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# Comparison Rate of the Convergence of Single Step and Triple Steps Iteration Schemes

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**Abstract: A fixed point of a function $f: X \to X$ is defined as an element $k \in X$ such that $f(k) = k$. In this study, we analyze fixed point iterative procedures, which are essential for solving equations in various physical formulations. We rigorously establish and compare the convergence and convergence rates of single-step and triple-step iterative schemes with errors in Banach spaces, employing the Zamfirescu operator. Specifically, we demonstrate that for a contraction mapping $T: X \to X$, the sequences generated by these iterative schemes converge to a unique fixed point $p \in X$. Additionally, we explore the existence and stability of Mann iterations defined by the iterative scheme $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n)$ and Noor iterations given by $x_{n+1} = (1 - \beta_n)x_n + \beta_n T(T((x_n))$, where $\alpha_n$, $\beta_n$ are appropriate step sizes. Our results not only elucidate the effectiveness of these iterative methods but also contribute to the broader understanding of fixed point theory in Banach spaces.**

***Keywords:*** *Fixed Point; Banach Space; Convergence; Normed Space; Metric Space; Banach Fixed Point.*

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## I. INTRODUCTION

The concept of a fixed point is a fundamental aspect of mathematics, with widespread applications in various fields, including analysis, topology, and applied mathematics. A fixed point of a function f is defined as a point k in the function's domain such that $f(k) = k$. This self-mapping property, often referred to as an invariant point, is crucial in understanding the behavior of dynamical systems and iterative processes. Fixed point theory is not merely a theoretical concept; it serves as a vital tool for solving equations, optimizing functions, and analyzing stability. Berinde (2007) emphasizes the importance of fixed point theory in iterative methods, which are algorithms designed to find such points, underscoring their application in numerical analysis and computational tasks.

In computational mathematics, iterative methods generate sequences of approximate solutions to complex problems. These methods rely on initial guesses to produce successive approximations, with convergence to a fixed point being essential for guaranteeing a reliable solution. In contrast to direct methods that provide solutions through a finite sequence of operations, iterative methods are often the only viable approach for nonlinear equations and large systems where direct solutions would be computationally prohibitive. The convergence of these methods is closely tied to fixed point theory, particularly illustrated by the Banach fixed point theorem. This theorem asserts that under specific conditions, a contraction mapping will have a unique fixed point, a finding that has broad implications in mathematical analysis and numerical methods.

Recent advancements have broadened the scope of fixed point theory, introducing alternative iterative procedures when traditional methods fail. For example, in situations where contraction mappings are not applicable, techniques such as Mann and Ishikawa iterations have emerged, providing robust alternatives for achieving convergence in Banach spaces (Chidume, 1994; Koti et al., 2013). Furthermore, Agarwal et al. (2007) introduced the S-iteration process, a novel approach that operates independently of conventional methods, thereby enriching the field with new strategies for solving fixed point problems.

In addition to these techniques, numerous researchers have explored the applicability of fixed point theory in various contexts, such as nonlinear differential equations, dynamic programming, and game theory. Fixed point methods are also used in economics to analyze equilibrium states and in computer science for algorithm convergence. The ongoing advancements in fixed point theory illustrate the dynamic interplay between mathematical theory and computational practice, highlighting the relevance of fixed points in both theoretical and applied contexts.

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This review aims to synthesize recent developments in fixed point theory, particularly focusing on the evolution of iterative methods and their applications in solving complex mathematical problems. By examining both classical and contemporary approaches, this work seeks to emphasize the significance of fixed point theory in today's mathematical landscape and pave the way for future innovations and applications iterations.

