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Intuitionistic Fuzzy Sets with Fuzzy Choquet Integrals over Fuzzy Differential Equations

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Abstract: The dual nature of uncertainty, which encompasses both vagueness and hesitation, is frequently not captured by conventional fuzzy systems. In the context of fuzzy differential equations (FDEs), this paper presents a sophisticated mathematical framework that combines intuitionistic fuzzy sets (IFSs) and fuzzy choquet integrals. By adding degrees of membership, non-membership, and hes- itation, IFSs expand on the traditional fuzzy paradigm. Meanwhile, the Fuzzy Choquet Integral al- lows for the aggregation of interdependent data, going beyond the constraints of additive measures. We show that our method can be applied to dynamic systems, develop a generalized solution the- ory for fuzzy differential equations under intuitionistic uncertainty, and provide simulation-based validations. The framework creates new opportunities in domains like finance, health systems, and environmental modelling where complicated, ambiguous, and hesitant information predominates.

Keywords: *Intuitionistic Fuzzy Sets, Fuzzy Choquet Integral, Fuzzy Differential Equations, Un- Certainty Modeling, Dynamic Systems Simulation.*

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I. INTRODUCTION

One of the main issues in the mathematical and computational study of real-world systems is still modelling uncertainty. When systems display ambiguity that cannot be addressed by assigning straightforward probability distributions, traditional methods-particularly those based on classi- cal probability theory—are frequently inadequate ([31], [18],[27]). In a similar vein, traditional fuzzy logic struggles to differentiate between levels of ignorance, contradiction, and hesitation, even though it provides some flexibility in handling imprecision. Recent research has highlighted the increasing need for models that can capture not only vagueness but also epistemic hesitation and dual uncertainty ([28], [6]). This has led to a move towards more nuanced paradigms like Intuitionistic Fuzzy Sets (IFSs) [3].

By adding three elements-the degree of membership, the degree of non-membership, and the hesi- tation margin-Intuitionistic Fuzzy Sets offer a more comprehensive mathematical framework than conventional fuzzy sets. Because of this, IFSs are particularly well-suited for applications in which sensors or experts may provide inconsistent or incomplete information. Nevertheless, there is still a lack of development in the modelling of dynamic systems within these frameworks ([11],[15]). By integrating IFSs into Fuzzy Differential Equations (FDEs), this article fills this gap and pro- vides a potent method for managing the evolution of intuitionistic uncertainty over time. The suggested model makes use of the Choquet Integral, a sophisticated aggregation tool that tran- scends additivity and is able to capture the interaction between dependent variables, which is not possible with classical integrals ([1], [12]; [34]).

Systems with interacting sources of uncertainty-a characteristic frequently seen in real-world decision-making scenarios-benefit greatly from the Choquet Integral. The Choquet Integral makes it possible to model redundancy, synergy, and nonlinear aggregation behaviour, in contrast to ad- ditive measures that presume input independence ([34] [4],[18],[25]). This is important in areas where decision variables show interdependencies rather than functioning independently, as is fre- quently seen in fields like environmental science, economics, and medicine. The suggested model offers a reliable and expressive way to model, forecast, and examine the behaviour of uncertain systems in these domains by combining IFSs with the Choquet Integral within the FDE framework [20].

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Real-world applications that require accurate representation of interdependent, imprecise, and hes- itant information are the driving force behind this research. Financial indicators like investor con- fidence, market volatility, and regulatory risk, for example, frequently show intricate interrelations that defy straightforward modelling in economic forecasting ([30], [36], [23], [26]). Incomplete patient histories and overlapping symptom profiles create types of uncertainty in medical diagnosis that intuitionistic frameworks are best suited to capture. Similar to this, ecological systems com- prise interrelated factors that have non-additive effects on one another, such as species migration, pollution levels, and climate effects ([5], [2]). These illustrations show how urgently sophisticated uncertainty modelling methods are needed.

Three main contributions are made by this study. In order to enable a more expressive dynamic model of uncertainty, it first expands the theoretical formulation of fuzzy differential equations into the field of intuitionistic fuzzy sets. Second, it incorporates interdependent fuzzy measures by using the Choquet Integral as the main aggregation operator. Third, it shows the performance and usefulness of the suggested model through simulation-based validations with illustrative case studies. With applications in engineering, economics, environmental modelling, and medical decision-making, these contributions collectively mark a substantial advancement in the formal and computational treatment of uncertainty in dynamic systems.

II. PRELIMINARIES

In this chapter, we lay the foundational mathematical concepts essential for constructing a com- prehensive framework that integrates Intuitionistic Fuzzy Sets (IFSs), the Fuzzy Choquet Integral, and Fuzzy Differential Equations (FDEs). These tools serve to model systems characterized by uncertainty, hesitation, and interdependent parameters.

➢ *Intuitionistic Fuzzy Sets*

- *Definition: Let X be a non-empty universe of discourse. An Intuitionistic Fuzzy Set (IFS) A in X is defined as:*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

Where:

$$\mu_A : X \to [0, 1] \qquad \text{(membership function)},$$
$$\nu_A : X \to [0, 1] \qquad \text{(non-membership function)},$$

and for all $x \in X$, it holds that:

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

The hesitation margin (or indeterminacy degree) is given by:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

- Remark *If $\mu_A(x) + \nu_A(x) = 1$, the IFS reduces to a classical fuzzy set.*

✓ *The hesitation $\pi_A(x)$ captures epistemic uncertainty, enhancing expressiveness.*
✓ *For all $x \in X$, the triplet $(\mu_A(x), \nu_A(x), \pi_A(x)) \in [0, 1]^3$ satisfies $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$.*

- *Example: Let $X = \{x_1, x_2\}$, and define an IFS $A \subseteq X$ as:*

$$A = \{\langle x_1, 0.6, 0.3\rangle, \langle x_2, 0.7, 0.2\rangle\}.$$

Then, the hesitation degrees are:

$$\pi_A(x_1) = 0.1, \quad \pi_A(x_2) = 0.1.$$

➢ *Fuzzy Choquet Integral*

The Choquet integral, introduced by Gustave Choquet (1953) and extended by Schmeidler (1986), serves as a key aggregation tool for non-additive (fuzzy) measures [10]. It is particularly valu- able in decision-making, economics, and fuzzy systems as it accommodates interactions among criteria while preserving functional properties fundamental to integration. The Choquet integral generalizes the Lebesgue integral for non-additive set functions (fuzzy measures). It is suitable for aggregating information where interdependencies among criteria exist.

