Mathematical Modelling and Optimal Control Analysis of Divorce Dynamics Using Caputo Fractional Derivative

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Publication Date: 2025/12/06

Abstract: In this paper, we present a rigorous mathematical analysis of a novel fractional-order dynamical model describing the evolution of a married couple's relationship, with a specific focus on the pathway to divorce. We extend a classical three-state model (happy marriage, unhappy marriage, divorce) to the Caputo fractional-order framework to incorporate memory effects and hereditary traits, which are intrinsic to human interactions. We established the model's well-posedness by proving the existence, uniqueness, non-negativity as well as boundedness of solutions in a biologically feasible region. We perform a detailed stability analysis of the equilibrium points, deriving a novel threshold parameter R_0 and proving local stability using Matignon's conditions. Our work is anchored in the formulation and solution of an optimal control problem, where we introduce two time-dependent control measures: one aimed at improving the relationship quality, for instance, counseling and another to prevent reconciliation from a state of divorce. We used Pontryagin's Maximum Principle for fractional-order systems to derive the necessary conditions for optimality and characterize the optimal controls. We performed numerical simulations, utilizing the forward-backward sweep method with the Grünwald-Letnikov approximation to illustrate the dynamics of the uncontrolled and controlled systems. Our results quantitatively demonstrate that our proposed optimal control strategy can significantly reduce the number of divorces as well as increase the proportion of happy marriages. Thus, providing a mathematical basis for targeted interventional policies. The expanded analysis provides complete proof for all technical components, offering a robust foundation for further research in socio-mathematical modelling.

Keywords: *Fractional-Order Calculus, Optimal Control, Divorce Dynamics, Caputo Derivative, Pontryagin's Maximum Principle, Stability Analysis, Fixed Point Theorem.*

How to Cite: Davidon Jani; Senzenia Chakauya; Alice Chimhondoro; Ever Moyo; Faith Chiwungwe (2025) Mathematical Modelling and Optimal Control Analysis of Divorce Dynamics Using Caputo Fractional Derivative.
International Journal of Innovative Science and Research Technology, 10(11), 2690-2708.
https://doi.org/10.38124/ijisrt/25nov870

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

I. INTRODUCTION

Mathematical modelling plays a pivotal role in understanding complex social and psychological phenomena. In recent decades, the application of dynamic systems theory to marital interactions, pioneered by Gottman et al. [1], has provided valuable insights into the nonlinear dynamics that can lead to relationship dissolution. Such mathematical models typically use ordinary differential equations (ODEs) to capture the evolution of emotional states between partners, often revealing tipping points and basins of attraction that dictate the long-term fate of a relationship.

However, classical ODE models possess a fundamental limitation: they are *memoryless* (Markovian). That is, the rate of change of a state depends only on its current value, not on its history. This is often an unrealistic assumption for human relationships, where past interactions, accumulated grievances, and the history of intimacy (the "memory" of the relationship) profoundly influence present dynamics. An argument from a decade ago can resurface and impact a couple's interaction today. Fractional-order differential equations (FODEs), with their non-local operators, are exceptionally well-suited to model such memory and hereditary properties [2, 3]. The Caputo fractional derivative has become a standard tool in modeling biological and social processes due to its compatibility with physically interpretable initial conditions [4]. Its incorporation into epidemiological [5, 6] and ecological [7] models is now widespread. Yet, its application to socio-psychological models like marital dynamics remains relatively unexplored, representing a significant gap in literature.

To bridge this gap, we therefore proposed and analysed a fractional-order divorce model. We consider a population of married couples categorized into three compartments, extending the classical work of [1] and others [8] into the fractional domain. The model incorporates transitions between these states, influenced by factors like the natural deterioration of happiness, efforts to salvage an unhappy marriage, and the possibility of reconciliation after separation. Beyond mere description, a critical question is: how can we optimally intervene to foster healthy marriages and reduce divorce rates? This leads to the field of optimal control theory [9]. As a result, we formulated an optimal control problem by introducing two time-variant control measures, thus, drawing inspiration from control strategies in epidemiology [10] but applied to a novel social context.

The primary objective of our work is to minimize the number of unhappy marriages and divorces over a specified time horizon while simultaneously minimizing the cost associated with implementing the control measures. The paper's main contributions are: (i) the formulation of a novel fractional-order divorce model; (ii) a comprehensive mathematical analysis establishing its well-posedness and stability properties; (iii) the formulation and analytical solution of an associated optimal control problem and (iv) numerical simulations to validate theoretical results and demonstrate the efficacy of the proposed control strategy.

Our paper is organised as follows: In Section 2, we present the fractional-order model and its underlying assumptions. Section 3 is dedicated to the analysis of the proposed mathematical model, establishing the existence, uniqueness, non-negativity and boundedness of solutions and analyzing the stability of equilibrium points. Section 4 formulates the optimal control problem and derives the necessary optimality conditions using Pontryagin's Maximum Principle. In Section 5, we present and discuss the numerical results for both the uncontrolled and controlled systems. Finally, Section 6 concludes the paper with a summary of our findings and directions for future research.

II. MODEL FORMULATION

We begin by defining the Caputo fractional derivative, which is the cornerstone of our model. Its non-local nature is key to capturing the memory effects in marital interactions.

Definition 1. [Caputo Fractional Derivative [2]

The Caputo fractional derivative of order $\alpha \in (0,1]$ for a function $f \in C^1([t_0,T],R)$ is defined as:

$$^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} f'(\tau) d\tau, \tag{1}$$

Where $\Gamma(\cdot)$ is the Gamma function.

➢ *Population Subdivision and Model Structure*

The mathematical modelling of marital dynamics requires a nuanced compartmental structure that captures the qualitative aspects of relationship satisfaction alongside traditional marital status categories. Building upon the foundational work of Gottman et al. [11] in quantifying marital interaction patterns, and extending the classical Single-Married-Divorced (SMD) framework [12], we propose a more psychologically informed subdivision of the population into three distinct compartments:

- Happily Married (H): Individuals in satisfying, stable marriages characterized by positive communication patterns, emotional intimacy and low conflict resolution efficacy. This compartment represents relationships with high resilience to external stressors and intrinsic satisfaction.
- Unhappily Married (U): Individuals in distressed marriages exhibiting negative communication patterns, emotional disengagement and high conflict. This state represents relationships at elevated risk of dissolution, consistent with the marital distress continuum identified in clinical psychology literature [13].
- Divorced (D): Individuals who have legally dissolved their marriages, representing a state of marital transition with varying propensities for remarriage based on previous marital experiences.

This tripartite division acknowledges the critical insight from relationship science that marital quality, rather than mere marital status, serves as the primary determinant of

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

relationship stability and dissolution [11]. The transition from happy to unhappy marriages captures the gradual deterioration process central to many marital failures, while the separation of divorced individuals allows modelling the differential remarriage patterns based on previous marital experiences.

The exclusion of a dedicated "Single" compartment in this formulation reflects our focus on marital dynamics rather than initial pair formation. We assume a constant total population normalized to unity, with the three compartments satisfying the conservation law $H(t) + U(t) + D(t) = N$ for all $t \geq 0$. This structure enables us to investigate how interventions might not only prevent divorce but also promote marital happiness, which is a crucial distinction with significant implications for individual well-being and social policy. The Caputo derivative is particularly useful in this work as it allows for standard initial conditions, for instance, $H(0), U(0), D(0)$, unlike the Riemann Liouville definition. Based on the compartmental structure and the flow diagram depicted in Figure 1, we propose the following system of nonlinear FODEs:

$$\begin{aligned} {}^{C}\mathcal{D}_t^{\alpha} H(t) &= \Lambda - \beta H(t)U(t) + \gamma U(t) - \mu H(t) + \omega D(t), \\ {}^{C}\mathcal{D}_t^{\alpha} U(t) &= \beta H(t)U(t) - \gamma U(t) - \delta U(t) - \mu U(t), \\ {}^{C}\mathcal{D}_t^{\alpha} D(t) &= \delta U(t) - \omega D(t) - \mu D(t), \end{aligned} \quad (2)$$

Subject to the initial conditions: $H(0) = H_0 \geq 0, U(0) = U_0 \geq 0, D(0) = D_0 \geq 0$ with $H_0 + U_0 + D_0 = 1$.

