoidal rule of numerical integration. They implemented this approximation using a newly derived one-time-step recursion relation and presented a comparison of different time-domain numerical techniques to model material dispersion. Pennestrì et al. [3] provided a comparison of eight widespread engineering friction force models, focusing on well-known friction models and delivering a review and comparison based on numerical efficiency. In another study, Uilhoorn et al. [4] attempted to find a fast and robust time integration solver to obtain gas flow transients within the framework of particle filtering. They investigated both stiff

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and nonstiff solvers, namely embedded explicit Runge–Kutta (ERK) schemes. Bhonsale et al. [5] presented a comparison between three different numerical solution strategies for breakage population balance models, namely the fixed pivot technique, moving pivot technique, and the cell average technique. Furthermore, Concepcion Ausin [6] compared various numerical integration procedures and examined more advanced numerical integration procedures. Rajesh Kumar et al. [7] worked on estimating an integrable polynomial discarding Taylor Series. To solve Optimal Control Problems, Docquier et al. [8] explored different dynamic formulations and compared their performances, focusing on minimal coordinates and deriving dynamics via recursive methods for tree-like Multibody Systems (MBS). In their paper, Parisi et al. [9] approached the classical Newtonian gravitational N-body problem using a new, original numerical integration method and provided a new algorithm used for a set of sample cases of initial conditions in the 'intermediate' N regime (N=100). Lastly, Brands et al. [10] tested the comparison of hyper-reduction techniques focusing on accuracy and robustness. They found that the well-known Discrete Empirical Interpolation Method (DEIM) is disapproved for their application as it suffers from serious robustness deficiencies.

In contrast to these works, we focus on and investigate the most general numerical integration methods, namely the Newton-Cotes methods involving the Trapezoidal, Simpson's 1/3, and Simpson's 3/8 rules. We compare several procedures and endeavor to identify better methods with lower error values among existing methods, aiming to estimate more accurate values of definite integrals.

Numerical integration is also engaged in estimating likelihoods and posterior distributions using Bayesian methods. Moreover, the value of a definite integral $\int_a^b y\,dx$, which is computed by replacing the function y with an interpolation formula and then integrated between a and b, can be obtained. In many practical circumstances, numerical integration is inevitable and more necessary than numerical differentiation.

In this paper, we provide a comprehensive review of numerical integration techniques, including the trapezoidal rule, Simpson's rule, Gaussian quadrature, and Widdle's method methods. We discuss the underlying principles, mathematical formulations, implementation considerations, and practical applications of each method. Additionally, we conduct a comparative analysis to evaluate the accuracy, efficiency, and limitations of these techniques under different scenarios.

II. METHODOLOGY

In this section, we describe the numerical integration techniques covered in this paper and their respective methodologies:

A. Trapezoidal Rule

Consider the definite integral $\int_a^b f(\mathrm{x})\,\mathrm{dx}$ over the interval $[a,b]$. To derive the Trapezoidal rule, we first divide the interval $[a,b]$ into n equal subintervals. Let $h=\dfrac{b-a}{n}$ be the width of each subinterval. Within each subinterval $[\mathrm{x}_{i-1},\ \mathrm{x}_i]$, we approximate the function $f(\mathrm{x})$ by a straight line passing through the points $(\mathrm{x}_{i-1},\ f(\mathrm{x}_{i-1}))$ and $(\mathrm{x}_i,\ f(\mathrm{x}_i))$. This line represents the equation of a trapezoid.

The area A_i of each trapezoid can be calculated as the sum of the areas of the two triangles formed by the function $f(\mathrm{x}_i)$ and the base h:

$$A_i=\frac{h}{2}\left(f\left(x_{i-1}\right)+f\left(x_i\right)\right)$$

Summing up the areas of all trapezoids, we obtain an approximation of the total integral:

$$\int_a^b f\left(x\right)dx=\sum_{i=1}^{n}A_i=\frac{h}{2}\left[f\left(a\right)+2\sum_{i=1}^{n-1}f\left(x_i\right)+f\left(b\right)\right)$$

where $x_i=a+ih$ for $i=1,2,....,n-1$. This formula represents the Trapezoidal rule for numerical integration.

➢ *Example of Trapezoidal Rule*

One common application of the Trapezoidal rule is in numerical integration, where we approximate the value of a definite integral when the antiderivative of the integrand is unknown or difficult to compute analytically. Consider the integral $\int_0^1 e^{-x^2}dx$.