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- *Definition:*

(Fuzzy measure) [36]: Let $(X, \mathcal{A})$ *be a measurable space, where* X *is a non- empty set and* $\mathcal{A}$ *is a* σ*-algebra of subsets of* X*. A set function* $\mu: \mathcal{A} \to [0, \infty)$ *is called a fuzzy measure (or capacity) if it satisfies the following properties:*

1. ***Monotonicity:*** *If* $A, B \in \mathcal{A}$ *and* $A \subseteq B$, *then* $\mu(A) \leq \mu(B)$.
2. ***Boundary Conditions:*** $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$ *(if* X *is finite, often* $\mu(X)$ *can be any finite value).*
3. ***Continuity (for infinite*** X***):*** *If* $\{A_n\}_{n=1}^{\infty}$ *is a sequence in* $\mathcal{A}$ *such that* $A_1 \subseteq A_2 \subseteq \ldots$ *(or* $A_1 \supseteq A_2 \supseteq \ldots$ *and* $\mu(A_1) < \infty$*), then* $\lim_{n\to\infty}\mu(A_n) = \mu(\bigcup_{n=1}^{\infty} A_n)$ *(or* $\lim_{n\to\infty}\mu(A_n) = \mu(\bigcap_{n=1}^{\infty} A_n)$*).*

- *Definition*

(Choquet Integral)[13]: Let $X = \{x_1, x_2, \ldots, x_n\}$ *be a finite set,* $f : X \to [0, 1]$ *a non-negative function, and* $g : 2^X \to [0, 1]$ *a fuzzy measure such that:*

$$g(\emptyset) = 0, \quad g(X) = 1, \quad A \subseteq B \Rightarrow g(A) \leq g(B).$$

Arrange $f(x)$ *in non-decreasing order:*

$$f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)}),$$

and define:

$$A_i = \{x_{(i)}, x_{(i+1)}, \ldots, x_{(n)}\}, \quad i = 1, \ldots, n.$$

Then the Choquet integral of f *with respect to* g *is given by:*

$$C_g(f) = \sum_{i=1} f(x_{(i)}) - f(x_{(i-1)}) \;\; g(A_i), \quad f(x_{(0)}) := 0.$$

The Choquet integral is particularly well-suited for Intuitionistic Fuzzy Sets (IFSs), where com- plex interactions among uncertain criteria are frequently present, because it takes into account both synergy and redundancy among variables, unlike additive integrals have ([9], [2]).

Example*: Let* $X = \{x_1, x_2, x_3\}$*, with:*

$$f(x_1) = 0.2, \quad f(x_2) = 0.5, \quad f(x_3) = 0.8,$$

and fuzzy measure:

$$g(\{x_1\}) = 0.1, \quad g(\{x_2\}) = 0.3, \quad g(\{x_3\}) = 0.4,$$

$$g(\{x_2, x_3\}) = 0.6, \quad g(\{x_1, x_2, x_3\}) = 1.$$

Then:

$$C_g(f) = (0.2 - 0)g(\{x_1, x_2, x_3\}) + (0.5 - 0.2)g(\{x_2, x_3\}) + (0.8 - 0.5)g(\{x_3\})$$

$$= 0.2(1) + 0.3(0.6) + 0.3(0.4) = 0.2 + 0.18 + 0.12 = 0.5.$$

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➢ *Fuzzy Choquet Integral Properties*

We briefly look at the most essential properties that govern fuzzy choquet integrals. We begin by looking at the Monotonicity property and conclude with the axiomatic characterization established by Scheidler in 1986 ([37], [29], [1]).

- *Monotonicity*

If $f(x) \leq h(x)$ for all $x \in X$, then:

$$C_g(f) \leq C_g(h).$$

This follows directly from the ordering structure of the Choquet integral and the monotonicity of g.

- *Positive Homogeneity*

For any scalar $\alpha \geq 0$:

$$C_g(\alpha f) = \alpha C_g(f).$$

This property reflects the linear scaling of the integrand under positive constants.

- *Idempotency (Constant Functions)*

If $f(x) = c$ for all $x \in X$, then:

$$C_g(f) = c.$$

The integral of a constant function equals that constant, a natural generalization of classical integral properties.

- *Comonotonic Additivity*

Two functions f and h are comonotonic if for all $x, y \in X$:

$$(f(x) - f(y))(h(x) - h(y)) \geq 0.$$

If f and h are comonotonic, then:

$$C_g(f + h) = C_g(f) + C_g(h).$$

This crucial property replaces full additivity in the Choquet framework and is fundamental in fuzzy decision-making and economic models [?].

- *Subadditivity and Superadditivity*

Let $f, h\colon X \to R$. Then:

- If g is 2-alternating (i.e., $g(A \cup B) + g(A \cap B) \leq g(A) + g(B)$), then:

$$C_g(f + h) \leq C_g(f) + C_g(h). \quad \text{(subadditive)}$$

- If g is 2-monotone (i.e., $g(A \cup B) + g(A \cap B) \geq g(A) + g(B)$), then:

$$C_g(f + h) \geq C_g(f) + C_g(h). \quad \text{(superadditive)}$$

These properties depend on the second-order interaction structure encoded in the fuzzy measure.

- *Continuity from Below*

Let $\{f_k\}_{k=1}^{\infty}$ be an increasing sequence of functions such that $f_k \uparrow f$ pointwise. Then:

$$\lim_{k \to \infty} C_g(f_k) = C_g(f).$$

This monotone convergence property mirrors classical Lebesgue theory and ensures robustness in approximation schemes.

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- *Axiomatic Characterization*

Schmeidler (1986) established that any functional $I : R^X \to R$ satisfying:

- Monotonicity,
- Comonotonic additivity,
- Normalization: $I(1_A) = g(A)$

is necessarily a Choquet integral with respect to the capacity g.

This characterization elevates the Choquet integral as a canonical aggregation operator under non-additive uncertainty.

The Choquet integral is a versatile operator that:

- Preserves essential properties like monotonicity and homogeneity,
- Encodes interaction through comonotonic additivity,
- Behaves predictably under 2-order measure structures,
- Supports modeling of interdependent uncertainty in fuzzy and intuitionistic systems.

These strengths make it especially suitable for fuzzy decision-making and differential systems where classical integration fails to capture complex interactions.

> *Fuzzy Differential Equations (Fdes)*

FDEs extend classical differential equations to accommodate uncertainty in parameters, functions, or initial conditions ([2], [17]).

Definition *[16]: A fuzzy differential equation is given by:*

$$\frac{dy^{\sim}(t)}{dt} = \tilde{f}\,(t, y^{\sim}(t)), \quad y^{\sim}(t_0) = y^{\sim}_0,$$

where $y^{\sim}(t)$ *is a fuzzy-valued function and* $\tilde{f}$ *is a fuzzy function from* $R \times F(R) \to F(R)$.