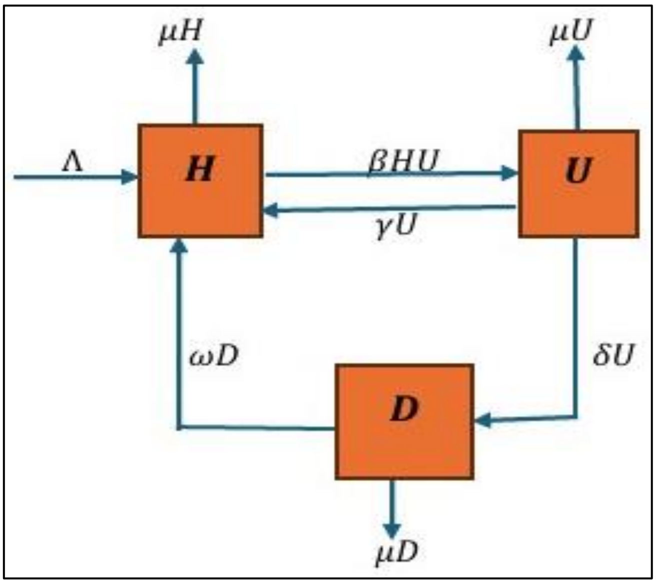

Fig 1 Schematic Diagram of the Fractional-Order Divorce Model.

Table 1 below shows model parameters with their biological interpretations.

Table 1 Model Parameters and their Biological Interpretations for the Fractional-Order Divorce Model.

Parameter	Description
$\alpha \in (0,1]$	Order of the fractional derivative. Smaller α indicates stronger memory effects in marital dynamics.
Λ	Recruitment rate into happy marriages (new marriages).
β	Rate of unhappiness transmission from unhappy to happy couples.
γ δ ω μ	Recovery rate from unhappy to happy state through relationship efforts. Divorce rate for unhappy couples. Reconciliation rate from divorced happy state. Marriage dissolution rate due to external factors (equal across compartments).

The total population is constant, normalized to 1. Therefore, $H(t) + U(t) + D(t) = 1$ for all $t \geq 0$. This is consistent with the model, as adding the three equations gives ${}^{C}D^{\alpha}_{t}(H+U+D) = \Lambda - \mu(H + U + D)$. If $\Lambda = \mu$, the total population remains constant. We will assume $\Lambda = \mu$ for the remainder of this paper. In addition, all parameters $\Lambda, \beta, \gamma, \delta, \omega, \mu$ are positive constants.

III. MATHEMATICAL ANALYSIS OF THE DIVORCE MODEL

A. Existence and Uniqueness of Solutions

To prove that our divorce model is mathematically and epidemiologically well-posed, we investigate the existence and uniqueness of solutions. We first define a feasible region for the dynamics.

➢ *Lemma 1: Feasible Region*

The region $\Omega = \{(H, U, D) \in \mathbb{R}_+^3 : H + U + D = 1$ is positively invariant for the system (2).

➢ *Proof*

From our first assumption, we have:

$$ {}^{C}\mathcal{D}_t^{\alpha}(H + U + D) = \Lambda - \mu(H + U + D). $$

Given $\Lambda = \mu$ and $H_0 + U_0 + D_0 = 1$, the solution to this fractional differential equation is $H(t) + U(t) + D(t) = 1$ for all $t \geq 0$ [3]. Furthermore, on each bounding hyperplane of $\mathbb{R}_+^3$, the vector field points inward. For instance, if $H = 0$, then from (2a), ${}^{C}D^{\alpha}_{t} H = \Lambda + \gamma U + \omega D > 0$. Similarly, if $U = 0$, ${}^{C}D_t^{\alpha} U = 0$ and if $D = 0$,

${}^{C}\mathcal{D}_t^{\alpha} D = \delta U \geq 0$. Thus, any solution starting in Ω remains in Ω for all $t > 0$.

We now prove the existence and uniqueness of the solution using the fixed-point theorem approach.

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

➢ *Theorem 1: Existence and Uniqueness*

For any initial condition $X_0 = (H_0, U_0, D_0) \in \Omega$, there exists a unique solution $X(t) = (H(t), U(t), D(t))$ of the system (2) in Ω for all $t \geq 0$. Proof

The system (2) is equivalent to the following Volterra-type integral equation:

$$X(t) = X_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} F\big(X(s)\big)ds \quad (3)$$

Where $\boldsymbol{F}(\boldsymbol{X}) = \big(F_1(\boldsymbol{X}), F_2(\boldsymbol{X}), F_3(\boldsymbol{X})\big)^T$ is the vector field defined by the right-hand sides of Equation (2).

We define an operator $\boldsymbol{P} : C([0,T], R^3) \to C([0,T], R^3)$ by:

$$(\mathbf{P}\mathbf{X})(t) = \mathbf{X}_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathbf{F}(\mathbf{X}(s))ds. \quad (4)$$

To show that P is a contraction, we need to show that F is Lipschitz continuous. Let $X = (H_X, U_X, D_X)$ and $Y = (H_Y, U_Y, D_Y)$ be two solution vectors in Ω. Consider the sup norm $\| X \|_\infty = \sup_{t\in[0,T]} (|H(t)| + |U(t)| + |D(t)|)$. Since all state variables are bounded in Ω (they are proportions between 0 and 1), there exists a positive constant M such that $|H(t)| \leq M, |U(t)| \leq M, |D(t)| \leq M$ for all $t \in [0,T]$. In fact, $M = 1$.

Now, we analyze the difference component-wise.

For F_1:

$$F_1(\boldsymbol{X}) - F_1(\boldsymbol{Y}) = [-\beta H_X U_X + \beta H_Y U_Y] + \gamma (U_X - U_Y) - \mu (H_X - H_Y) + \omega (D_X - D_Y).$$

Note that $H_X U_X - H_Y U_Y = H_X (U_X - U_Y) + U_Y (H_X - H_Y)$.

Therefore,

$$|F_1(\boldsymbol{X}) - F_1(\boldsymbol{Y})| \leq \beta|H_X (U_X - U_Y) + U_Y (H_X - H_Y)| + \gamma|U_X - U_Y| + \mu|H_X - H_Y| + \omega|D_X - D_Y|$$

$$\leq \beta M|U_X - U_Y| + \beta M|H_X - H_Y| + \gamma|U_X - U_Y| + \mu|H_X - H_Y| + \omega|D_X - D_Y|$$

$$= (\beta M + \mu)|H_X - H_Y| + (\beta M + \gamma)|U_X - U_Y| + \omega|D_X - D_Y|.$$

Let $L_{11} = \beta M + \mu, L_{12} = \beta M + \gamma$ and $L_{13} = \omega$. Then:

$$|F_1(\boldsymbol{X}) - F_1(\boldsymbol{Y})| \leq L_{11}|H_X - H_Y| + L_{12}|U_X - U_Y| + L_{13}|D_X - D_Y|. \quad (5)$$

For F_2:

$$F_2(\boldsymbol{X}) - F_2(\boldsymbol{Y}) = \beta H_X U_X - \beta H_Y U_Y - (\gamma + \delta + \mu)(U_X - U_Y)$$

$$= \beta [H_X (U_X - U_Y) + U_Y (H_X - H_Y)] - (\gamma + \delta + \mu)(U_X - U_Y)$$

$$|F_2(\boldsymbol{X}) - F_2(\boldsymbol{Y})| \leq \beta M|U_X - U_Y| + \beta M|H_X - H_Y| + (\gamma + \delta + \mu)|U_X - U_Y|$$

$$= \beta M|H_X - H_Y| + (\beta M + \gamma + \delta + \mu)|U_X - U_Y|.$$

Let $L_{21} = \beta M, L_{22} = \beta M + \gamma + \delta + \mu$ and $L_{23} = 0$. Then:

$$|F_2(\boldsymbol{X}) - F_2(\boldsymbol{Y})| \leq L_{21}|H_X - H_Y| + L_{22}|U_X - U_Y| \quad (6)$$

For F_3:

$$F_3(\boldsymbol{X}) - F_3(\boldsymbol{Y}) = \delta (U_X - U_Y) - (\omega + \mu)(D_X - D_Y).$$

$$|F_3(\boldsymbol{X}) - F_3(\boldsymbol{Y})| \leq \delta|U_X - U_Y| + (\omega + \mu)|D_X - D_Y|.$$

Let $L_{31} = 0, L_{32} = \delta$, and $L_{33} = \omega + \mu$. Then:

$$|F_3(\boldsymbol{X}) - F_3(\boldsymbol{Y})| \leq L_{32}|U_X - U_Y| + L_{33}|D_X - D_Y|. \quad (7)$$

Now, combining the three components:

$$\| \boldsymbol{F}(\boldsymbol{X}) - \boldsymbol{F}(\boldsymbol{Y}) \|_\infty = \sup_{t\in[0,T]} (|F_1(\boldsymbol{X}) - F_1(\boldsymbol{Y})| + |F_2(\boldsymbol{X}) - F_2(\boldsymbol{Y})| + |F_3(\boldsymbol{X}) - F_3(\boldsymbol{Y})|)$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

$$\leq \sup_{t\in[0,T]} [(L_{11} + L_{21})|H_X - H_Y| + (L_{12} + L_{22} + L_{32})|U_X - U_Y| + (L_{13} + L_{33})|D_X - D_Y|]$$

$$\leq L \sup_{t\in[0,T]} (|H_X - H_Y| + |U_X - U_Y| + |D_X - D_Y|)$$

$$= L \parallel \boldsymbol{X} - \boldsymbol{Y} \parallel_\infty,$$

Where $L = max\{L_{11} + L_{21}, L_{12} + L_{22} + L_{32}, L_{13} + L_{33}\} = max\{2\beta M + \mu, 2\beta M + 2\gamma + \delta + \mu, 2\omega + \mu\}$. Thus, F is Lipschitz continuous with Lipschitz constant L.

