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Einstein’s Field Equation Extension for Spherical Fields with Tensor Variation in time and Radial Distance

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Abstract: In this article, we applied the Riemannian Geometry of space-time and obtained the affine connection coefficient, Riemann Christofell tensor, Ricci tensor and an extended Einstein’s field equation for spherical fields. The obtained result reduces to corresponding pure Newtonian indicating that an agreement with the equivalence principle in Physics. It contains correction term that not in the Newton’s dynamical theory or Einstein’s geometrical theory of gravitation. The consequence of the correction term is that it can be applicable to the determination of the existence of gravitational waves.

Keywords: *Riemann Christofell Tensor, Ricci Tensor, Riemannian Theory, Newton’s Theory, Einstein’ Theory*.

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I. INTRODUCTION

Einstein's field equations stand as the cornerstone of modern gravitational theory, describing how matter and energy curve space time, thereby influencing the paths of particles and light. (Davis and Philip, 2006). n standard applications, these equations govern static or stationary gravitational fields around spherically symmetric masses, such as stars or black holes. (*Fock,1966).* however, in scenarios where space time metrics vary both with time and radial distance, a more intricate formulation of these equations becomes necessary. (Rynasiewicz,*2004)*

This article explores the extension of Einstein's field equations to accommodate such variations in a spherically symmetric context. Specifically, it investigates how the metric tensor components, which depend on both radial distance and time contribute to the curvature of space time. These variations introduce a tonsorial complexity, reflecting how gravitational fields evolve dynamically over both spatial and temporal scales (*Rynasiewicz, 2004)*

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Understanding these extensions is crucial for modeling scenarios where gravitational sources are not static or where the gravitational field itself evolves over time. Such scenarios are pertinent in cosmological contexts, during gravitational collapse, or when considering time-dependent phenomena near compact objects. (Davis and Philip, 2006)

This article delves into the mathematical formalism of Einstein's field equations under these conditions, highlighting the coupled partial differential equations that govern the curvature of space time.

By elucidating the theoretical framework and its implications, this article aims to provide a comprehensive overview of Einstein's field equations in the context of spherical symmetry with time and radial distance variations, offering insights into the fundamental nature of gravitational interactions in dynamic space time environments and understanding how gravitational fields evolve over time and in different radial directions is crucial for cosmological models and understanding the large-scale structure of the universe, spherical symmetry is often used to study black holes, where the gravitational collapse and the resulting singularities are analyzed under the framework of general relativity (Lawden,1982).

II. METHODOLOGY

To formulate the exterior field equation, we consider astrophysical body in spherical geometry in which the tensor field varies with time and radial distance. The covariant

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metric tensors for this distribution of mass or pressure is given by (Howusu, 2009; Howusu, 2007)

$$g_{00} = -\left[1 + \frac{2}{c^2} f(t,r)\right] \quad (1)$$

$$g_{11} = [1 + \frac{2}{c^2} f(t,r)]^{-1} \quad (2)$$

$$g_{22} = r^2[1 + \frac{2}{c^2} f(t,r)]^{-1} \quad (3)$$

$$g_{33} = r^2 \sin^2\theta[1 + \frac{2}{c^2} f(t,r)]^{-1} \quad (4)$$

$$g_{\mu\nu} = 0, \text{ Otherwise} \quad (5)$$

where $f(t,r)$ is a gravitational scalar potential determined by mass or pressure and has symmetries of the later/ where, $f(t,r)$ is a gravitational scalar potential, determined by the mass or pressure and possess symmetries of the latter. It is equivalent to Newton's gravitational scalar potential exterior to the spherical mass distribution in approximate gravitational field.

The Golden Riemannian metric tensors in spherical polar coordinate in the form of contravariant metric tensors are given by (Gupta, 2010)

$$g^{00} = -[1 + \frac{2}{c^2} f(t,r)]^{-1} \quad (6)$$

$$g^{11} = [1 + \frac{2}{c^2} f(t,r)] \quad (7)$$

$$g^{22} = \frac{1}{r^2}[1 + \frac{2}{c^2} f(t,r)] \quad (8)$$

$$g^{33} = \frac{1}{r^2 \sin^2\theta}[1 + \frac{2}{c^2} f(t,r)] \quad (9)$$

$$g^{\mu\nu} = 0, \text{ Otherwise} \quad (10)$$

To obtain the affine connection coefficient, the covariant and contravariant metric tensors as well as the affine connection coefficient were used .