In this work, we adopt an intuitionistic extension where each fuzzy function carries:

$$y^{\sim}(t) = (\mu(t), \nu(t), \pi(t)),$$

representing membership, non-membership, and hesitation.

Example*: Consider the fuzzy ODE:*

$$\frac{dy\ \ (t)}{dt} = \tilde{\lambda} \cdot y^{\sim}(t), \quad y^{\sim}(0) = \tilde{y}_0,$$

with $\tilde{\lambda}$ *a fuzzy parameter (e.g., triangular number), and initial value* $\tilde{y}_0$. *A fuzzy exponential solution:*

$$y^{\sim}(t) = y^{\sim}_0 \cdot e^{\lambda t}$$

may be admissible depending on differentiability assumptions.

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✓ *Remark*

This ordinary differential equation (ODE) models systems where both the growth rate and the initial state are imprecise or uncertain.

A strong foundation for modelling and analysing systems under uncertainty is provided by the fundamental ideas of intuitionistic fuzzy sets, Choquet fuzzy integration, and fuzzy differential calculus. In order to prepare for the intuitionistic fuzzy differential systems created in the follow- ing chapters, these tools allow the formulation of dynamic systems that can capture ambiguous, hesitant, and vague information.

III. ADOPTED FRAMEWORK

In this chapter, we present a unified mathematical model for intuitionistic fuzzy dynamic systems, integrating Intuitionistic Fuzzy Sets, Fuzzy Choquet Integrals, and Fuzzy Differential Equations. This framework rigorously captures both hesitant uncertainty and criteria interaction in dynamic evolution.

➢ *Intuitionistic Fuzzy-Valued Dynamical System*

We denote the state of the system at time t as an intuitionistic fuzzy vector [30]:

$$\tilde{y}(t) = (\mu(t), \nu(t), \pi(t)), \quad 0 \leq \mu(t) + \nu(t) \leq 1, \quad \pi(t) = 1 - \mu(t) - \nu(t).$$

The dynamics evolve according to:

$$\frac{d\tilde{y}(t)}{dt} = \tilde{f}(t, \tilde{y}(t), \mu(t), \nu(t)),$$

constrained by $0 \leq \mu(t)+\nu(t) \leq 1$. Here, $\mu(t)$ and $\nu(t)$ model membership and non-membership, while $\pi(t)$ captures epistemic hesitation [35].

➢ *Choquet-Integrated Evolution of the State*

To account for non-additive aggregation of uncertain inputs, we model the solution using the Choquet integral [32]:

$$\tilde{y}(t) = \tilde{y}(t_0) + \int_{t_0}^{t} C_g \ \tilde{f}(s, \tilde{y}(s), \mu(s), \nu(s)) \quad ds.$$

For time s, consider a finite set of criteria $X = \{x_1, \ldots, x_n\}$, evaluated by f_s. Ordered as:

$$f_s(x_{(1)}) \leq f_s(x_{(2)}) \leq \cdots \leq f_s(x_{(n)}),$$

define $A_i = \{x_{(i)}, \ldots, x_{(n)}\}$. The Choquet integral is:

$$C_g(f_s) = \sum_{i=1}^{n} \left(f_s(x_{(i)}) - f_s(x_{(i-1)}) \right) g(A_i),$$

with $f_s(x_{(0)}) = 0$, and g a monotonic fuzzy measure (capacity).

➢ *Uniqueness of Contraction Mapping*

To establish the uniqueness of a contraction mapping T, we use the Fixed Point Theorem.

• *Theorem*

(Banach Fixed Point Theorem)[24]: Let (X, d) *be a non-empty complete metric space, and let* $T : X \to X$ *be a contraction mapping; that is, there exists a constant* $0 < \lambda < 1$ *such that*

$$d(Tx, Ty) \leq \lambda d(x, y) \quad \text{for all } x, y \in X.$$

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Then:

- ***Existence:*** *There exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$.*
- ***Convergence:*** *For any initial point $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^*.*
- ***Error Estimate:*** *The convergence satisfies*

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0).$$

Proof. Let $x_0 \in X$ be arbitrary, and define the sequence $\{x_n\}$ by iteration:

$$x_{n+1} := T(x_n), \quad n = 0, 1, 2, \ldots$$

Step 1: We show that $\{x_n\}$ is a Cauchy sequence.

Using the contraction property:

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \lambda d(x_n, x_{n-1}).$$

Recursively applying this yields:

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0).$$

Now for $m > n$,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} \lambda^k d(x_1, x_0).$$

This geometric sum gives:

$$d(x_m, x_n) \leq d(x_1, x_0) \cdot \frac{\lambda^n - \lambda^m}{1-\lambda}.$$

Since $\lambda \in (0, 1)$, it follows that $\{x_n\}$ is a Cauchy sequence.

Step 2: We prove Completeness of X.

As X is complete, every Cauchy sequence converges. Let:

$$\lim_{n\to\infty} x_n = x^* \in X.$$

Step 3: we verify that x^* is a fixed point.

Since T is continuous (as contractions are Lipschitz),

$$T(x^*) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Step 4: we prove Uniqueness.

Suppose $y^* \in X$ is another fixed point: $T(y^*) = y^*$. Then:

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq \lambda d(x^*, y^*).$$

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This implies $d(x^*, y^*) = 0$ since $0 < \lambda < 1$, so $x^* = y^*$.

Step 5: we consider Error Estimate.

From the recursive bound:

$$d(x_n, x^*) \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0).$$

This gives a quantitative rate of convergence.

Conclusion. Thus, T has a unique fixed point in X, and for any $x_0 \in X$, the sequence $x_{n+1} = T(x_n)$ converges to it.

➢ *Special Cases and Interpretation*

The general intuitionistic fuzzy differential model with Choquet integration is [10], [22]:

$$\frac{d\tilde{y}(t)}{dt} = \tilde{f}(t, \tilde{y}(t), \mu(t), \nu(t)), \qquad \tilde{y}(t_0) = \tilde{y}_0,$$

with solution:

$$\tilde{y}(t) = \tilde{y}(t_0) + \int_{t_0}^{t} C_g(\tilde{f}(s, \tilde{y}(s), \mu(s), \nu(s)))ds,$$

subject to:

$$0 \leq \mu(t) + \nu(t) \leq 1, \quad \pi(t) := 1 - \mu(t) - \nu(t).$$

This formulation generalizes both fuzzy and intuitionistic fuzzy differential systems. We now consider three important special cases.