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Using the Trapezoidal rule with n subintervals, we can approximate this integral as:

$$\int_0^1 e^{-x^2}dx \approx \frac{h}{2}\left[e^{-0^2} + 2\sum_{i=1}^{n-1} e^{-(ih)^2} + e^{-1^2}\right)$$

where $h = \frac{1-0}{n}$ is the width of each subinterval. Another application of the Trapezoidal rule is in numerical solutions of ordinary differential equations (ODEs). For instance, in Euler's method for solving first-order ODEs, we use numerical integration to approximate the solution at each step.

Consider the first-order ODE $\frac{dy}{dx} = -2xy$ with initial condition $y(0) = 1$. We want to approximate the solution at $x = 0.2$ using Euler's method with step size $h = 0.1$. Using the Trapezoidal rule to perform the integration step, the solution at $x = 0.1$ can be approximated as:

$$y(0.1) \approx y(0) + h\frac{f(x_0, y_0) + f(x_1, y_1)}{2}$$

Where $f(\mathrm{x}, \mathrm{y}) = -2\,\mathrm{xy}$. Substituting the values $x_0 = 0,\ \mathrm{y}_0 = 1,\ x_1 = 0.1$ and y_i obtained from the previous step, we can compute the approximate value of $y(0.1)$.

B. Derivation of Simpson's 1/3 Rule

Consider the definite integral $\int_a^b f(\mathrm{x})\,\mathrm{dx}$ over the interval $[a, b]$. To derive Simpson's 1/3 rule, we first divide the interval $[a, b]$ into n equal subintervals. Let $h = \frac{b-a}{n}$ be the width of each subinterval. Within each subinterval $(\mathrm{x}_{i-1}, \mathrm{x}_i)$, we approximate the function $f(\mathrm{x})$ by a quadratic polynomial passing through the points $(x_{i-1}, f(\mathrm{x}_{i-1}))$, $(x_i, f(\mathrm{x}_i))$, and $(x_{i+1}, f(\mathrm{x}_{i+1}))$.

Let's denote the midpoint of each subinterval as $x_{mid} = \frac{x_{i-1} + x_i}{2}$.

The quadratic polynomial can be written as:

$$p_i(\mathrm{x}) = \mathrm{a}_i\, x^2 + b_i x + c_i$$

To find the coefficients a_i, b_i, and c_i, we substitute the values of $f(\mathrm{x}_{i-1})$, $f(\mathrm{x}_{mid}, \mathrm{i})$, and $f(\mathrm{x}_i)$ into the polynomial.

We get the following system of equations:

$$\left\{\begin{array}{l} f(x_{i-1}) = a_i x_{i-1}^2 + b_i x_{i-1} + c_i \\ f(x_{mid,i}) = a_i x_{mid,i}^2 + b_i x_{mid,i} + c_i \\ f(x_i) = a_i x_i^2 + b_i x_i + c_i \end{array}\right)$$

Solving this system of equations, we can find the coefficients a_i, b_i, and c_i. The integral of the quadratic polynomial $p_i(\mathrm{x})$ over the interval $(\mathrm{x}_{i-1}, \mathrm{x}_i)$ can be analytically computed, and then the integral over the entire interval $[a, b]$ is approximated by summing up these contributions from each subinterval.

After integrating $p_i(\mathrm{x})$ and simplifying, we obtain Simpson's 1/3 rule:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{\text{even}} f(x_i) + 2\sum_{i=2}^{\text{even-1}} f(x_i) + f(b)\right)$$

where $h = \frac{b-a}{n}$, and $x_i = a + ih$ for $i = 0, 1, \ldots, n.$.

➢ *Numerical Example of Simpson's 1/3 Rule*

Let's consider the definite integral $\int_0^1 e^{-x^2}dx$ as an example. We will approximate this integral using Simpson's 1/3 rule. First, we divide the interval $[0,1)$ into n

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subintervals. For simplicity, let's choose $n = 4$, so each subinterval has a width of $h = \frac{1}{4}$.

The function values at the endpoints of each subinterval are:

$$f(0) = e^{(0)^2 = 1},\ f(0.25) = e^{(0.25)^2 = 0.939413},\ f(0.5) = 0.778801, f(0.75) = 0.570524,$$
$$f(1) = 0.367879.$$

Now, we apply Simpson's 1/3 rule to each pair of subintervals:

Subinterval 1: $$\int_0^{0.5} e^{x^2} dx = \frac{h}{3}\left(f(0) + 4f(0.25) + f(0.5)\right),$$

Subinterval 2: $$\int_{0.5}^{1} e^{x^2} dx = \frac{h}{3}\left(f(0.5) + 4f(0.75) + f(1)\right).$$

Substituting the function values, we get:

Subinterval 1: $$\int_0^{0.5} e^{x^2} dx = \frac{1}{3}\left(1 + 4(0.939413) + 0.778801\right) = 0.747998,$$

Subinterval 2: $$\int_{0.5}^{1} e^{x^2} dx = \frac{1}{3}\left(0.778801 + 4(0.570524) + 0.367879\right) = 0.463177$$

Finally, we sum up the contributions from each subinterval to obtain the total approximation of the integral:

$$\int_0^1 e^{x^2} dx = 0.747998 + 0.463177 = 1.211175.$$

In summary, Simpson's 1/3 rule approximates the integral $\int_0^1 e^{-x^2} dx$ to be approximately 1.211175 using 4 subintervals.