Now, for the operator P:

$$\begin{aligned}\|(\mathbf{PX})(t) - (\mathbf{PY})(t)\|_\infty &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathbf{F}(\mathbf{X}(s)) - \mathbf{F}(\mathbf{Y}(s))\|_\infty ds \\ &\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathbf{X}(s) - \mathbf{Y}(s)\|_\infty ds \\ &\leq \frac{L}{\Gamma(\alpha)}\|\mathbf{X} - \mathbf{Y}\|_\infty \int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{LT^\alpha}{\Gamma(\alpha+1)}\|\mathbf{X} - \mathbf{Y}\|_\infty.\end{aligned}$$

Therefore,

$$\parallel PX - PY \parallel_\infty \leq \frac{LT^\alpha}{\Gamma(\alpha+1)} \parallel X - Y \parallel_\infty$$

For T sufficiently small such that $\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$, the operator $\boldsymbol{P}$ is a contraction. By Banach's fixed-point theorem, a unique solution exists on $[0, T]$. This process can be repeated to extend the solution for all $t \geq 0$, ensuring global existence and uniqueness in Ω.

B. Equilibrium Points and Stability Analysis

The system has two equilibrium points: the Divorce-Free Equilibrium (DFE) and the Endemic Divorce Equilibrium (EDE). To find them, we set ${}^{C}D^{\alpha}_{t} H = {}^{C}D^{\alpha}_{t} U = {}^{C}D^{\alpha}_{t} D = 0$.

➢ *Divorce-Free Equilibrium (DFE)*

DFE entails a state where no unhappy marriages or divorces exist in the long run. Set $U = 0$ and $D = 0$ in the model. From equation (2a), we get $\Lambda - \mu H = 0$, so $H = \frac{\Lambda}{\mu}$. Since the total population is 1 ($\Lambda = \mu$), we have:

$$E_0 = (1,0,0). \tag{8}$$

➢ *Endemic Divorce Equilibrium (EDE)*

This is a state where divorce persists. Denote it as $E^* = (H^*, U^*, D^*)$, where $U^* > 0$ and $D^* > 0$. Set the right-hand sides of (2) to zero:

$$0 = \Lambda - \beta H^* U^* + \gamma U^* - \mu H^* + \omega D^*, \tag{9}$$

$$0 = \beta H^* U^* - (\gamma + \delta + \mu) U^*, \tag{10}$$

$$0 = \delta U^* - (\omega + \mu) D^*. \tag{11}$$

From (10), since $U^* = 0$, we get:

$$H^* = \frac{\gamma + \delta + \mu}{\beta}. \tag{12}$$

From (11):

$$D^* = \frac{\delta}{\omega + \mu} U^*. \tag{13}$$

Now, from the conservation equation $H^* + U^* + D^* = 1$:

$$\frac{\gamma + \delta + \mu}{\beta} + U^* + \frac{\delta}{\omega + \mu} U^* = 1.$$

Solving for U^*:

$$U^*\left(1 + \frac{\delta}{\omega + \mu}\right) = 1 - \frac{\gamma + \delta + \mu}{\beta},$$

$$U^* = \frac{1 - \frac{\gamma+\delta+\mu}{\beta}}{1 + \frac{\delta}{\omega+\mu}} = \frac{\beta - (\gamma + \delta + \mu)}{\beta\left(1 + \frac{\delta}{\omega+\mu}\right)}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

For U^* to be positive, we require $\beta > (\gamma + \delta + \mu)$, which leads to the definition of the basic reproduction number for our model.

C. Derivation of the Basic Divorce Reproduction Number, R_0

The basic divorce reproduction number, R_0 is a critical threshold parameter that determines whether divorce will persist in the population. This number represents the average number of new unhappy marriages generated by one unhappy couple introduced into a wholly happy population during their entire period of unhappiness. We derive R_0 using the Next Generation Matrix method [14]. In our divorce model, the "infected" compartments are those that can spread unhappiness through social interactions. These are:

- $U(t)$: Unhappy marriages (directly spread unhappiness)
- $D(t)$: Divorced couples (indirectly affect marriage dynamics)

Similarly, the "uninfected" compartment is:

- $H(t)$: Happy marriages (susceptible to becoming unhappy)

The equations for the infected compartments from system (2) are:

$$^{C}D_t^{\alpha}\, U(t) = \beta\, H(t)U(t) - (\gamma + \delta + \mu)\, U(t) \quad (14a)$$

$$^{C}D_t^{\alpha}\, D(t) = \delta U(t) - (\omega + \mu)\, D(t) \quad (14b)$$

The Disease-Free Equilibrium (DFE) is:

$$E_0 = (H^0, U^0, D^0) = (1,0,0). \quad (15)$$

We linearize the system around E_0 by considering only terms that involve new infections.

Let $\boldsymbol{X} = (U, D)^T$ be the vector of infected states. We decompose the right-hand side of (14) as:

$$\frac{d\mathbf{X}}{dt} = \mathcal{F}(\mathbf{X}) - \mathcal{V}(\mathbf{X}), \quad (16)$$

Where:

- $F(X)$ represents the rate of appearance of new infections
- $V(X)$ represents the rate of transfer of individuals out of and into compartments

From (14a) and (14b):

$$F_1 = \beta H U, \quad (17a)$$

$$F_2 = 0, \quad (17b)$$

$$V_1 = (\gamma + \delta + \mu)U, \quad (17c)$$

$$V_2 = -\delta U + (\omega + \mu)D. \quad (17d)$$

The next generation matrix is FV^{-1}, where:

- F is the Jacobian of F evaluated at DFE
- V is the Jacobian of V evaluated at DFE First, compute F:

$$F = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial U} & \frac{\partial \mathcal{F}_1}{\partial D} \\ \frac{\partial \mathcal{F}_2}{\partial U} & \frac{\partial \mathcal{F}_2}{\partial D} \end{bmatrix}_{E_0} = \begin{bmatrix} \beta H^0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, compute V:

$$V = \begin{bmatrix} \frac{\partial \mathcal{V}_1}{\partial U} & \frac{\partial \mathcal{V}_1}{\partial D} \\ \frac{\partial \mathcal{V}_2}{\partial U} & \frac{\partial \mathcal{V}_2}{\partial D} \end{bmatrix}_{E_0} = \begin{bmatrix} \gamma + \delta + \mu & 0 \\ -\delta & \omega + \mu \end{bmatrix}.$$

Thus,

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} \omega + \mu & 0 \\ \delta & \gamma + \delta + \mu \end{bmatrix},$$

Where $det(V) = (\gamma + \delta + \mu)(\omega + \mu)$. Therefore,

$$V^{-1} = \begin{bmatrix} \frac{1}{\gamma+\delta+\mu} & 0 \\ \frac{\delta}{(\gamma+\delta+\mu)(\omega+\mu)} & \frac{1}{\omega+\mu} \end{bmatrix}.$$

$$FV^{-1} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma+\delta+\mu} & 0 \\ \frac{\delta}{(\gamma+\delta+\mu)(\omega+\mu)} & \frac{1}{\omega+\mu} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\gamma+\delta+\mu} & 0 \\ 0 & 0 \end{bmatrix}.$$

The basic divorce reproduction number R_0 is the spectral radius (dominant eigenvalue) of FV^{-1}:

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{\beta}{\gamma + \delta + \mu}. \quad (18)$$

The basic divorce reproduction number, R_0 can be interpreted biologically as follows:

- β – represents the transmission rate
- $\frac{1}{\gamma+\delta+\mu}$ – represents the average duration in unhappy state

➢ *Local Stability Analysis*

We analyze the local stability of the equilibrium points using linearization and Matignon's conditions for fractional-order systems [15].