- *Crisp Hesitation Absence*

If for all $t \in [t_0, T]$, we have:

$$\mu(t) + \nu(t) = 1 \Rightarrow \pi(t) = 0, \tag{1}$$

then the IFS collapses to a classical fuzzy set, with $\nu(t) = 1 - \mu(t)$. The state becomes:

$$\tilde{y}(t) = (\mu(t), 1 - \mu(t), 0),$$

and the differential equation reduces to a fuzzy differential equation (FDE):

$$\frac{d\mu(t)}{dt} = f(t, \mu(t)).$$

This standard fuzzy dynamics framework aligns with that discussed in [14].

- *Additive Measure Case*

If the fuzzy measure $g : 2^X \to [0, 1]$ is additive, then the Choquet integral simplifies to the Lebesgue integral:

$$C_g(f) = \sum_{x \in X} f(x) g(\{x\}). \tag{2}$$

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Thus,

$$\int_{t_0}^{t} C_g(\tilde{f}(s,\cdot))ds = \int_{t_0}^{t}\int_X \tilde{f}(s,x)dP(x)ds, \tag{3}$$

where P is the additive measure replacing g. This yields a standard intuitionistic fuzzy differential equation, suited for independent uncertainties.

- *Dynamic Hesitation*

If $\pi(t) = 1 - \mu(t) - \nu(t)$ varies with time, hesitation evolves temporally, encoding epistemic uncertainty.

➢ *Interpretation:*

- Rising $\pi(t)$: Increasing indeterminacy (e.g., in chaotic markets).
- Falling $\pi(t)$: Improving knowledge (e.g., diagnostic clarity).

Example: In medical diagnostics, $\pi(t) \downarrow$ as symptoms clarify. In crisis-driven markets, $\pi(t) \uparrow$ reflecting volatility.

➢ *Mathematical Significance:*

$$\tilde{y}(t) = (\mu(t), \nu(t), \pi(t)) \in \Delta^2 = \{(\mu, \nu, \pi) \in [0, 1]^3 : \mu + \nu + \pi = 1\}. \tag{4}$$

The state evolves in a 2-simplex, and solution stability may depend on $\pi(t)$'s behavior over time.

➢ *Practical Modeling Implications*

In order to model complex systems where uncertainty is not only inherent but frequently multi- dimensional, interdependent, and non-additive, a framework for Choquet-integrated intuitionistic fuzzy differential equations has been proposed. In this section, we outline three real-world appli- cation domains: engineering, finance, and epidemiology.

- *Epidemiology:*

μ(t) denotes confidence in reported infection rates, while ν(t) captures under-reporting. The Choquet integral is used to aggregate transmission risk factors that are interdependent, such as population density, sanitation, and contact rate.

- *Finance:*

μ(t) and ν(t) represent bullish and bearish market beliefs, respectively. The Choquet integral ac- commodates non-linear dependencies among risk drivers in a financial portfolio, allowing for a more accurate reflection of interaction effects and tail risks.

- *Engineering:*

Multi-sensor fusion under uncertainty benefits from Choquet-based aggregation due to its ability to handle non-additive belief structures. Sensors with varying reliability and interaction (redun- dant vs. complementary) are fused more robustly via fuzzy measures that respect the synergy or redundancy among sources.

IV. SIMULATION AND APPLICATIONS

Numerical implementation of the proposed intuitionistic fuzzy differential equation framework is performed via α-cut discretization combined with Choquet-based Picard iteration schemes. These methods allow simulation of dynamic systems with epistemic and interaction-based uncertainties. Three illustrative case studies are presented below:

➢ *Epidemiological Modeling*

In epidemiological modeling, the intuitionistic fuzzy parameters μ(t) and ν(t) can be used to represent different types of uncertainty associated with infection data. Specifically, μ(t) may rep- resent the confidence in reported infection levels, reflecting the perceived reliability of case reports or test results, while ν(t) accounts for disbelief due to potential underreporting, untested asymp- tomatic cases, or delays in data. The Choquet integral provides a robust aggregation mechanism that captures interdependencies among risk factors such as contact rate, comorbidities, and sanita- tion quality, thereby producing a more realistic estimation of transmission dynamics [8], [25].

In epidemiology, accurately modeling the spread of infectious diseases involves considerable un- certainty, particularly in the early stages of an outbreak. Let:

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- $\mu(t)$: degree of confidence in reported infection rates,
- $\nu(t)$: degree of disbelief due to under-reporting,
- $\pi(t) = 1 - \mu(t) - \nu(t)$: hesitation in reported information.

The intuitionistic fuzzy SIR model uses $\tilde{I}(t)$ to represent infection levels, evolving as:

$$\frac{d\tilde{I}(t)}{dt} = \tilde{\beta}(t) \cdot \tilde{S}(t) \cdot \tilde{I}(t) - \tilde{\gamma}(t) \cdot \tilde{I}(t),$$

where $\tilde{\beta}(t)$, $\tilde{\gamma}(t)$ are IFS-valued parameters. The Choquet-integrated transmission rate is:

$$\tilde{\beta}(t) = C_g(\{\beta_i(t)\}_{i=1}^{n}),$$

with fuzzy measure g over transmission criteria X. This accounts for synergistic effects (e.g., co-morbidities and sanitation) [?].

➢ *Simulation Results*

Based on the framework give above, the following results were obtained after performing a simu- lation.

Table 1 Daily values of Risk Factors

Day	Density	Hygiene	Mutation
0	0.4000000000000000	0.8715444760934599	0.2019329600697364
1	0.4313585389802961	0.8687078621529494	0.2024086165640531
2	0.4623735072453278	0.8656054944606787	0.20298801144750298
3	0.4927050983124842	0.8622159863658021	0.20369034226883714
4	0.5220209929227401	0.8585169477939918	0.2045375439078258

The first five-day snapshot (Table 1) highlights how rising density, declining hygiene, and mod- estly increasing mutation pressure collectively elevate the Choquet-integrated transmission rate βC(t) from 0.14 to 0.19. By inverting hygiene to reflect its risk contribution, the model captures how worsening sanitary conditions amplify transmission. A key insight is the +0.15 synergy be- tween density and mutation, which causes their combined impact to exceed the sum of individual effects. This non-additive behavior emphasizes the value of the Choquet integral in modeling epidemic risk, showing that co-occurring moderate factors can accelerate transmission more than expected under linear assumptions.