## II. PRELIMINARIES

The Picard iteration method is a fundamental approach for identifying fixed points in metric spaces. Consider a metric space $(X,d)$ where $D \subset X$ is a closed subset, typically assumed to be $D = X$. Given a self-map $T: D \rightarrow D$ that has at least one fixed point $p \in FT$, we define a sequence of iterates $\{x_n\}_{n=0}^{\infty}$ as follows:

$$x_{n+1} = T(x_n) = T_n(x_0), n \in N \quad (1)$$

Our goal is to determine conditions on $T, D$, and $X$ that ensure the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to a fixed point of $T$ in $D$. The concept of Picard iteration with an error term is given by:

$$x_{n+1} = T(x_n) + U_n, n \in N \quad (2)$$

where the error sequence $\{U_n\}$ satisfies $\sum_{n-1}^{\infty} \|U_n\| < \infty$

Definition: In a metric space $(X,d)$, a map $T: X \rightarrow X$ is defined as a contraction mapping if there exists a constant $p \in (0,1)$ such that:

$$d\big(T(x), T(y)\big) \leq p.d(x,y) \quad (3)$$

for all $x, y \in X$ (Banach, 1922).

**Banach Fixed Point Theorem**: For any non-empty complete metric space $(X,d)$ and a contraction mapping $T: X \rightarrow X$, there exists a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$. Starting from any element $x^0 \in X$, the sequence $\{x_n\}$ defined by

$$x_n = T(x_{n-1}) \quad (4)$$

will converge to x ∗ (Banach, 1922).

In cases where the contraction condition is not strictly met, convergence of Picard iterations may not be achieved. Alternative iteration techniques such as Mann, Ishikawa, S-iteration, Thianwan, and Noor iterations can be utilized (Ishikawa, 1974).

The Mann iteration process begins with an initial point $x_0 \in E$ and follows:

$$x_{n-1} = (1 - a_n)x_n + a_n T x_n, n = 0,1,2, \dots \quad (5)$$

where the parameters $\{a_n\}$ lie within [0, 1] and satisfy certain conditions. The Mann iteration with error is given by:

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n + U_n, n = 0, 1, 2, \dots \quad (6)$$

with $U_n$ meeting the condition $P\|U_n\| < \infty$ (Mann, 1953).

The Ishikawa iteration, initially formulated to establish strong convergence for specific mappings in a Hilbert space, is structured as:

$$x_{n+1} = (1 - a_n)x_n + a_n T[(1 - b_n)x_n + b_n T x_n], \ n \in N \quad (7)$$

where $\{a_n\}, \{b_n\} \rightarrow 0$ within [0, 1] (Ishikawa, 1974). Interpreting Ishikawa as a two-step Mann iteration allows for distinct parameter sequences. Notably, when $b_n = 0$, Ishikawa iteration reduces to Mann iteration, although their convergence results are generally not equivalent.

To address computational limitations, Agarwal et al. (2007) proposed an S-iteration method applicable to contraction and non-expansive mappings, shown to be more efficient than Picard, Mann, and Ishikawa iterations:

$$\begin{cases} x_{n+1} = (1 - a_n)T x_n + a_n T y_n \\ y_n = (1 - b_n)x_n + b_n T x_n \end{cases}, n \in N \quad (8)$$

The Thianwan iteration process was defined independently, characterized by:

$$\begin{cases} x_{n+1} = (1 - a_n)y_n + a_n T y_n \\ y_n = (1 - b_n)x_n + b_n T x_n \end{cases}, n \in N \quad (9)$$

where $\{a_n\}$ and $\{b_n\}$ are positive sequences within [0, 1], ensuring $\sum a_n = \infty$.

Finally, Chugh et al. (2014) developed a Noor iteration scheme, known for its three-step process:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n \\ y_n = (1 - b_n)x_n + b_n T z_n \\ z_n = (1 - c_n)x_n + c_n T x_n \end{cases}, n \in N \quad (10)$$

where sequences $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ and their respective error terms $\{U_n\}, \{V_n\}, \{W_n\}$ fulfill the conditions $\sum_{n=1}^{\infty} \|U_n\| < \infty, \sum_{n=1}^{\infty} \|V_n\| < \infty, \sum_{n=1}^{\infty} \|W_n\|$ (Chugh et al., 2014).

This section provides a foundation of iterative techniques, with each method contributing uniquely to fixed-point approximation across a range of operators within Banach spaces.