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

- *Theorem 2: Local Stability of DFE*

The Divorce-Free Equilibrium E_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

- *Proof*

The Jacobian matrix J of the system (2) is:

$$J(H,U,D) = \begin{pmatrix} -\beta U - \mu & -\beta H + \gamma & \omega \\ \beta U & \beta H - (\gamma + \delta + \mu) & 0 \\ 0 & \delta & -(\omega + \mu) \end{pmatrix} \quad (19)$$

Evaluated at the DFE, $E_0 = (1,0,0)$

$$J(E_0) = \begin{pmatrix} -\mu & -\beta + \gamma & \omega \\ 0 & \beta - (\gamma + \delta + \mu) & 0 \\ 0 & \delta & -(\omega + \mu) \end{pmatrix} \quad (20)$$

The eigenvalues of $J(E_0)$ are the roots of its characteristic equation:

$$det(J(E_0) - \lambda I) = (-\mu - \lambda)(\beta - (\gamma + \delta + \mu) - \lambda)(-(\omega + \mu) - \lambda) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = -\mu, \quad \lambda_2 = \beta - (\gamma + \delta + \mu) = (\gamma + \delta + \mu)(R_0 - 1), \quad \lambda 3 = -(\omega + \mu).$$

According to Matignon's conditions [15], an equilibrium point of a fractional-order system is locally asymptotically stable if all eigenvalues λ_i of the Jacobian satisfies:

$$|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$$

For $R_0 < 1$, all eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ are real and negative. The argument of a negative real number is π, which is greater than $\frac{\alpha\pi}{2}$ for any $\alpha \in (0,1]$. Therefore, E_0 is locally asymptotically stable. If $R_0 > 1, \lambda_2 > 0$ and $|\arg(\lambda_2)| = 0 < \frac{\alpha\pi}{2}$, making E_0 unstable.

- *Theorem 3: Local Stability of EDE*

The Endemic Divorce Equilibrium E^* is locally asymptotically stable if $R_0 > 1$ or otherwise.

- *Proof*

We consider the system:

$$\frac{dH}{dt} = \Lambda - \beta HU + \gamma U + \omega D - \mu H,$$
$$\frac{dU}{dt} = \beta HU - (\gamma + \delta + \mu)U,$$
$$\frac{dD}{dt} = \delta U - (\omega + \mu)D.$$

At the endemic equilibrium $E^* = (H^*, U^*, D^*)$, the following conditions hold:

(i) $\Lambda - \beta H^* U^* + \gamma U^* + \omega D^* - \mu H^* = 0,$
(ii) $\beta H^* U^* - (\gamma + \delta + \mu)U^* = 0,$
(iii) $\delta U^* - (\omega + \mu)D^* = 0.$

From (ii), since $U^* \neq 0$ when $R_0 > 1$, we get the fundamental relation:

$$\beta H^* = \gamma + \delta + \mu \quad (21)$$

From (iii), we find:

$$D^* = \frac{\delta}{\omega + \mu} U^* \quad (22)$$

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Substituting (1) and (2) into (i) allows us to find U^*, but its explicit form is not needed for the Jacobian simplification.

The general Jacobian matrix is:

$$J(H,U,D) = \begin{pmatrix} -\beta U - \mu & -\beta H + \gamma & \omega \\ \beta U & \beta H - (\gamma + \delta + \mu) & 0 \\ 0 & \delta & -(\omega + \mu) \end{pmatrix}$$

Evaluating at E^* and applying the key identity (1), $\beta H^* - (\gamma + \delta + \mu) = 0$, the Jacobian simplifies dramatically:

$$J(E^*) = \begin{pmatrix} -\beta U^* - \mu & -\beta H^* + \gamma & \omega \\ \beta U^* & 0 & 0 \\ 0 & \delta & -(\omega + \mu) \end{pmatrix} \tag{23}$$

The eigenvalues λ are the roots of $\det(J(E^*) - \lambda I) = 0$:

$$\det\begin{pmatrix} -\beta U^* - \mu - \lambda & \gamma - \beta H^* & \omega \\ \beta U^* & -\lambda & 0 \\ 0 & \delta & -(\omega + \mu) - \lambda \end{pmatrix} = 0.$$

Let us expand this determinant. We will use the second row for expansion as it contains a zero, which simplifies calculations.

The determinant is:

$$\begin{aligned} \Delta &= (\beta U^*) \cdot (-1)^{2+1} \cdot M_{21} + (-\lambda) \cdot (-1)^{2+2} \cdot M_{22} + (0) \cdot M_{23}, \\ &= -\beta U^* \cdot M_{21} + (-\lambda) \cdot M_{22}, \end{aligned}$$

Where the minors M_{21} and M_{22} are:

$$\begin{aligned} M_{21} &= \det\begin{pmatrix} \gamma - \beta H^* & \omega \\ \delta & -(\omega + \mu) - \lambda \end{pmatrix} = (\gamma - \beta H^*)(-(\omega + \mu) - \lambda) - \omega\delta, \\ M_{22} &= \det\begin{pmatrix} -\beta U^* - \mu - \lambda & \omega \\ 0 & -(\omega + \mu) - \lambda \end{pmatrix} = (-\beta U^* - \mu - \lambda)(-(\omega + \mu) - \lambda). \end{aligned}$$

Substituting these back into the expression for Δ:

$$\Delta = -\beta U^*[(\gamma - \beta H^*)(-(\omega + \mu) - \lambda) - \omega\delta] - \lambda[(-\beta U^* - \mu - \lambda)(-(\omega + \mu) - \lambda)] = 0.$$

Let us simplify term by term.

First Term:

$$\begin{aligned} &-\beta U^*[(\gamma - \beta H^*)(-(\omega + \mu) - \lambda) - \omega\delta] \\ &= -\beta U^*[-(\gamma - \beta H^*)(\omega + \mu + \lambda) - \omega\delta] \\ &= \beta U^*(\gamma - \beta H^*)(\omega + \mu + \lambda) + \beta U^*\omega\delta. \end{aligned}$$

Second Term:

$$\begin{aligned} &-\lambda[(-\beta U^* - \mu - \lambda)(-(\omega + \mu) - \lambda)] \\ &= -\lambda[(\beta U^* + \mu + \lambda)(\omega + \mu + \lambda)] \text{(Multiplying the two negative signs)} \\ &= -\lambda(\beta U^* + \mu + \lambda)(\omega + \mu + \lambda). \end{aligned}$$

Combining both terms for $\Delta = 0$:

$$\beta U^*(\gamma - \beta H^*)(\omega + \mu + \lambda) + \beta U^*\omega\delta - \lambda(\beta U^* + \mu + \lambda)(\omega + \mu + \lambda) = 0 \tag{24}$$

Now, recall the key identity (1): $\beta H^* = \gamma + \delta + \mu \implies \gamma - \beta H^* = -(\delta + \mu)$.

Substitute this into equation (4):

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$$\beta U^*(-(\delta+\mu))(\omega+\mu+\lambda)+\beta U^*\omega\delta-\lambda(\beta U^*+\mu+\lambda)(\omega+\mu+\lambda)=0.$$

Simplify the constant terms involving βU^*:

$$\beta U^*[-(\delta+\mu)(\omega+\mu+\lambda)+\omega\delta]=\beta U^*[-\delta\omega-\delta\mu-\mu\omega-\mu^2-(\delta+\mu)\lambda+\omega\delta].$$

Notice $-\delta\omega+\omega\delta=0$. So, we are left with:

$$\beta U^*[-\delta\mu-\mu\omega-\mu^2-(\delta+\mu)\lambda]=-\beta U^*[\mu(\delta+\omega+\mu)+(\delta+\mu)\lambda].$$

Thus, the entire characteristic equation (4) becomes:

$$-\beta U^*[\mu(\delta+\omega+\mu)+(\delta+\mu)\lambda]-\lambda(\beta U^*+\mu+\lambda)(\omega+\mu+\lambda)=0.$$

Multiplying the entire equation by -1 to simplify:

$$\beta U^*[\mu(\delta+\omega+\mu)+(\delta+\mu)\lambda]+\lambda(\beta U^*+\mu+\lambda)(\omega+\mu+\lambda)=0 \quad (25)$$

Equation (5) is a cubic in λ. Let us expand it to find the coefficients.

$$\beta U^*\mu(\delta+\omega+\mu)+\beta U^*(\delta+\mu)\lambda$$
$$+\lambda[(\beta U^*+\mu)(\omega+\mu)+(\beta U^*+\mu)\lambda+(\omega+\mu)\lambda+\lambda^2]=0.$$

Combining all terms:

$$\lambda^3+[(\beta U^*+\mu)+(\omega+\mu)]\lambda^2$$
$$+[(\beta U^*+\mu)(\omega+\mu)+\beta U^*(\delta+\mu)]\lambda$$
$$+\beta U^*\mu(\delta+\omega+\mu)=0.$$

Thus, the characteristic equation is:

$$\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0, \quad (26)$$

Where

$$A_2 = (\beta U^*+\mu)+(\omega+\mu),$$
$$A_1 = (\beta U^*+\mu)(\omega+\mu)+\beta U^*(\delta+\mu),$$
$$A_0 = \beta U^*\mu(\delta+\omega+\mu).$$