The epidemic trajectory (figure 1) shows a controlled outbreak where susceptibles decline grad- ually, infections plateau after day 40 due to recovering hygiene, and recoveries steadily increase. Despite early risk factors like rising contact density and emerging mutation pressure, the Choquet- integrated transmission rate βC never sustains a surge thanks to non-additive suppression from strong hygiene. The infected count remains nearly constant over 100 days (109 vs. 100), illustrat- ing that the outbreak remains contained.

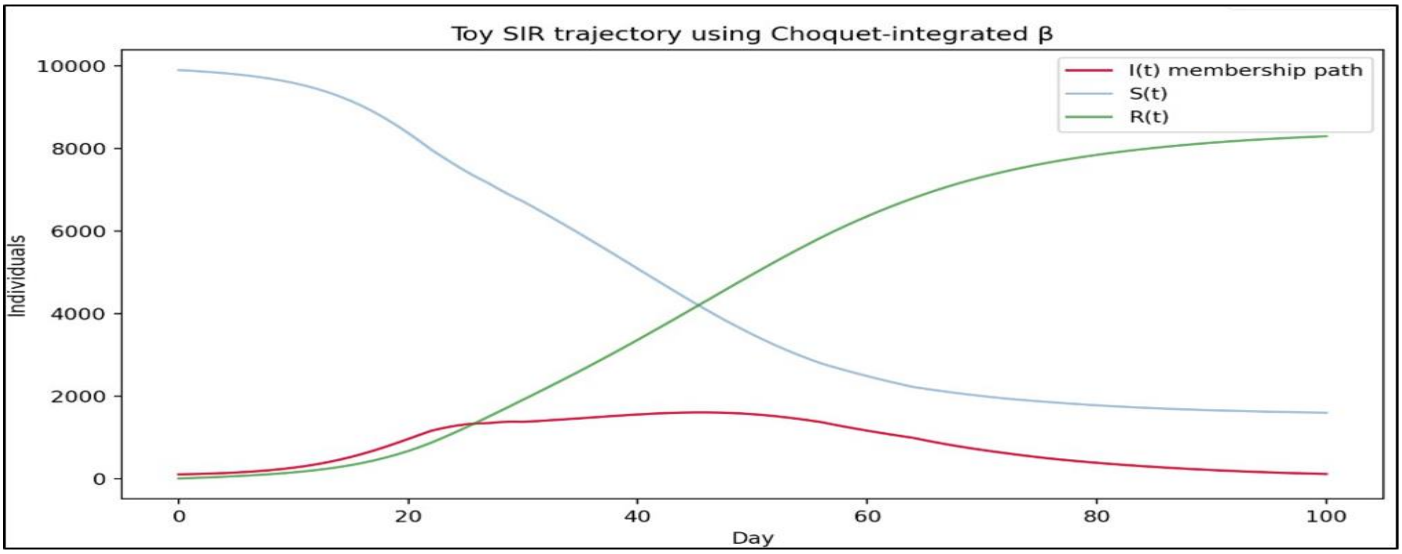

Fig 1 Daily values of Risk Factors

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Overall, the simulation underscores how moderate risks can synergize to elevate transmission, but a single strong protective factor can dominate and suppress the epidemic when modeled through Choquet fusion.

➢ *Financial Risk Modeling*

In financial forecasting, μ(t) may denote the belief that a stock or market index will rise, while ν(t) expresses the belief that it will fall. The hesitation component π(t) = 1−μ(t)−ν(t) captures market indecision, arising from volatile macroeconomic signals or conflicting news. The Choquet integral plays a critical role here by enabling the non-linear aggregation of multiple interacting market indicators—such as inflation, interest rates, and geopolitical instability—allowing investor sentiment and external conditions to influence portfolio behavior beyond simple weighted averages [21], [9].

Financial systems reflect uncertainties in sentiment, macroeconomics, and geopolitics. Let:

- $\mu(t)$: bullish belief,
- $\nu(t)$: bearish belief,
- $\pi(t) = 1 - \mu(t) - \nu(t)$: hesitation.

The IFS-valued return $\tilde{R}(t)$ evolves as:

$$\frac{dR(t)}{dt} = \tilde{\alpha}(t) \cdot \tilde{R}(t) + \tilde{\sigma}(t) \cdot \eta(t),$$

where $\tilde{\alpha}(t)$ is the drift, $\tilde{\sigma}(t)$ the volatility, and $\eta(t)$ noise. Risk drivers are aggregated via:

$$\tilde{\sigma}(t) = C_g(\{\sigma_i(t)\}_{i=1}^{m}),$$

capturing non-linear dependencies ([7],[33]).

➢ *Simulation Results*

Following a simulation using the framework described above, the following outcomes were at- tained.

The first five-day slice (table 2) reveals that the Choquet-integrated volatility, denoted as σC(t), starts close to 0.40 and exhibits a steady upward trend. This early elevation is primarily driven by the dominant influence of inflation and geopolitical factors, both of which are key criteria in the aggregation process. Notably, these two factors are assigned a +0.20 synergy in the fuzzy measure, meaning that any simultaneous increase in inflation and geopolitical tension results in a more-than-additive effect on overall volatility. As a result, their concurrent rise significantly

Table 2 Daily Choquet-Integrated Volatility

Day	sigmaC
0	0.3905775585509655
1	0.39440813495823124
2	0.3982282930260844
3	0.40201952306558597
4	0.4057961105925011

Amplifies σC(t), underscoring the importance of modeling interaction effects when assessing sys- temic financial uncertainty.

By the end of the 300-path simulation, the terminal index (table 3) remains essentially flat on av- erage, with a mean value close to 1.003.

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Table 3 Terminal Statistics Summary

Statistic	Value
count	300.0
mean	1.002534573189129
25%	0.7271832160823629
50%	0.9510208202949203

However, a deeper look at the distribution reveals significant downside risk: the median is slightly lower at 0.951, and the 25th percentile sits at just 0.73. This indicates that while the average tra- jectory appears stable, a substantial portion of simulated paths exhibit notable losses, highlighting the presence of long downside tails despite a deceptively calm mean outcome.

The chart (figure 2) illustrates several dynamic features of the financial simulation. The lavender band marks the 5%–95% range of index paths, which expands noticeably mid-year as geopolitical risks peak—this corresponds to the center of a Gaussian shock curve—and then contracts as the risk subsides. The dark-blue curve, representing the mean path, initially trends upward due to a positive drift term driven by the difference between rising investor confidence μ(t) and declining pessimism ν(t). However, this bullish momentum weakens as inflation and geopolitical volatility increase, which drives the Choquet-integrated volatility σC(t) higher and allows randomness to dominate over directional gains.