$$\Gamma^0_{00} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \quad (11)$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \quad (12)$$

$$\Gamma^0_{11} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \quad (13)$$

$$\Gamma^0_{22} = -\frac{r^2}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \quad (14)$$

$$\Gamma^0_{33} = -\frac{r^2 \sin^2\theta}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-3} \frac{\partial f}{\partial t} \quad (15)$$

$$\Gamma^1_{00} = \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right) \frac{\partial f}{\partial r} \quad (16)$$

$$\Gamma^1_{01} = \Gamma^1_{10} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \quad (17)$$

$$\Gamma^1_{11} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \quad (18)$$

$$\Gamma^1_{22} = -r + \frac{r^2}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \quad (19)$$

$$\Gamma^2_{02} = \Gamma^2_{20} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \quad (20)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} - \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \quad (21)$$

$$\Gamma^3_{03} = \Gamma^3_{30} = -\frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial t} \quad (22)$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} - \frac{1}{c^2}\left(1 + \frac{2}{c^2} f(t,r)\right)^{-1} \frac{\partial f}{\partial r} \quad (23)$$

$$\Gamma^{\mu}_{\alpha\beta} = 0; \text{ Otherwise} \quad (24)$$

Equations (11) - (24) are the affine connection coefficients of the gravitational field. They are fourteen in number unlike that the Schwarzchild's field that are nine in number. It then means that there is an expectation that in this article some features of the gravitational field may not be found in the Scharzchild's field.

The Ricci tensors were obtained as (23) – (34) from the construction of the Riemann-Christoffel tensor for the gravitational field.

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$$R_{00}=\frac{12}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2-\frac{3}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r)}{\partial t^2}-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{2}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r}+\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2$$
(25)

$$R_{11}=\frac{6}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-4}\left(\frac{\partial f}{\partial t}\right)^2+\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\frac{\partial^2 f}{\partial t^2}-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f}{\partial r^2}-\frac{2}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial f}{\partial r}+\frac{4}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f}{\partial r}\right)^2 \quad (26)$$

$$R_{11}=\frac{6}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-4}\left(\frac{\partial f}{\partial t}\right)^2+\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\frac{\partial^2 f}{\partial t^2}-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f}{\partial r^2}-\frac{2}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial f}{\partial r}+\frac{4}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f}{\partial r}\right)^2 \quad (27)$$

$$R_{33}=-\frac{6r^2\sin^2\theta}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-4}\left(\frac{\partial f(t,r)}{\partial t}\right)^2+\frac{r^2\sin^2\theta}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\frac{\partial^2 f(t,r)}{\partial t^2}+\frac{2r^2\sin^2\theta}{c^4}$$

$$\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)-\frac{r^2\sin^2\theta}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{2r\sin^2\theta}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\frac{\partial f(t,r)}{\partial r}+\sin^2\theta \quad (2.28)$$

Equation (4.66) to (4.70) are the Ricci tensors for spherical bodies that their tensor field varies with time and radial distance. spherical bodies whose tensor field varies with time and radial distance. It will also interest you to note that our Ricci tensor R_{22} and R_{33} contains time component which are not found in Chifu and Howusu

From the Ricci tensor, the Riemann scalar, R was constructed by the substitution of the contravariant metric tensor and the Ricci curvature tensor as

$$R=-\frac{30}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2+\frac{6}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\frac{\partial^2 f(t,r)}{\partial t^2}+\frac{4}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2-\frac{2}{c^2}\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{4}{c^2 r}\frac{\partial f(t,r)}{\partial r}+\frac{2}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2+\frac{2}{r^2}\left[1+\frac{2f(t,r)}{c^2}\right] \quad (29)$$

Einstein's gravitational field for a region exterior to the spherical mass distribution was constructed with the Ricci tensor and the Rieman curvature scalar. The field equations are generally as

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=0 \quad (30)$$

Consider the exterior field equation (31)

$$G_{00}=-\frac{2}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial^2 f(t,r)}{\partial r^2}-\frac{4}{c^2 r}\left[1+\frac{2f(t,r)}{c^2}\right]\frac{\partial f(t,r)}{\partial r}+\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial r}\right)^2+\frac{1}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}$$
$$\left(\frac{\partial f(t,r)}{\partial r}\right)^2\frac{2}{c^4}\left(\frac{\partial f(t,r)}{\partial t}\right)^2-\frac{3}{c^4}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial t}\right)^2+\frac{1}{r^2}\left[1+\frac{2f(t,r)}{c^2}\right]^2 \quad (31)$$

$$\nabla^2 f(t,r)-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2-\frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2$$
$$-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2+\frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2-\frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right]=0 \quad (32)$$

Equation (32) reduces to (33) in the weak field limit of order c^0.

$$\nabla^2 f(t,r)=0 \quad (33)$$

Equation (33) agrees with the concept of general relativity, reducing to Laplacian equation and has a gravitational scalar potential of two functions.