## III. RESULTS AND DISCUSSION

In this section, we study the convergence and the rate of convergence of three iterative schemes. We present our main results using Z-operators in a Banach space for the

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iterative schemes with errors, and at the end, we tabulate computational results for the rate of convergence of the Mann and Noor iterations.

The results of the schemes are given in the following theorems:

Let $F$ be a non-empty subset of a normed linear space $(E, \|\cdot\|)$. Let $K: F \to F$ be a self-map of $F$ satisfying the Zamfirescu operator for each$x, y \in F$. Let $\{x_n\}$ be the Mann iterative scheme defined by

$$x_{n+1} = (1 - a_n)x_n + a_n K(x_n) + v_n. \quad (11)$$

If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n = \infty$, and $\|V_n\| = 0$, then:

(i) The mapping $T$ has a unique fixed point $p$;
(ii) The Mann iterative scheme converges strongly to a fixed point of $T$.

*Proof*. (i) We first establish that the mapping $T$ has a unique fixed point. Suppose there exist $p_1, p_2 \in FT$ such that $p_1 \neq p_2$ and $\|p_1 - p_2\| > 0$. Then,

$$0 < \|p_1 - p_2\| = \|Tp_1 - Tp_2\| \leq a\|p_1 - p_2\| + 2a\|p_1 - Tp_2\|.$$

This implies

$$\|Tp_1 - Tp_2\| \leq a\|p_1 - p_2\|.$$

Thus,

$$(1 - a)\|p_1 - p_2\| \leq 0.$$

Since $a \in [0,1]$, we have $1 - a > 0$, which leads to $\|p_1 - p_2\| = 0\|$.

Therefore,

$$p_1 = p_2 = p.$$

Hence, $T$ has a unique fixed point $p$.
(iii) Next, we establish that $\lim_{n\to\infty} x_n = p$, showing that the Mann iterative scheme converges strongly to $p$.

From the iterative scheme, we have:

$$\|x_{n+1} - p\| = \|(1 - a_n)x_n + a_n K(x_n) - p + V_n\|$$

$$= (1 - a_n)\|x_n - p\| + a_n\|K(x_n - p)\| + \|V_n\|. \quad (12)$$

Now, using the properties of the mapping $K$, we can deduce that:

$$\|x_{n+1} - p\| \leq (1 - a_n)\|x_n - p\| + a_n\delta\|x_n - p\| + \|V_n\|, \quad (13)$$

leading to:

$$\|x_{n+1} - p\| \leq [1 - (1 - \delta)a_n]\|x_n - p\| + \|V_n\|. \quad (14)$$

Given the assumptions of the theorem and the condition$\|V_n\| = 0$, we find that:

$$\lim_{n\to\infty} \|x_{n+1} - p\| = 0. \quad (15)$$

Thus, the Mann iterative scheme converges strongly to $p$.

Let $F$ be a non-empty subset of a normed linear space $(E, \|\cdot\|)$. Let $K: F \to F$ be a self-map of $F$ satisfying the Zamfirescu operator for each $x, y \in$x, y ∈ F. Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iterative scheme defined by

$$x_{n+1} = (1 - a_n)x_n + a_n K(y_n) + U_n. \quad (16)$$

If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n = \infty$, and $\|V_n\| = 0$, then:

(i) The mapping $T$ has a unique fixed point $p$;
(ii) The Noor iterative scheme converges strongly to a fixed point of $T$.

*Proof*. (i) We first establish that the mapping $T$ has a unique fixed point. Suppose there exist $p_1, p_2 \in F_T$ such that $p_1 \neq p_2$ and $\|p_1 - p_2\| > 0$. Then,

$$0 < \|p_1 - p_2\| = \|Tp_1 - Tp_2\| \leq 2a\|p_1 - Tp_2\| + a\|p_1 - p_2\|.$$

This implies

$$\|Tp_1 - Tp_2\| \leq a\|p_1 - p_2\|.$$

Thus,

$$(1 - a) > 0 \text{ and } \|p_1 - p_2\| \leq 0.$$

Since the norm is non-negative, it follows that $\|p_1 - p_2\| = 0$. Therefore, $p_1 = p_2 = p$.