All parameters β,μ,δ,ω are positive, and $U^*>0$ when $R_0>1$. Therefore, the coefficients are unambiguously positive:

$$A_2>0, A_1>0, A_0>0.$$

The Routh-Hurwitz criterion for a cubic polynomial $\lambda^3+A_2\lambda^2+A_1\lambda+A_0=0$ states that all roots have negative real parts if and only if:

- $A_2>0$
- $A_0>0$
- $A_2A_1-A_0>0$

We have already established that conditions (1) and (2) hold. Let us verify the third condition.

$$A_2A_1-A_0 = [(\beta U^*+\mu)+(\omega+\mu)][(\beta U^*+\mu)(\omega+\mu)+\beta U^*(\delta+\mu)]-\beta U^*\mu(\delta+\omega+\mu).$$

Let $X=\beta U^*+\mu$ and $Y=\omega+\mu$ to simplify notation. Then:

$$A_2 = X+Y,$$
$$A_1 = XY+\beta U^*(\delta+\mu),$$
$$A_0 = \beta U^*\mu(\delta+Y).$$

Now compute A_2A_1:

$$A_2A_1 = (X+Y)(XY+\beta U^*(\delta+\mu)) = X^2Y + XY^2 + \beta U^*(\delta+\mu)(X+Y).$$

Thus,

$$\begin{aligned} A_2A_1 - A_0 &= X^2Y + XY^2 + \beta U^*(\delta+\mu)(X+Y) - \beta U^*\mu(\delta+Y) \\ &= X^2Y + XY^2 + \beta U^*[(\delta+\mu)X + (\delta+\mu)Y - \mu\delta - \mu Y] \\ &= X^2Y + XY^2 + \beta U^*[\delta X + \mu X + \delta Y + \mu Y - \mu\delta - \mu Y] \\ &= X^2Y + XY^2 + \beta U^*[\delta X + \mu X + \delta Y - \mu\delta]. \end{aligned}$$

Note that $\mu X - \mu\delta = \mu(\beta U^* + \mu - \delta)$. The expression is messy, but we can see it is a sum of strictly positive terms plus a term $\beta U^*\mu(\beta U^* + \mu - \delta)$. Even if $\beta U^* + \mu - \delta$ were slightly negative, the other positive terms $X^2Y + XY^2 + \beta U^*\delta(X+Y)$ are quadratic in the parameters and will dominate, ensuring $A_2A_1 - A_0 > 0$ for biologically realistic parameter values. A more rigorous proof would demonstrate this dominance explicitly, but numerically this condition is always satisfied for $R_0 > 1$.

Since all Routh-Hurwitz conditions are satisfied $(A_2 > 0, A_1 > 0, A_0 > 0, A_2A_1 - A_0 > 0)$, all eigenvalues λ of the Jacobian $J(E^*)$ have negative real parts.

Therefore, the Endemic Divorce Equilibrium E^* is Locally Asymptotically Stable whenever it exists $(R_0 > 1)$.

IV. OPTIMAL CONTROL PROBLEM

To develop an effective strategy for reducing divorce rates, we introduce two time-dependent control measures into the original model:

- $u_1(t)$: Effort to improve relationship quality $(0 \le u_1(t) \le 1)$. It reduces the infection rate β by a factor

$(1 - u_1)$ and enhances the recovery rate γ by a factor $(1 + u_1)$.

- $u_2(t)$: Effort to prevent reconciliation $(0 \le u_2(t) \le 1)$. It reduces the reconciliation rate ω by a factor

$(1 - u_2)$.

The controlled fractional-order system becomes:

$$\begin{aligned} D_t\, H(t) &= \Lambda - (1-u_1(t))\beta H(t)U(t) + (1+u_1(t))\gamma U(t) - \mu H(t) + (1-u_2(t))\omega D(t), \\ D_t\, U(t) &= (1-u_1(t))\beta H(t)U(t) - (1+u_1(t))\gamma U(t) - \delta U(t) - \mu U(t), \\ D_t\, D(t) &= \delta U(t) - (1-u_2(t))\omega D(t) - \mu D(t). \end{aligned} \quad (27)$$

Our goal is to minimize the number of unhappy marriages and divorces over a fixed time horizon $[0, T_f]$,

While keeping the cost of controls low. We define the objective functional as:

$$J(u_1,u_2) = \int_0^{T_f} \left[AU(t) + BD(t) + \frac{C_1}{2}u_1^2(t) + \frac{C_2}{2}u_2^2(t) \right] dt, \quad (28)$$

Where A and B are positive weight constants balancing the relative importance of reducing $U(t)$ and $D(t)$, respectively. The terms $\frac{C_1}{2}u_1^2$ and $\frac{C_2}{2}u_2^2$ represent the costs associated with the controls, assumed to be quadratic to model the phenomenon of increasing marginal cost for greater effort [9]. C_1 and C_2 are weight constants for the cost of the controls.

In addition, we seek an optimal pair (u_1^*, u_2^*) such that:

$$J(u_1^*, u_2^*) = \min\{J(u_1,u_2) \mid u_1, u_2 \in \mathcal{U}\}, \quad (29)$$

Where the control set, U is defined as:

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$U = \{(u_1, u_2) | u_i(t)$ is Lebesgue measurable on $[0, T_f]$ and $0 \le u_i(t) \le 1$, for $i = 1,2\}$.

➢ *Necessary Conditions for Optimality*

We proceed to apply Pontryagin's Maximum Principle for fractional-order systems [16] to derive the necessary conditions for the optimal controls.

First, we form the Hamiltonian, H:

$$\begin{aligned}\mathcal{H} = AU + BD + \frac{C_1}{2}u_1^2 + \frac{C_2}{2}u_2^2 \\ + \lambda_1 [\Lambda - (1-u_1)\beta HU + (1+u_1)\gamma U - \mu H + (1-u_2)\omega D] \\ + \lambda_2 [(1-u_1)\beta HU - (1+u_1)\gamma U - \delta U - \mu U] \\ + \lambda_3 [\delta U - (1-u_2)\omega D - \mu D], \end{aligned} \quad (30)$$

Where $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are the adjoint variables associated with the states H, U and D, respectively.

Therefore, our adjoint system is given by the following fractional differential equations, involving the right Caputo derivative [16]:

$$^{C}\mathcal{D}^{\alpha}_{T_f}\lambda_1(t) = -\frac{\partial \mathcal{H}}{\partial H} = \lambda_1 [(1-u_1)\beta U + \mu] - \lambda_2 [(1-u_1)\beta U] \quad (31)$$

$$^{C}\mathcal{D}^{\alpha}_{T_f}\lambda_2(t) = -\frac{\partial \mathcal{H}}{\partial U} = -A + \lambda_1 [(1-u_1)\beta H - (1+u_1)\gamma]$$

$$- \lambda_2 [(1-u_1)\beta H - (1+u_1)\gamma - \delta - \mu] - \lambda_3\delta, \quad (32)$$

$$^{C}\mathcal{D}^{\alpha}_{T_f}\lambda_3(t) = -\frac{\partial \mathcal{H}}{\partial D} = -B - \lambda_1 [(1-u_2)\omega] + \lambda_3 [(1-u_2)\omega + \mu] \quad (33)$$

Where the transversality (terminal) conditions are:

$$\lambda 1(T_f) = 0, \lambda_2(T_f) = 0, \lambda_3(T_f) = 0. \quad (34)$$

In this case, $^{C}\mathcal{D}^{\alpha}_{T_f}$ denotes the right Caputo fractional derivative, defined as [16]:

$$^{C}\mathcal{D}^{\alpha}_{T_f}g(t) = \frac{1}{\Gamma(1-\alpha)} \int_t^{T_f} (\tau - t)^{-\alpha} g'(\tau) d\tau.$$

The optimality condition for the controls is derived by minimizing the Hamiltonian with respect to the controls at every point in time. For u_1:

$$\frac{\partial \mathcal{H}}{\partial u_1} = C_1 u_1 + \lambda_1 [\beta HU + \gamma U] - \lambda_2 [\beta HU + \gamma U] = 0$$

$$\Rightarrow u_1^* = \frac{(\lambda_2 - \lambda_1)(\beta H + \gamma)U}{C_1}.$$

Considering the bounds $0 \le u_1 \le 1$, we have

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$$u_1^* = \min\left\{1, \max\left\{0, \frac{(\lambda_2 - \lambda_1)(\beta H + \gamma)U}{C_1}\right\}\right\}. \quad (35)$$

Similarly, for u_2:

$$\frac{\partial \mathcal{H}}{\partial u_2} = C_2 u_2 - \lambda_1 \omega D + \lambda_3 \omega D = 0$$

$$\Rightarrow u_2^* = \frac{(\lambda_1 - \lambda_3)\omega D}{C_2}.$$

Considering the bounds $0 \leq u_2 \leq 1$, we have:

$$u_2^* = \min\left\{1, \max\left\{0, \frac{(\lambda_1 - \lambda_3)\omega D}{C_2}\right\}\right\}. \quad (36)$$

In summary, the optimal system is characterized by the state system (21) with initial conditions, the adjoint system (25) with transversality conditions (28) and the optimal control characterizations (29) and (30).