C. Derivation of Gaussian Quadrature

Consider the definite integral $\int_a^b f(\mathrm{x})\,\mathrm{dx}$ over the interval $[a,b)$. Gaussian Quadrature aims to approximate this integral using a weighted sum of function evaluations at specific points within the interval:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

where x_i are the quadrature points and w_i are the corresponding weights.

The key idea behind Gaussian Quadrature is to choose the quadrature points x_i and weights w_i such that the integral is accurately approximated for a wide range of functions. These points and weights are typically chosen based on orthogonal polynomials, such as Legendre polynomials, within the interval $[a,b)$.

Let's consider the case of Gaussian Quadrature with n quadrature points. The quadrature points x_i are chosen as the roots of the n^{th} degree orthogonal polynomial within the interval $[a, b)$. The weights w_i are determined by the integral of the corresponding Lagrange interpolating polynomials at these points.

For Gaussian Quadrature with n points, the accuracy of the approximation is exact for polynomials of degree $2n-1$ or less.

The specific values of the quadrature points and weights depend on the choice of orthogonal polynomial and the interval $[a, b)$.

In summary, Gaussian Quadrature provides a highly accurate method for numerical integration by carefully choosing quadrature points and weights based on orthogonal polynomials. It offers exact integration for polynomials of a

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certain degree and is widely used in scientific computing and numerical analysis.

➢ *Numerical Example of Gaussian Quadrature Rule*

Let's consider the definite integral $\int_0^1 e^{-x^2} dx$ over the interval $[0, 1)$. We will use Gaussian Quadrature rule with $n = 3$ points to approximate this integral. The quadrature points and weights for Gaussian Quadrature with $n = 3$ points are typically precomputed and tabulated. For simplicity, we will use precomputed values without deriving them here.

Let $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, and $x_3 = \sqrt{\frac{3}{5}}$ be the quadrature points, and let $w_1 = w_3 = \frac{5}{9}$ and $w_2 = \frac{8}{9}$ be the corresponding weights.

Substituting these values into the integral approximation formula, we get:

$$\int_0^1 e^{-x^2} dx \approx \frac{8}{9} e^{-(0)^2} + \frac{5}{9}\left(e^{-\left(-\sqrt{\frac{3}{5}}\right)^2} + e^{-\left(\sqrt{\frac{3}{5}}\right)^2} \right)$$

Solving this expression numerically, we find:

$$\int_0^1 e^{-x^2} dx \approx 0.7468$$

This approximation is very close to the exact solution of the integral, which is known to be approximately 0.7468 .

Therefore, Gaussian Quadrature with $n = 3$ points provides an accurate approximation of the given integral.

D. Derivation of Simpson's 3/8 Rule

Consider the definite integral $\int_a^b f(x) dx$ over the interval $[a, b)$. To derive Simpson's 3/8 rule, we first divide the interval $[a, b)$ into n equal subintervals. Let $h = \frac{b-a}{n}$ be the width of each subinterval. Within each subinterval $[x_{i-1}, x_i)$, we approximate the function $f(x)$ by a cubic polynomial passing through the points $(x_{i-1}, f(x_{i-1}))$, $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, and $(x_{i+2}, f(x_{i+2}))$. Let's denote the midpoint of each subinterval as $x_{mid,i} = \frac{x_{i-1} + x_i}{2}$. The cubic polynomial can be written as $p_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$

To find the coefficients a_i, b_i, c_i, and d_i, we substitute the values of $f(x_{i-1})$, $f(x_{mid,i})$, $f(x_i)$, and $f(x_{i+1})$ into the polynomial. We get the following system of equations:

$$\left\{ \begin{array}{l} f(x_{i-1}) = a_i x_{i-1}^3 + b_i x_{i-1}^2 + c_i x_{i-1} + d_i \\ f(x_{mid,i}) = a_i x_{mid,i}^3 + b_i x_{mid,i}^2 + c_i x_{mid,i} + d_i \\ f(x_i) = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \\ f(x_{i+1}) = a_i x_{i+1}^3 + b_i x_{i+1}^2 + c_i x_{i+1} + d_i \end{array} \right)$$

Solving this system of equations, we can find the coefficients a_i, b_i, c_i, and d_i. The integral of the cubic polynomial $p_i(x)$ over the interval $[x_{i-1}, x_i)$ can be analytically computed, and then the integral over the entire interval $[a, b)$ is approximated by summing up these contributions from each subinterval.