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Considering limiting equation (32) to the order $c^{-2,}$ the field equation (32) becomes

$$\nabla^2 f(t,r) - \frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial r}\right)^2 - \frac{1}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-2}\left(\frac{\partial f(t,r)}{\partial r}\right)^2$$

$$-\frac{1}{c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-1}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 + \frac{3}{2c^2}\left[1+\frac{2f(t,r)}{c^2}\right]^{-3}\left(\frac{\partial f(t,r)}{\partial t}\right)^2 - \frac{c^2}{2r^2}\left[1+\frac{2f(t,r)}{c^2}\right] = 0 \quad (34)$$

Seeking a solution of (34) in the form

$$f(t,r) = \sum_{n=0}^{\infty} R_n(r)\exp n\left(t-\frac{r}{c}\right) \quad (35)$$

Where $R_n(r)$ is a function of r only

Now taking the partial derivative of equation (35) twice w.r.t to (r) yields equation (36)

$$\frac{\partial^2 f}{\partial r^2} = R_0^{11}(r) + \left[R_1^{11}(r) - \frac{2}{c}R_1^{1}(r) + \frac{1}{c^2}R_1\right]\exp(t-\frac{r}{c}) + \left[R_2^{11}(r) - \frac{2.2}{c}R_2^{1}(r) + \frac{2^2}{c^2}R_2\right]\exp 2(t-\frac{r}{c}) +$$

$$\left[R_3^{11}(r) - \frac{2.3}{c}R_3^{1}(r) + \frac{3^2}{c^2}R_3\right]\exp 3(t-\frac{r}{c}) \quad (36)$$

Taking the partial derivative of equation (35) w.r.t to (r) yields equation (37)

$$\frac{2}{r}\frac{\partial f}{\partial r} = \frac{2}{r}R_0^{1}(r) + \frac{2}{r}R_1^{1}(r)\exp(t-\frac{r}{c}) - \frac{2}{cr}R_1(r)\exp(t-\frac{r}{c}) + \frac{2}{r}R^1{}_2(r)\exp 2(t-\frac{r}{c}) - \frac{2.2}{cr}R_2(r)\exp 2(t-\frac{r}{c}) + \quad (37)$$

$$\frac{2}{r}R_3^1(r)\exp 3(t-\frac{r}{c}) - \frac{2.3}{cr}R_3(r)\exp 3(t-\frac{r}{c})$$

Again partially differentiating equation (35) w.r.t to (r) and squaring yields equation (38)

$$\frac{1}{c^2}\left(\frac{\partial f}{\partial r}\right)^2 = \frac{1}{c^2}\left(R_0^{1}\right)^2(r) + \frac{1}{c^2}\left(R^1{}_1(r)\right)^2\exp(t-\frac{r}{c}) - \frac{2}{c^3}R^1{}_1(r)R_1(r)\exp 2(t-\frac{r}{c}) +$$

$$\frac{1}{c^2}\left(R^1{}_3\right)^2(r)\exp 6(t-\frac{r}{c}) - \frac{6}{c^3}R^1{}_3(r)R_3(r)\exp 6(t-\frac{r}{c}) + \frac{3}{c^4}R^1{}_3(r)\exp 6(t-\frac{r}{c})$$

$$\frac{1}{c^4}R^2{}_1(r)R_1(r)\exp 2(t-\frac{r}{c}) + \frac{1}{c^2}\left(R^1{}_2\right)^2(r)\exp 4(t-\frac{r}{c}) - \frac{4}{c^3}R^1{}_2(r)R_2(r)\exp 4(t-\frac{r}{c}) + \quad (38)$$

$$\frac{1}{2c^2}\frac{\partial f}{\partial t} = \frac{1}{2c^2}R_1(r)\exp(t-\frac{r}{c}) + \frac{2}{c^2}R_2(r)\exp 2(t-\frac{r}{c}) + \frac{3}{2c^2}R_3(r)\exp 3(t-\frac{r}{c}) \quad (39)$$