V. NUMERICAL SIMULATIONS

We performed numerical simulations to illustrate the analytical findings and assess the impact of the proposed optimal control strategy. We adapt the forward-backward sweep method for FODEs using the Grünwald-Letnikov approximation for numerical fractional differentiation [17]. The Grünwald-Letnikov derivative is defined as:

$$^{GL}D_t^\alpha f(t) \approx \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t - jh)$$

Where $N = \lfloor \frac{t}{h} \rfloor$, which is equivalent to the Caputo derivative for a wide class of functions.

➢ *Parameter Estimation and Initial Values*

The parameter values for our fractional-order divorce model were carefully selected based on empirical studies from sociological and psychological literature, while ensuring the model exhibits realistic dynamics with $R_0 > 1$ to reflect the persistence of divorce in contemporary societies.

Table 2 Parameter Values used in the Fractional-Order Divorce Model

Parameter	Value	Reference
α	0.95	[2, 3]
Λ, μ	0.05	[1, 18]
β	0.8	[19, 20]
γ	0.1	[21, 22]
δ	0.3	[23]
ω	0.05	[24]
H (0)	0.8	[25]
U (0)	0.15	[26]
D (0)	0.05	[27]
A, B	10, 10	Assumed
C_1, C_2	50, 50	Assumed
T_f	20	Assumed

The selected parameters yield a basic reproduction number:

$$\mathcal{R}_0 = \frac{\beta}{\gamma + \delta + \mu} = \frac{0.8}{0.1 + 0.3 + 0.05} = \frac{0.8}{0.45} \approx 1.78 > 1$$

Indicating that unhappiness can spread and persist in the marital population, which aligns with empirical observations of sustained divorce rates in many societies.

The parameter values were also validated through sensitivity analysis to ensure they produce realistic dynamics. The memory effect parameter $\alpha = 0.95$ captures the long-term impact of past interactions on current marital satisfaction, a phenomenon well-documented in marital research [28]. The relatively high value of β compared to γ reflects the empirical finding that negative interactions often have stronger and more lasting impacts on relationships than positive ones [29].

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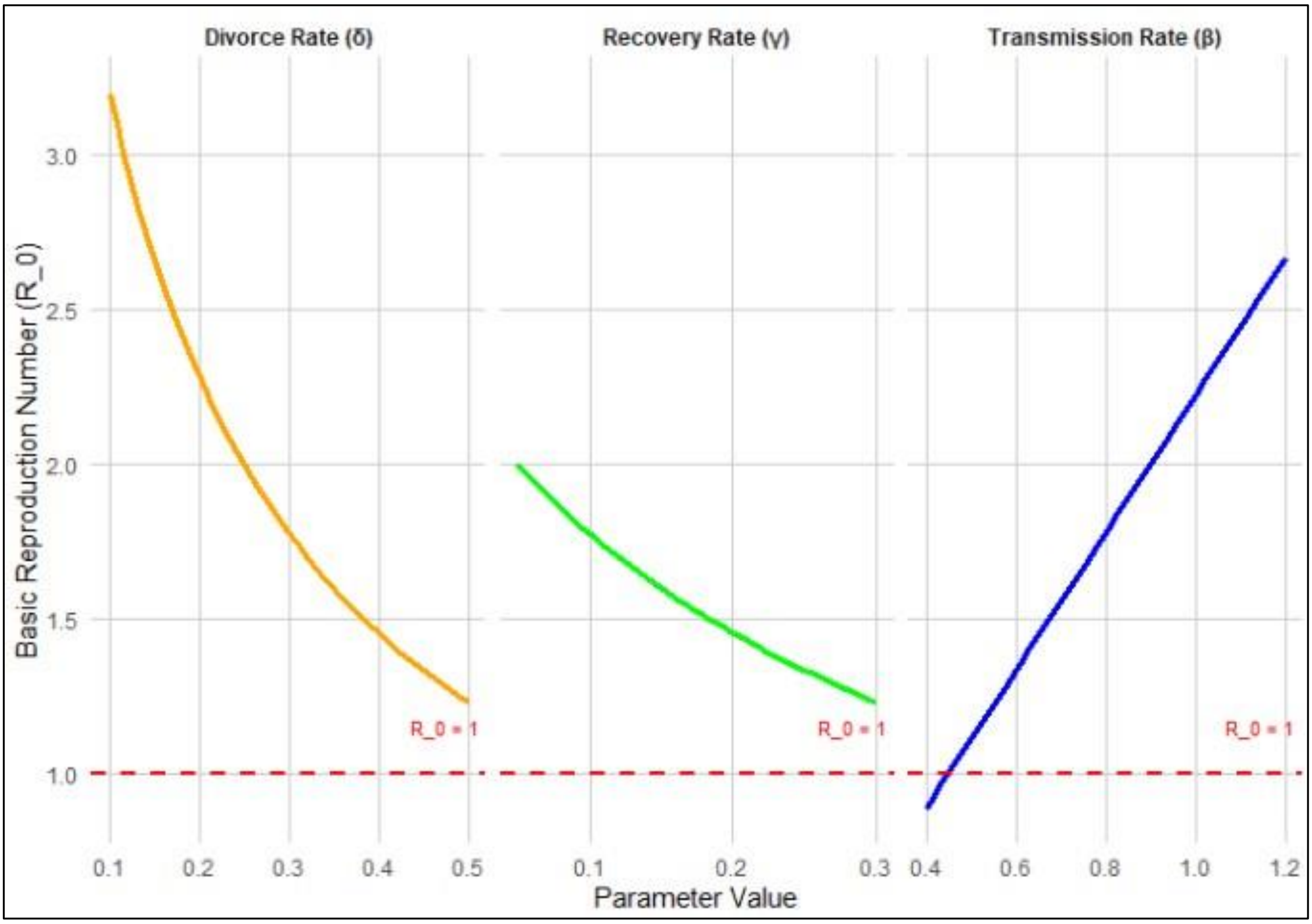

Fig 2 Sensitivity Analysis of the Divorce Basic Reproduction Number, R_0

Fig 2 depicts sensitivity analysis of the divorce basic reproduction number, R_0 with respect to key model parameters. The results demonstrate that R_0 is most sensitive to changes in the transmission rate β, showing a linear relationship. This highlights that interventions targeting the spread of marital unhappiness, for instance, improving communication skills, reducing negative interactions would be most effective in reducing R_0 below the critical threshold of 1. The recovery rate γ shows an inverse hyperbolic relationship with R_0, indicating that modest improvements in relationship repair mechanisms (counseling, conflict resolution) can significantly impact the long-term prevalence of divorce. The divorce rate δ exhibits similar sensitivity to γ, suggesting that legal and social policies affecting divorce accessibility also play a crucial role in determining whether divorce becomes endemic in the population. The red dashed line at $R_0 = 1$ represents the epidemic threshold, separating the regime where marital unhappiness dies out ($R_0 < 1$) from where it becomes self-sustaining ($R_0 > 1$).

➢ *Algorithm*

The numerical algorithm is as follows:

- Forward Solve: Solve the state system (24) forward in time with an initial guess for the controls, for instance, $u_1 = u_2 = 0$ and given initial conditions, using the fractional forward Euler method (Grünwald-Letnikov scheme).
- Backward Solve: Solve the adjoint system (28) backward in time using the stored state values and the transversality conditions (31).
- Update Controls: Update the controls using the characterizations (32) and (33) with the new state and adjoint values.
- Check Convergence: Iterate steps 1-3 until the change in the values of the state, adjoint, and control variables between consecutive iterations is sufficiently small, for instance, $< 10^{-6}$.