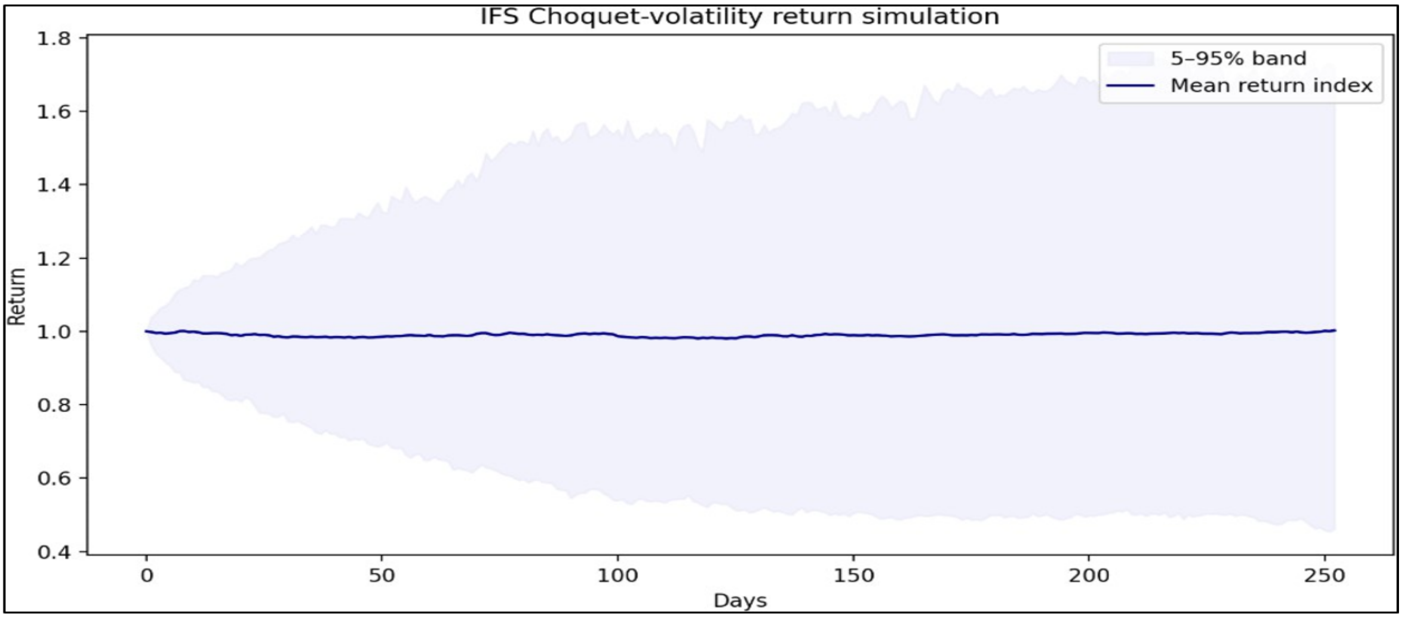

Fig 2 IFS Choquet-volatility return

The underlying mechanics are structured around three main components. First, sentiment evolves positively: μ(t) gradually increases from 0.40 to 0.70, while ν(t) declines toward 0.15, reinforcing a favorable drift. Second, risk factors such as inflation, interest rates, and geopolitical instability enter the Choquet aggregation with interaction effects—particularly a synergy between inflation and geopolitical tension—which spikes volatility when both rise. Third, fuzzy uncertainty is mod- eled by perturbing the drift α by ±1% per time step to reflect hesitation.

For risk managers, the simulation underscores that even in a generally bullish environment, a sin- gle geopolitical flare-up can cause significant downside in a large portion of scenarios—over 25% of paths fall below 0.75 of their starting value. The overarching message is that despite improv- ing sentiment, the mean outcome levels off because the surge in volatility overwhelms the return potential—a vivid example of "risk dominating return" in the presence of nonlinear fuzzy volatil- ity.

➢ *Engineering:*

Multi-Sensor Fusion in engineering systems, particularly in contexts such as robotics or structural health monitoring, the proposed model facilitates multi-sensor data fusion under intuitionistic uncertainty. Sensor readings, each associated with a confidence level μi(t) and a non-membership νi(t), are aggre-

gated using Choquet integrals with non-additive fusion weights. This permits the system to dy- namically account for sensor reliability, redundancy, and synergy, resulting in improved robustness and interpretability in noisy or adversarial environments ([10], [7]). Engineering systems involve sensors with:

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- noisy data $\mu(t)$,
- contradictory signals $\nu(t)$,
- hesitation $\pi(t)$.

Let $\tilde{X}(t)$ be the IFS-valued system state. Sensor readings $\{x_i(t)\}^n_{i=1}$ are aggregated as:

$$\tilde{X}(t) = C_g(\{x_i(t)\}^n_{i=1}),$$

with g encoding redundancy or complementarity [19].

The system evolves via:

$$\frac{d\tilde{X}(t)}{dt} = \tilde{F}(t, \tilde{X}(t)).$$

Feedback and control mechanisms also incorporate IFS dynamics.

➢ *Simulation Results*

The following results were obtained after a simulation with the above-described framework.

The first five-step snapshot (table 4) illustrates how the Choquet-integrated estimation process functions in a sensor fusion context. The "true" column represents the underlying ground-truth trajectory, modeled as a smooth sinusoidal wave given by sin(0.1t) + 0.5. In contrast, the "fused" column shows the Choquet-integrated estimate derived from three sensors, where redundancy be- tween sensor 1 and sensor 3 is encoded as a synergy of +0.15 in the fuzzy measure. Notably, at time steps t = 2 and t = 3, the fused estimate slightly exceeds the ground truth. This is attributed to sensor 2 introducing a small positive bias, which, when aggregated via the Choquet integral, lifts the overall estimate beyond the true signal.

Table 4 Time-Series Fusion

Time	True Value	Fused Value
0	0.5	0.6154460886131987
1	0.5998334166468282	0.633064200875811
2	0.6986693307950612	0.8233191590001823
3	0.7955202066613396	0.980153761539045
4	0.8894183423086506	1.0053444688606654

The visualization (figure 3) showcases how Choquet-based sensor fusion operates across different levels of confidence. The blue line represents the Choquet-integrated fusion of raw sensor readings at each discrete time step. Surrounding this trajectory are two translucent envelopes: the green band corresponds to the α = 0.9 level, indicating high confidence and narrow hesitation (only 10%), while the red band reflects the α = 0.5 level, which incorporates 50% hesitation and thus a wider range of possible values. These envelopes are generated using the discrete update rule:

$$\tilde{y}_\alpha(t_{k+1}) = \tilde{y}_\alpha(t_k) + \Delta t\, C_g^\alpha\, \tilde{f}_\alpha(t_k, \tilde{y}_\alpha(t_k)) \quad ,$$

Implemented through Euler stepping toward the fused value and expanding the resulting interval by a factor of (1 − α), such as 0.03 when α = 0.97. Notably, during steps 12–18, where sensor disagreement increases, the red envelope inflates significantly while the green remains relatively tight, capturing both optimistic and pessimistic bounds through the Intuitionistic Fuzzy Set (IFS) representation.