Partially differentiating equation (35) w.r.t to (t) and squaring yields

$$\frac{1}{c^2}\left[\frac{\partial f}{\partial t}\right]^2 = \frac{1}{c^2}R_1^2(r)\exp 2(t-\frac{r}{c}) + \frac{4}{c^2}R^2{}_2(r)\exp 4(t-\frac{r}{c}) + \frac{9}{c^2}R^2{}_3(r)\exp 6(t-\frac{r}{c}) \quad (40)$$

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Partially differentiating equation (35) w.r.t to (t) and squaring yields

$$\frac{3}{2c^2}\left[\frac{\partial f}{\partial t}\right]^2 = \frac{3}{2c^2}R_1^2(r)\exp 2(t-\frac{r}{c}) + \frac{6}{c^2}R^2{}_2(r)\exp 4(t-\frac{r}{c}) + \frac{27}{2c^2}R^2{}_3(r)\exp 6(t-\frac{r}{c}) \quad (41)$$

Equating the coefficient of exp (0) we get

$$R_0{}^{11}(r) + \frac{2}{r}R_0{}^1 + \frac{1}{c^2}\left(R_0{}^1\right)^2(r) = 0 \quad (42)$$

To the limit of c^0 equation (42) becomes

$$R_0{}^{11}(r) + \frac{2}{r}R_0{}^1 = 0 \quad (43)$$

Thus, the best astrophysical solution of equation (43) as conveniently chosen as

$$R_0(r) = -\frac{2}{r} \quad (44)$$

According to Newton's dynamical theory, Newton's gravitational scalar potential exterior to a distribution of mass or pressure is given by.

$$f(r) = -\frac{Gm_0}{r} \quad (45)$$

Where,

G = universal gravitational constant

m_0 = total mass of the spherical body

r = Distance of the spherical body

Now comparing equation (44) with equation (45)

We see that the most perfect astrophysical solution for equation (44) is give as

$$R_0 \approx -\frac{k}{r} \quad (46)$$

Then our gravitational scalar potential obtained is given as

$$f(t,r) \approx -\frac{k}{r} \quad (47)$$

Comparing the coefficient of $\exp(t-\frac{r}{c})$ yields

$$R_1{}^{11}(r) + 2\left[\frac{1}{r}-\frac{1}{c}\right]R^1{}_1(r) + \frac{1}{c}\left[1-\frac{2}{r}+\frac{1}{2c}\right]R_1(r) = 0 \quad (48)$$

This is our exact differential equation for R_1 which is our solution for R_0 Thus, the solution in the order of , c^0 , reduces to

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$$f(t,r) \approx -\frac{k}{r}\exp(t - \frac{r}{c}) \qquad (49)$$

III. DISCUSSION

The obtained result in equation (49) differs from that of azimuthal angle only (Chifu and howusu 2009), exterior to a spherical mass with varying potential whose tensor varies with time, radial distance and polar angle (Maisalatee *et al*. 2020) and for a static astrophysical system which varies with radial distance and azimuthal angle only (Sarki *et al* 2018). The fields (such as the metric tensor component) vary with time, indicating dynamical changes in the gravitational field over time and Solutions are considered in spherical symmetry, where the gravitational effects depend on the radial distance from a central mass or source, adhering to the spherical symmetry of the problem. Our function $f(t,r)$ which serves as our most satisfactory physical and mathematical solution, includes unknown post-Newtonian or pure Einsteinian gravitational terms. This research demonstrates that Einstein's geometrical field equations can be viewed as a generalization or completion of Newton's dynamic gravitational field equations.

IV. CONCLUSION

This study has expanded Einstein's field equations to include variations in tensors over time and radial distance within a spherical framework, providing significant insights into gravitational dynamics. By examining the evolution of gravitational fields in different spatial directions and over time, we have revealed new theoretical predictions and interpretations. Our findings enhance the theoretical framework of general relativity, offering fresh perspectives on phenomena such as black hole dynamics and cosmological evolution. While our research offers valuable theoretical insights, further exploration could investigate the implications of these variations in different physical contexts. Overall, this work highlights the importance of extending Einstein's field equations to include nuanced tensor variations, paving the way for a deeper understanding of gravity and its implications for the universe. The results demonstrate that it is possible to extend Einstein's geometrical field equations to predict post-Newtonian and post-Einstein correctional terms of all orders in c^{-2} for the gravitational field of a static homogeneous spherical geometry of space-time.

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