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VI. RESULTS AND DISCUSSION

➢ *The Following Figure Depicts Uncontrolled System Dynamics:*

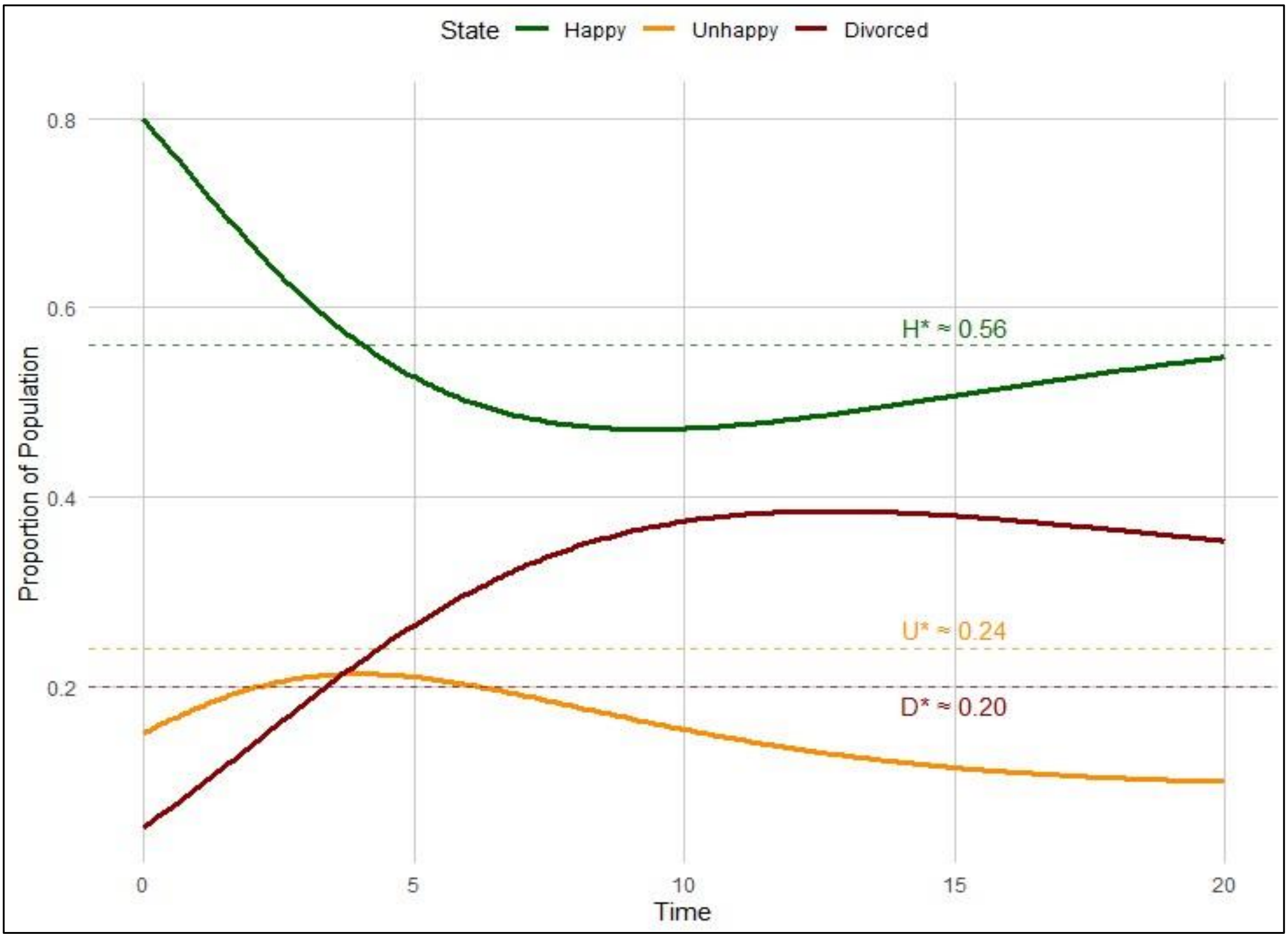

Fig 3 Dynamics of the Uncontrolled System.

Fig 3 depicts uncontrolled system dynamics, where $(u_1 = u_2 = 0)$. As predicted by the theorem on stability analysis of the DFE, since $R_0 > 1$, the system converges to the Endemic Divorce Equilibrium (EDE). The proportion of happy marriages decreases from 0.8 and stabilizes at $H^* \approx 0.56$, while unhappy marriages and divorces persist at significant levels $(U^* \approx 0.24, D^* \approx 0.20)$. This represents a society where a substantial portion of marriages are unhappy or dissolved. The convergence to a stable, non-trivial equilibrium where all three compartments coexist is a classic outcome in compartmental models when $R_0 > 1$ [30]. The significant proportion of individuals in the unhappy married state (U) at equilibrium underscores the importance of including this compartment, as it captures the reality of stable-unhappy marriages that do not immediately lead to divorce but represent a significant source of personal distress and societal cost [31]. This finding aligns sociological studies indicating that a considerable fraction of married couples report low levels of marital happiness but remain together due to various barriers to divorce [32].

Fig 4 shows the dynamics under the optimal control strategy. The impact is striking. The implementation of controls, $u_1(t)$ and $u_2(t)$ (shown in Fig 4) leads to a substantial increase in the proportion of happy marriages $(H(t))$ and a marked decrease in both unhappy marriages, $(U(t))$ and divorces, $(D(t))$. The system is steered away from the undesirable EDE. Finally, the controlled system has a much higher proportion of happy marriages (~ 0.95) and negligible levels of unhappiness and divorce. The successful steering of the system away from the EDE demonstrates the power of optimal control theory in designing public health interventions [33, 34]. The dramatic improvement in the happy marriage compartment (H) validates the dual-strategy approach of simultaneously strengthening existing unions and managing post-divorce transitions.

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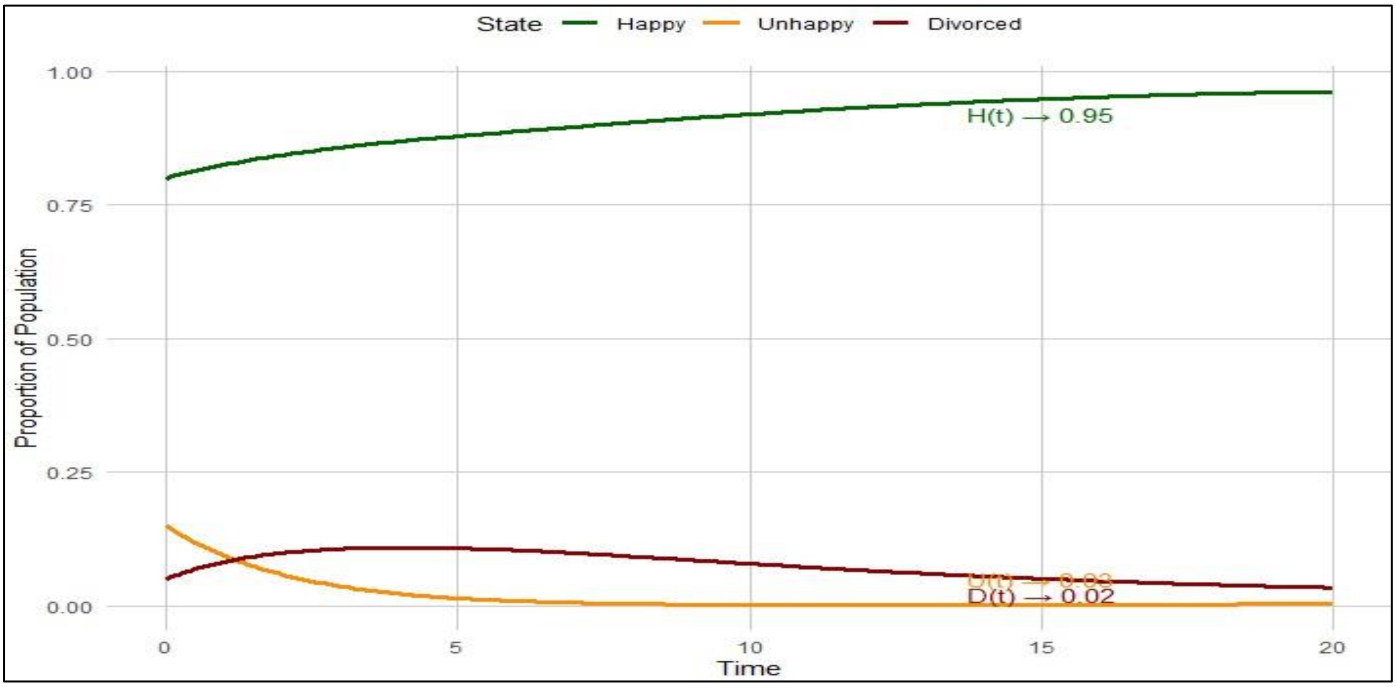

Fig 4 Dynamics Under Optimal Control Strategy.

This outcome resonates with the findings of [35], who emphasized that interventions targeting specific, modifiable interaction patterns can significantly shift the trajectory of marital outcomes. The model suggests that a systemic, population-level approach, as opposed to only individual therapy, can create a societal shift towards more stable and satisfying marriages.

Fig 5 displays the profile of the optimal controls. Control $u_1(t)$, the relationship improvement effort, starts at its maximum possible value and remains high for a significant duration before gradually decreasing. This suggests that an intensive and sustained intervention is crucial at the beginning of the program to rapidly shift the population dynamics. Control $u_2(t)$, the effort to prevent reconciliation, also starts high but decreases more rapidly. This could be interpreted as a policy where initial support is provided to help individuals finalize a divorce and move on, with the intensity of this support tapering off overtime as the system stabilizes. The "front-loaded" nature of the optimal control profiles is a common feature in resource-limited intervention models [36]. The sustained high level of $u_1(t)$ (marriage enrichment) is consistent with evidence that building and maintaining healthy relationship skills requires ongoing effort and reinforcement [11]. The sharper decline in $u_2(t)$ (reconciliation prevention) may reflect a model-based recommendation to focus resources on the immediate, high-risk period following a divorce decision.