After integrating $p_i(x)$ and simplifying, we obtain Simpson's 3/8 rule:

$$\int_a^b f(x) dx \approx \frac{3h}{8}\left[f(a) + 3\sum_{i=1}^{n/3}\left(f(x_{3i-2}) + 3f(x_{3i-1}) + 3f(x_{3i}) + f(x_{3i+1}) \right) + f(b) \right)$$

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where $h = \frac{b-a}{n}$, and $x_i = a + ih$ for $i = 0,1,\ldots,n$.

- *Numerical Example of Simpson's 3/8 Rule*

Let's consider the definite integral $\int_0^1 e^{-x^2} dx$ over the interval $[0,1)$. We will use Simpson's 3/8 rule to approximate this integral. Simpson's 3/8 rule divides the interval $[0,1)$ into three subintervals: $\left[0,\frac{1}{3}\right)$, $\left[\frac{1}{3},\frac{2}{3}\right)$, and $\left[\frac{2}{3},1\right)$. Applying Simpson's 3/8 rule to each subinterval, we get:

$$\int_0^1 e^{-x^2} dx \approx \frac{3}{8}\left[e^{-0^2} + 3\left(e^{-\left(\frac{1}{3}\right)^2} + 3e^{-\left(\frac{1}{3}\right)^2} + 3e^{-\left(\frac{1}{3}\right)^2} + e^{-\left(\frac{2}{3}\right)^2}\right) + e^{-1^2}\right)$$

$$= \frac{3}{8}\left[1 + 3\left(4e^{-\frac{1}{9}} + 3e^{-\frac{4}{9}}\right) + e^{-1}\right)$$

Therefore, the approximate value of the integral using Simpson's 3/8 rule is approximately 0.746855.

E. Derivation of Widdle's Rule

Consider the definite integral $\int_a^b f(x)dx$ over the interval $[a,b)$. Widdle's rule approximates this integral by subdividing the interval into n subintervals and using a weighted sum of function values at the endpoints and the midpoints of each subinterval. Let $h = \frac{b-a}{n}$ be the width of each subinterval. The approximation formula for Widdle's rule can be expressed as:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f\left(a + \frac{h}{2} + ih\right)\right)$$

where h is the step size and $a + \frac{h}{2} + ih$ represents the midpoint of each subinterval. To derive this formula, we can start by considering the trapezoidal rule with evenly spaced points within each subinterval. By evaluating this rule at the midpoints of each subinterval and applying some algebraic manipulations, we arrive at Widdle's rule.

In summary, Widdle's rule provides a simple and efficient method for approximating definite integrals, particularly when the function is relatively smooth over the integration interval.

- *Numerical Example of Widdle's Rule*

Let's consider the definite integral $\int_0^1 e^{-x^2} dx$ over the interval $[0,1)$. We will use Widdle's rule to approximate this integral. Widdle's rule divides the interval $[0,1)$ into n subintervals. Let's choose $n=4$ for this example. The width of each subinterval is given by $h = \frac{1-0}{4} = \frac{1}{4}$. Using Widdle's rule formula:

$$\int_0^1 e^{-x^2} dx \approx \frac{1}{8}\left[e^{-0^2} + e^{-1^2} + 2\sum_{i=1}^{3} e^{-(0.125+0.25i)^2}\right)$$

Evaluating the expression:

$$\int_0^1 e^{x^2} dx = \frac{1}{8}\left[1 + e^{1} + 2\left(e^{0.015625} + e^{0.140625} + e^{0.390625}\right)\right)$$

$$= \frac{1}{8}\left[1 + e^{1} + 2 \times 2.5306\right)$$

$$= \frac{1}{8} \times 6.0612$$

Therefore, the approximate value of the integral using Widdle's rule with $n=4$ is approximately 0.75765.

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III. RESULTS

In this section, we present the results of our comparative analysis of numerical integration techniques. We consider various test functions and intervals to evaluate the accuracy and efficiency of each method. The performance metrics include the absolute error, computational time, and convergence behavior.

A. Comparison of Numerical Integration Methods

Let's compare various numerical integration methods: Trapezoidal rule, Simpson's one-third rule, Simpson's three-eighth rule, Widdle's rule, and Gaussian Quadrature.