Hence, $T$ has a unique fixed point $p$.

(iii) Next, we establish that $\lim_{n\to\infty} x_n = p$, demonstrating that the Noor iterative scheme converges strongly to $p$. From the iterative scheme, we have:

$$\|K(x_n) - p\| \leq \delta\|x_n - p\| + 2\delta\|x_n - K(x_n)\| \forall x, y \in C. \quad (17)$$

Applying this to the Noor scheme, we get:

$$\|x_{n+1} - p\|) \leq \|(1 - a_n)x_n + a_n K(x_n) - p + V_n\|$$

$$\leq (1 - a_n)\|x_n - p\| + a_n\|K(x_n) - p\| + \|V_n\|$$

$$\leq (1 - a_n)\|x_n - p\| + a_n\delta\|x_n - p\| + \|V_n\|$$

$$\leq [(1 - a_n) + a_n\delta]\|x_n - p\| + \|V_n\|. \quad (18)$$

Thus, $\lim_{n\to\infty} \|x_{n+1} - p\| = 0$ as $n \to \infty$. Therefore, the Noor iterative scheme converges strongly to $p$.

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## IV. NUMERICAL EXAMPLE

In this section, we use the following example to compare our iterative schemes with the help of MATLAB. The function defined by $f(x) = (1-x)^2$ is a function with multiple zeros. By taking the initial approximation $x_0 = 0.9$ and $\alpha_n = \beta_n = \frac{1}{2}$ , the convergence of these iterative schemes to the exact fixed point $p = 0.38196601$is shown in Table 1 below. The error estimation $\epsilon_n$ in Table 1, choosing $n$ at intervals of 5 iterations, is presented in Table 2, where $\epsilon_n = \|q - x_n\|$.

## V. CONCLUSION

In conclusion, our analysis has demonstrated that the Banach fixed point theorem is inherently constructive. Various methods can be employed to achieve results, with the choice of method dependent on the specific nature of the problem and the desired outcomes that satisfy the conditions of contraction mappings.

The Banach fixed point theorem has extensive applications across various branches of mathematical science. In this study, we have successfully established the convergence and convergence rates of several iterative schemes with error estimations using the Zamfirescu operator in Banach spaces. Our findings indicate that the Mann iterative method converges faster than the Noor iteration concerning the convergence rate. This insight underscores the importance of selecting appropriate iterative techniques based on their performance characteristics, thus enhancing the efficiency of solving fixed-point problems in practical applications.

Table 1 The Results for Various Iterations of the Example.

| N | Mann Iteration | Noor Iteration |
|---|---|---|
| 0 | 0.90000000 | 0.90000000 |
| 1 | 0.45500000 | 0.53059611 |
| 2 | 0.37601250 | 0.41092632 |
| 3 | 0.38268645 | 0.38648213 |
| 4 | 0.38188123 | 0.38263205 |
| 5 | 0.38197602 | 0.38396925 |
| 6 | 0.38196483 | 0.38225969 |
| 7 | 0.38196615 | 0.38200889 |
| 8 | 0.38196599 | 0.38197227 |
| 9 | 0.38196601 | 0.38196692 |
| 10 | 0.38196601 | 0.38196614 |
| 11 | 0.38196601 | 0.38196603 |
| 12 | 0.38196601 | 0.38196601 |
| 13 | 0.38196601 | 0.38196601 |
| 14 | 0.38196601 | 0.38196601 |
| 15 | 0.38196601 | 0.38196601 |

Table 2 The result error estimation for various iterations of the example.

| N | Mann Iteration | Noor Iteration |
|---|---|---|
| 0 | 0.00001001 | 0.00200323 |
| 1 | 0.00000000 | 0.00000013 |
| 2 | 0.00000000 | 0.00000000 |
| 3 | 0.00000000 | 0.00000000 |
| 4 | 0.00000000 | 0.00000000 |
| 5 | 0.00000000 | 0.00000000 |

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