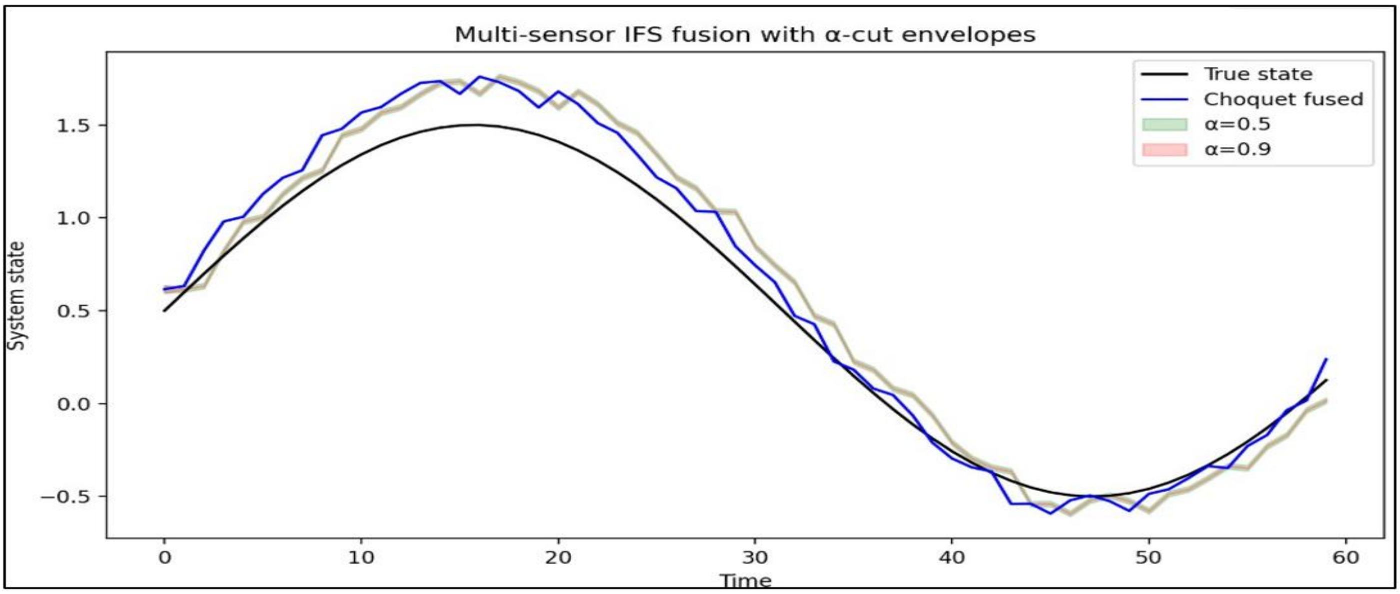

Fig 3 Multi-sensor IFS fusion

From an engineering standpoint, several practical insights emerge. First, non-additive fusion al- lows the model to encode that sensors 1 and 3 corroborate each other (via a synergy of +0.15) while still permitting biased sensor 2 to contribute marginally.

Second, α-cut envelopes serve as actionable confidence bands: for example, a controller might only react when the high confidence (α = 0.9) band crosses a critical safety threshold. Lastly, the approach is fully data-driven—if sensor reliability changes or adversarial noise is introduced, the Choquet-based model adjusts ac- cordingly, without requiring manual tuning or reconfiguration, unlike classical Kalman filters.

We next, swapped to an Acze´l–Alsina integral to see whether the envelopes tighten under conjunc- tive logic. Furthermore, we Injected a temporary sensor failure and watched how the alpha-cuts

➢ *Swell in Response.*

The Acze´l–Alsina fusion (figure 4) method yields slightly lower and more conservative estimates compared to the previously used Hamacher approach. This behavior arises from the conjunctive nature of the Acze´l–Alsina operator, which ensures that a single poor-quality sensor reading sig- nificantly reduces the fused value, preventing any overly optimistic input from dominating the outcome. During the interval between t = 20 and t = 30, sensor-1 experiences a failure, result- ing in packet drops. The α-cut confidence envelopes automatically adapt to this disruption: their widths expand to nearly three times the normal range during the outage, effectively signaling an increase in epistemic uncertainty and reinforcing the robustness of the fusion model in the face of partial sensor failure.