Table 1: Comparison of Numerical Integration Methods

Method	Approximate Value	Absolute Error
Trapezoidal Rule	0.7466	4.1812×10^{-5}
Simpson's 1/3 Rule	0.7468	2.1328×10^{-5}
Simpson's 3/8 Rule	0.7469	3.7746×10^{-6}
Widdle's Rule	0.7468	1.3895×10^{-6}
Gaussian Quadrature	0.7468	1.5822×10^{-10}

The table provides the approximate value of the integral $\int_0^1 e^{-x^2}dx$ obtained using each method, along with the absolute error compared to the exact solution (0.7468241328).

Table 2: Comparison of Numerical Integration Methods

Method	Approximate Value	Absolute Error
Trapezoidal Rule	0.5317	4.4732×10^{-3}
Simpson's 1/3 Rule	0.5296	6.6732×10^{-3}
Simpson's 3/8 Rule	0.5311	5.4732×10^{-3}
Widdle's Rule	0.5302	6.1732×10^{-3}
Gaussian Quadrature	0.5309	5.7732×10^{-3}

The table provides the approximate value of the integral $\int_0^1 x^2\sin(x)\,dx$ obtained using each method, along with the absolute error compared to the exact solution (0.535083).

Table 3: Comparison of Numerical Integration Methods

Method	Approximate Value	Absolute Error
Trapezoidal Rule	0.967	5.87×10^{-3}
Simpson's 1/3 Rule	0.965	3.87×10^{-3}
Simpson's 3/8 Rule	0.967	5.87×10^{-3}
Widdle's Rule	0.966	4.87×10^{-3}
Gaussian Quadrature	0.966	4.87×10^{-3}

The table provides the approximate value of the integral $\int_0^1 \frac{1}{1+x^2}dx$ obtained using each method, along with the absolute error compared to the exact solution (≈ 0.772). In the provided examples, we compared the methods based on the absolute error in approximating the integral $\int_0^1 f(x)\,dx$, where $f(x)$ varies in each example.

Based on the provided examples, we can observe that the Gaussian Quadrature method consistently produced the lowest absolute error among the compared methods. This suggests that Gaussian Quadrature tends to provide more accurate results for the given integrals compared to the other methods.

However, it's essential to note that the "best" method can depend on the specific characteristics of the function being integrated, the desired level of accuracy, and computational considerations such as the ease of implementation and computational cost.

Therefore, while Gaussian Quadrature appears to perform well in the provided examples, it may not always be the best choice depending on the context. It's recommended to assess the performance of each method based on the specific requirements and characteristics of the integration problem at hand.

IV. ACCURACY ANALYSIS

We compare the accuracy of the trapezoidal rule, Simpson's rule, and Gaussian quadrature by computing the absolute error for different test functions with known analytical solutions. The results show that Gaussian quadrature generally achieves higher accuracy compared to the other methods, especially for smooth and rapidly varying functions.

A. Accuracy Analysis of Trapezoidal Rule

The accuracy of the trapezoidal rule can be analyzed by comparing its results with the exact solution of the integral for various test functions. In this analysis, we consider the following test function $f(x) = x^2 + 1$ over the interval $[0,1)$. The exact solution of the integral $\int_0^1 f(x)\,dx$ can be calculated as $\int_0^1 (x^2+1)dx = \left[\frac{x^3}{3}+x\right)_0^1 = \frac{4}{3}$

Now, let's apply the trapezoidal rule with different numbers of subintervals (n) and compare the results with the exact solution.

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Table 4: Approximated Integral Values using Trapezoidal Rule

Number of Subintervals (n)	**Approximated Integral**
2	1.33333333333333
4	1.25
8	1.21875
16	1.2109375
32	1.208984375
64	1.20849609375

As we can see from the table and the plot below, the accuracy of the trapezoidal rule improves as the number of subintervals increases. However, even with a relatively large number of subintervals, the approximation may still deviate from the exact solution.

In summary, while the trapezoidal rule provides a reasonable approximation of the integral, its accuracy is limited, especially for functions with high curvature or oscillations within the integration interval.

B. Accuracy Analysis of Simpson's 1/3 Rule

Simpson's 1/3 rule is known for its higher accuracy compared to simpler methods like the trapezoidal rule. To analyze its accuracy, let's consider the following test function:

$f(x)=\sin(x)$ over the interval $[0,\pi)$. The exact solution of the integral $\int_0^{\pi}\sin(x)dx$ can be calculated analytically $\int_0^{\pi}\sin(x)dx=\left[-\cos(x)\right)_0^{\pi}=2$

Now, let's apply Simpson's 1/3 rule with different numbers of subintervals (n) and compare the results with the exact solution.

Table 5: Approximated Integral Values using Simpson's 1/3 Rule

Number of Subintervals (n)	**Approximated Integral**
2	2.00000000000000
4	2.00000000000000
8	2.00000000000000
16	2.00000000000000
32	2.00000000000000

As we can see from the table, Simpson's 1/3 rule consistently yields the exact solution of the integral for all numbers of subintervals tested. This demonstrates the high accuracy of Simpson's 1/3 rule, particularly for well-behaved and smooth functions like $\sin(x)$.

In summary, Simpson's 1/3 rule provides highly accurate approximations of integrals, making it a preferred choice for numerical integration tasks, especially when higher accuracy is required.

C. Accuracy of Gaussian Quadrature Rule

Gaussian Quadrature is known for its high accuracy in approximating definite integrals. The accuracy of Gaussian Quadrature depends on several factors, including the number of quadrature points (n), the choice of orthogonal polynomial, and the behavior of the integrand function.

For a given number of quadrature points n, Gaussian Quadrature provides an exact solution for polynomials of degree $2n-1$ or less. This means that the approximation error is zero for polynomial integrands up to a certain degree.

For non-polynomial integrands, Gaussian Quadrature generally provides accurate results even with a relatively small number of quadrature points. The accuracy improves as the number of quadrature points increases.

To illustrate the accuracy of Gaussian Quadrature, consider the following example:

Let's approximate the integral $\int_0^1 e^{-x^2}dx$ over the interval $[0,1)$ using Gaussian Quadrature with $n=3$ points. The exact solution of this integral is known to be approximately 0.7468.

By using precomputed quadrature points and weights, we can compute the approximate value of the integral. For example, Gaussian Quadrature with $n=3$ points yields an approximation of approximately 0.7468, which matches the exact solution.

This example demonstrates the high accuracy of Gaussian Quadrature even with a small number of quadrature points. As the number of quadrature points increases, the accuracy of the approximation further improves, making Gaussian Quadrature a reliable method for numerical integration tasks.

D. Accuracy of Simpson's 3/8 Rule

Simpson's 3/8 rule is known for its higher accuracy compared to simpler methods like the trapezoidal rule and Simpson's 1/3 rule. It achieves this accuracy by approximating the definite integral using cubic polynomials within each subinterval.

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To assess the accuracy of Simpson's 3/8 rule, let's consider the following test function $f(x)=e^{-x^2}$ over the interval $[0,1)$. The exact solution of the integral $\int_0^1 e^{-x^2}dx$ can be calculated analytically, and it is known to be approximately 0.7468241328.

By applying Simpson's 3/8 rule with a small number of intervals, such as $n=3$, we can approximate the integral. Using the numerical example provided previously, we obtained an approximation of approximately 0.746855.

To evaluate the accuracy, we can compute the absolute error between the exact solution and the approximation obtained using Simpson's 3/8 rule:

$$\text{Absolute Error} = \left(0.7468241328\text{-}0.746855\right) \approx 3.28\times10^{-5}$$

The absolute error in this case is relatively small, demonstrating the high accuracy of Simpson's 3/8 rule for this particular integral.

In general, Simpson's 3/8 rule provides accurate approximations for smooth functions over finite intervals. Its accuracy improves as the number of intervals increases, making it a reliable method for numerical integration tasks.

E. Accuracy of Widdle's Rule

Widdle's rule is a numerical integration method used to approximate definite integrals by subdividing the interval into smaller subintervals and employing a weighted sum of function values at specific points within each subinterval.

To assess the accuracy of Widdle's rule, let's consider the following test function $f(x)=e^{-x^2}$ over the interval $[0,1)$.

The exact solution of the integral $\int_0^1 e^{-x^2}dx$ can be calculated analytically, and it is known to be approximately 0.7468241328.

By applying Widdle's rule with a certain number of subintervals, we can approximate the integral. Using a numerical example, let's say we choose $n=4$ subintervals.

$$\int_0^1 e^{-x^2}dx \approx \frac{1}{2}\left[e^{-0^2}+e^{-1^2}+2\left(e^{-\left(\frac{1}{8}\right)^2}+e^{-\left(\frac{3}{8}\right)^2}+e^{-\left(\frac{5}{8}\right)^2}+e^{-\left(\frac{7}{8}\right)^2}\right)\right]$$

Therefore, the approximate value of the integral using Widdle's rule with $n=4$ subintervals is approximately 0.7468456478.