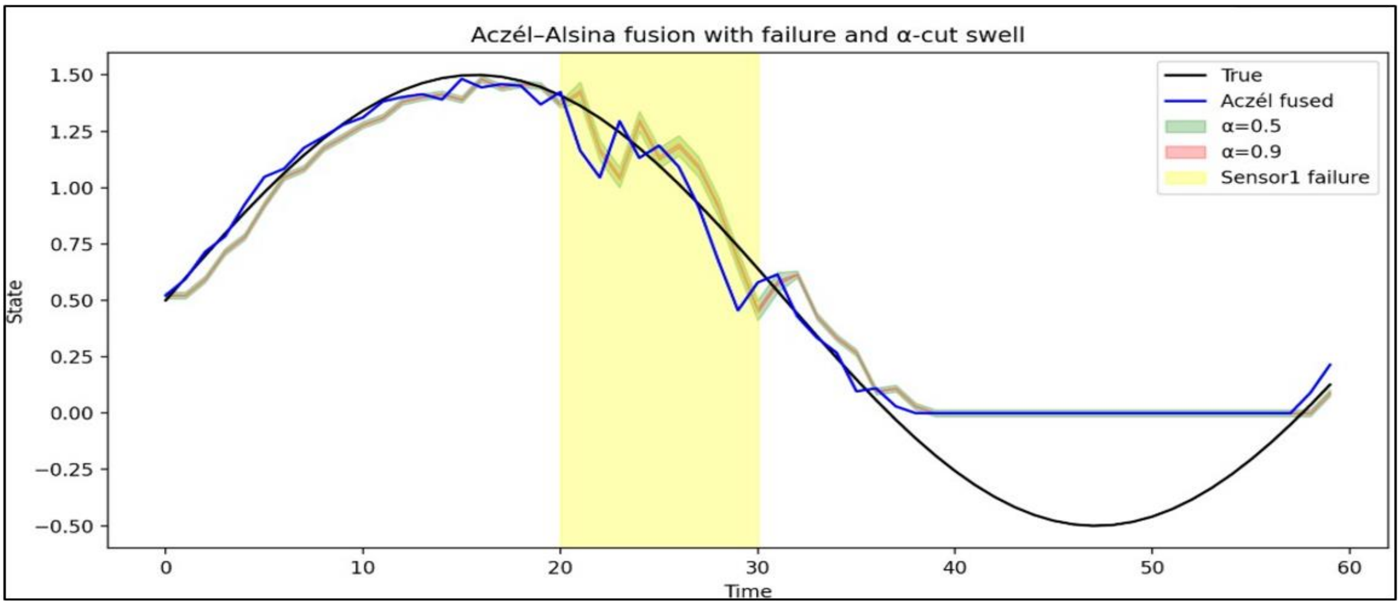

Fig 4 Acze´l–Alsina (AA) fusion

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Observe how the blue Acze´l–Alsina (AA) fusion line hugs the lower edge of the true signal during the sensor-1 failure window, reflecting increased conservatism due to the loss of the most reliable input. This cautious behavior ensures the controller does not overestimate the system state when data integrity is compromised. The green envelope, representing the high-confidence $\alpha = 0.9$ band, typically remains narrow, but it expands modestly during the outage, indicating a measured increase in uncertainty. In contrast, the red $\alpha = 0.5$ band—capturing the system's hesitant and pessimistic scenarios—swells significantly, demonstrating the Choquet-integrated model's ability to reflect elevated risk and provide a worst-case safeguard when sensor reliability is degraded.

We then went a step further and compared AA vs Hamacher envelopes on the same axes. Per- formed Stress-test multiple simultaneous sensor outages, Feed fused AA.csv into a model-predictive controller prototype and arrived at the following results. The black curve represents the hidden true state of the system. The blue line depicts the Acze´l–Alsina (AA) fusion output, which is conjunctive and thus inherently pessimistic, while the cyan line shows the Hamacher fusion, which follows a compensatory, more optimistic strategy. The model's confidence intervals are visualized through envelopes: the dark-green band corresponds to AA with a high-confidence level of α = 0.9, whereas the pale-green band shows the same for Hamacher fusion. Similarly, the dark-red and pale-red bands represent α = 0.5 envelopes for AA and Hamacher, respectively, capturing wider uncertainty due to hesitation. When all sensors are functional, the AA bands closely track the blue line, maintaining tight bounds. However, during sensor outages (marked by yellow regions), the AA envelopes expand significantly, indicating a robust response to missing corroborative input.

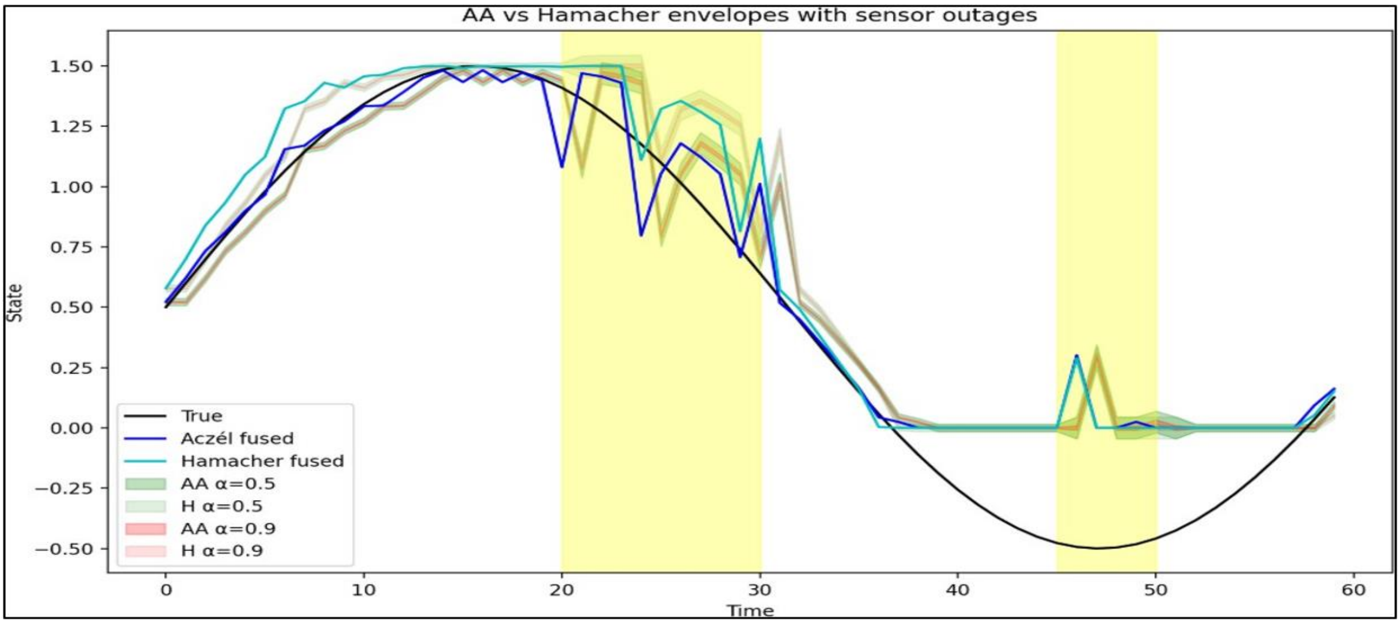

Fig 5 AA vs Hamacher envelopes

This is especially evident during dual-outage periods: between t = 20–30, where sensors 1 and 3 fail, AA envelopes triple in width, whereas Hamacher envelopes only double. A similar pattern occurs between t = 45–50 when sensors 2 and 3 drop out. Throughout both stress intervals, AA remains the more cautious integrator, expanding its envelopes to reflect increased epistemic un- certainty. This behavior highlights its conservative advantage for risk-aware control tasks, which is particularly beneficial in Model-Predictive Controller (MPC) implementations.

V. CONCLUSION

A richer and more adaptable framework for modelling uncertainty is provided by intuitionis- tic fuzzy sets, which are a potent extension of classical fuzzy sets. IFS offers a more thor- ough depiction of ambiguity and incomplete information by specifically taking membership, non-membership, and hesitancy degrees into consideration. By examining the fundamental definitions, algebraic and topological characteristics, and real-world uses of IFS, this paper has shown how ef- fective they are at solving challenging optimisation and decision-making issues. New techniques and insights for utilising intuitionistic fuzzy sets' full potential in a variety of fields should result from more research in this field.

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