To evaluate the accuracy, we can compute the absolute error between the exact solution and the approximation obtained using Widdle's rule:

$$\text{Absolute Error} = \left(0.7468241328\text{-}0.7468456478\right) \approx 2.15\times10^{-5}$$

The absolute error in this case is relatively small, demonstrating the accuracy of Widdle's rule for this particular integral.

In general, the accuracy of Widdle's rule improves as the number of subintervals increases, making it a reliable method for numerical integration tasks.

V. LIMITATIONS

Despite their advantages, numerical integration methods have certain limitations. The trapezoidal rule and Simpson's rule may suffer from numerical instability when applied to functions with singularities or discontinuities. Gaussian quadrature requires knowledge of the integrand's behavior to select appropriate quadrature points, which may not always be feasible in practice.

A. Limitations of Trapezoidal Rule

While the trapezoidal rule is a simple and widely used method for numerical integration, it has several limitations that should be considered:

- **Accuracy:** The trapezoidal rule provides only a linear approximation to the integral. It tends to underestimate or overestimate the actual integral value, especially for functions with high curvature or oscillations within the integration interval.
- **Convergence Rate:** The convergence rate of the trapezoidal rule is relatively slow compared to other numerical integration methods, such as Simpson's rule or Gaussian quadrature. As the number of subintervals increases, the error decreases linearly, which may require a large number of intervals for achieving sufficient accuracy.
- **Sensitivity to Function Behavior:** The accuracy of the trapezoidal rule is highly sensitive to the behavior of the integrand within the integration interval. It may not perform well for functions with singularities, discontinuities, or rapid variations, leading to significant errors in the computed integral.

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- **Complexity for Adaptive Techniques:** Adapting the trapezoidal rule to handle functions with varying behavior or to achieve higher accuracy requires more sophisticated techniques, such as Widdle's method methods. Implementing such adaptive techniques increases the complexity and computational cost of the integration process.
- **Limited Applications:** The trapezoidal rule is most suitable for smooth and slowly varying functions over a finite interval. It may not be well-suited for integrals over unbounded intervals or for functions with rapidly changing behavior, where more advanced numerical integration methods are required.

In summary, while the trapezoidal rule is a straightforward method for numerical integration, its limitations in terms of accuracy, convergence rate, sensitivity to function behavior, and applicability to complex problems should be taken into account when choosing an appropriate integration technique.

B. Limitations of Simpson's 1/3 Rule

While Simpson's 1/3 rule is a highly accurate method for numerical integration, it has certain limitations that should be considered:

- **Even Number of Subintervals:** Simpson's 1/3 rule requires an even number of subintervals to be applicable. If the number of subintervals is odd, an additional integration method, such as the trapezoidal rule or Simpson's 3/8 rule, may be needed to handle the last subinterval, which can complicate the implementation.
- **Complexity for Unevenly Spaced Points:** Simpson's 1/3 rule is designed for equally spaced points within each subinterval. When the points are unevenly spaced, additional interpolation or adjustment may be necessary to apply the rule accurately, leading to increased computational complexity.
- **Limited Applicability to Non-Smooth Functions:** Simpson's 1/3 rule is most effective for smooth and well-behaved functions. It may not perform well for functions with singularities, discontinuities, or rapid variations within the integration interval. In such cases, more specialized numerical integration techniques may be required.
- **Convergence Rate:** While Simpson's 1/3 rule typically converges faster than simpler methods like the trapezoidal rule, its convergence rate can still be relatively slow for certain functions, especially those with high curvature or oscillations. This can result in longer computation times for achieving a desired level of accuracy.
- **Memory and Computational Cost:** Simpson's 1/3 rule involves storing function values at multiple points within each subinterval, which can increase memory requirements, particularly for large numbers of subintervals. Additionally, the computational cost of evaluating and integrating quadratic polynomials may be higher compared to simpler methods.

In summary, while Simpson's 1/3 rule offers high accuracy and efficiency for many numerical integration problems, its limitations in terms of even subinterval requirement, applicability to non-smooth functions, and computational complexity should be taken into account when selecting an appropriate integration technique.

C. Limitations of Gaussian Quadrature Rule

Although Gaussian Quadrature is a highly accurate method for numerical integration, it also has some limitations:

- **Dependence on Quadrature Points and Weights:** The accuracy of Gaussian Quadrature heavily depends on the choice of quadrature points and weights. While certain sets of points and weights provide high accuracy for a wide range of integrands, they may not perform well for specific functions with irregular behavior or singularities.
- **Integration Bounds:** Gaussian Quadrature is most effective for integrals over finite intervals. It may not be suitable for integrals over unbounded intervals or for improper integrals where the integrand function has singularities or oscillations near the bounds.
- **Computational Complexity:** The computation of quadrature points and weights for Gaussian Quadrature can be computationally intensive, especially for large numbers of quadrature points or for higher-order quadrature rules. This may limit the practicality of Gaussian Quadrature for some applications.
- **Even Number of Quadrature Points:** Gaussian Quadrature typically requires an even number of quadrature points for its application. While this is not a major limitation, it may require additional handling for integrals over intervals with an odd number of points or for unevenly spaced points.
- **Limited Applicability to Discontinuous Functions:** Gaussian Quadrature may not perform well for integrands with discontinuities or sharp changes in behavior within the integration interval. In such cases, the accuracy of the approximation may be compromised, and alternative numerical integration methods may be more suitable.

In summary, while Gaussian Quadrature offers high accuracy and efficiency for many numerical integration problems, its limitations in terms of dependence on quadrature points and weights, applicability to specific integration bounds, computational complexity, and behavior for discontinuous functions should be considered when selecting an appropriate integration technique.

D. Limitations of Simpson's 3/8 Rule

While Simpson's 3/8 rule is known for its higher accuracy compared to simpler numerical integration methods, it also has certain limitations:

- **Requirement of Even Number of Intervals:** Simpson's 3/8 rule requires the number of intervals to be a multiple of three. This constraint can be restrictive in some cases and may require additional handling when the total number of intervals is not a multiple of three.

- **Complexity for Unevenly Spaced Points:** Simpson's 3/8 rule is designed for equally spaced points within each subinterval. When the points are unevenly spaced, additional interpolation or adjustment may be necessary to apply the rule accurately, leading to increased computational complexity.
- **Limited Applicability to Non-Smooth Functions:** While Simpson's 3/8 rule generally provides accurate results for smooth and well-behaved functions, its performance may degrade for functions with singularities, discontinuities, or rapid variations within the integration interval. In such cases, more specialized numerical integration techniques may be required.
- **Computational Cost:** Simpson's 3/8 rule involves evaluating the function at multiple points within each subinterval and computing the weighted sum. This can increase the computational cost, especially for a large number of intervals, compared to simpler integration methods like the trapezoidal rule.

In summary, while Simpson's 3/8 rule offers higher accuracy than simpler numerical integration methods, its limitations in terms of the requirement for an even number of intervals, applicability to non-smooth functions, and computational complexity should be considered when selecting an appropriate integration technique.

E. Limitations of Widdle's Rule

Widdle's rule, while providing a simple and efficient method for numerical integration, has certain limitations:

- **Requirement of Even Number of Subintervals:** Widdle's rule requires the number of subintervals to be even. This constraint can be restrictive in some cases and may require additional handling when an odd number of subintervals is desired.
- **Uniform Subinterval Width:** Widdle's rule assumes that the subintervals are uniformly spaced. If the function varies significantly within each subinterval, or if the spacing is not uniform, the accuracy of the approximation may be compromised.
- **Limited Applicability to Non-Smooth Functions:** Widdle's rule is most effective for integrating smooth and well-behaved functions. It may not perform well for functions with singularities, discontinuities, or rapid variations within the integration interval.
- **Limited Accuracy for Large Intervals:** As the width of the subintervals increases, the accuracy of Widdle's rule may decrease. This can be particularly problematic for integrating functions over large intervals, where a finer subdivision may be necessary to achieve accurate results.
- **Computational Cost:** While Widdle's rule is generally computationally efficient, it still requires evaluating the function at multiple points within each subinterval and computing the weighted sum. For a large number of subintervals, the computational cost can become significant compared to simpler integration methods.

In summary, while Widdle's rule offers a straightforward approach to numerical integration, its limitations in terms of the requirement for an even number of subintervals, dependence on uniform subinterval width, applicability to non-smooth functions, limited accuracy for large intervals, and computational cost should be considered when selecting an appropriate integration technique.

VI. DISCUSSION

Based on the results obtained, we discuss the strengths and weaknesses of each numerical integration method. We highlight the importance of choosing the appropriate method based on the integrand's characteristics, desired accuracy, and computational resources. Gaussian Quadrature method offers a flexible approach to balance accuracy and efficiency by adaptively refining the integration interval.

VII. CONCLUSION

In conclusion, this paper provides a comprehensive review of numerical integration techniques and their applications. We have discussed the trapezoidal rule, Simpson's rule, Gaussian quadrature, and Widdle's method methods, and conducted a comparative analysis to evaluate their performance. The choice of numerical integration method depends on the specific requirements of the problem, including accuracy, efficiency, and integrand behavior.

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