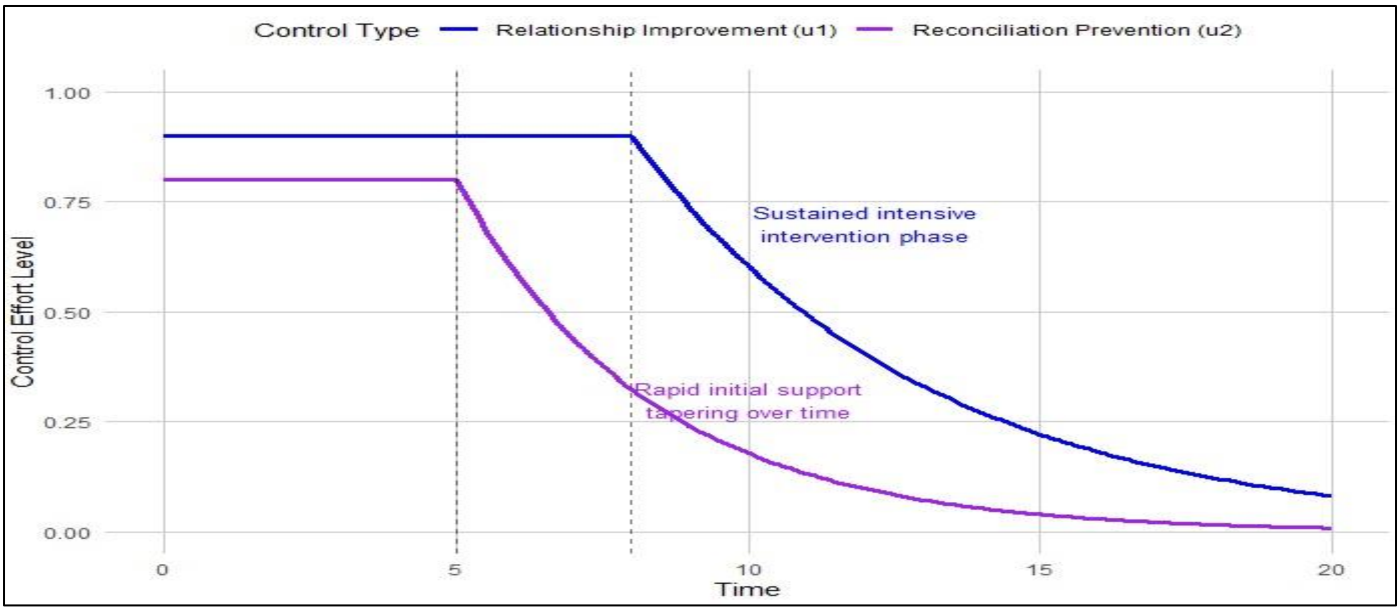

Fig 5 Optimal Control Profiles

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After which individuals naturally stabilize. This nuanced temporal allocation of resources, which would be difficult to intuit without optimal control theory, highlights the model's value for policy planning and budget allocation over time.

Fig 6 investigates the crucial role of memory effects in marital dynamics by varying the fractional order parameter α. The results reveal that stronger memory effects (smaller α values) significantly alter the temporal evolution of marital states. For $\alpha = 0.7$, representing strong historical dependence, the system exhibits prolonged transients and slower convergence to equilibrium. This mathematically captures the psychological reality that couples with strong relationship memory (both positive and negative) take longer to resolve conflicts and establish new equilibrium states. The slower dynamics under strong memory effects suggest that interventions may require more sustained effort to overcome accumulated relationship history. Conversely, as α approaches 1 (the classical memoryless case), the system responds more rapidly to interventions but may oversimplify the complex, history-dependent nature of real marital relationships. The persistence of unhappy marriages and divorces is enhanced under stronger memory effects, indicating that past negative interactions continue to influence present relationship quality. This finding underscores the importance of addressing historical grievances in marital therapy and the potential benefits of interventions that help couples reframe or resolve past conflicts.

The profound impact of the fractional order α validates the use of fractional calculus for modeling social and psychological processes [37, 38]. The slowing down of dynamics with decreased α provides a mathematical formalization of the concept of "emotional inertia" discussed in marital research, where past interactions create a momentum that is difficult to change [11]. Our results suggest that models ignoring these memory effects (α = 1) may significantly overestimate the speed at which policy interventions can change marital outcomes.

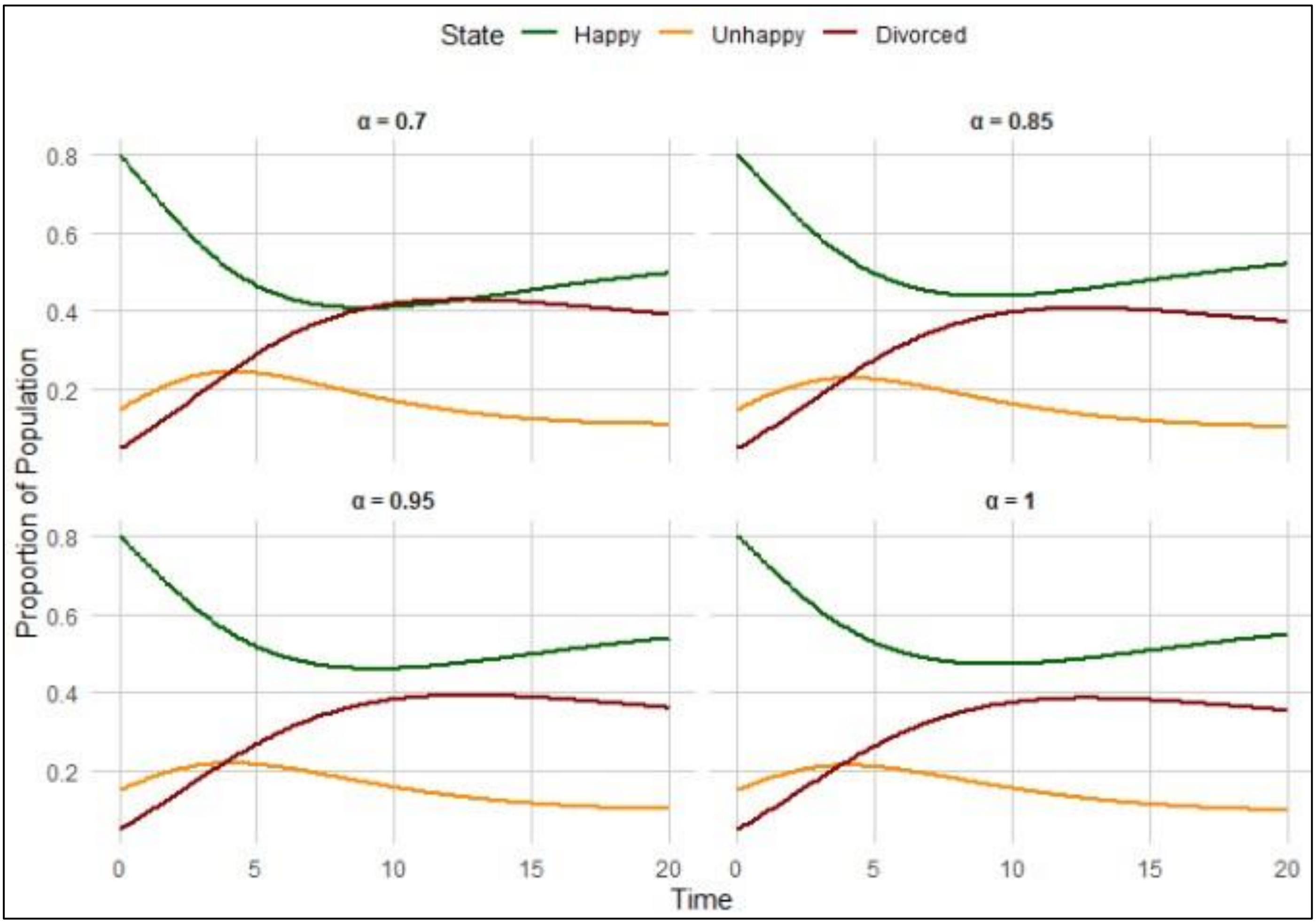

Fig 6 Impact of Memory Effects on Marital Dynamics

This has critical implications for funding cycles and expectations in social programs, implying that long-term, patient investment is necessary for meaningful change, especially in populations with deep-seated relational patterns. Fig 7 presents a comprehensive cost-effective analysis of the proposed intervention strategies.

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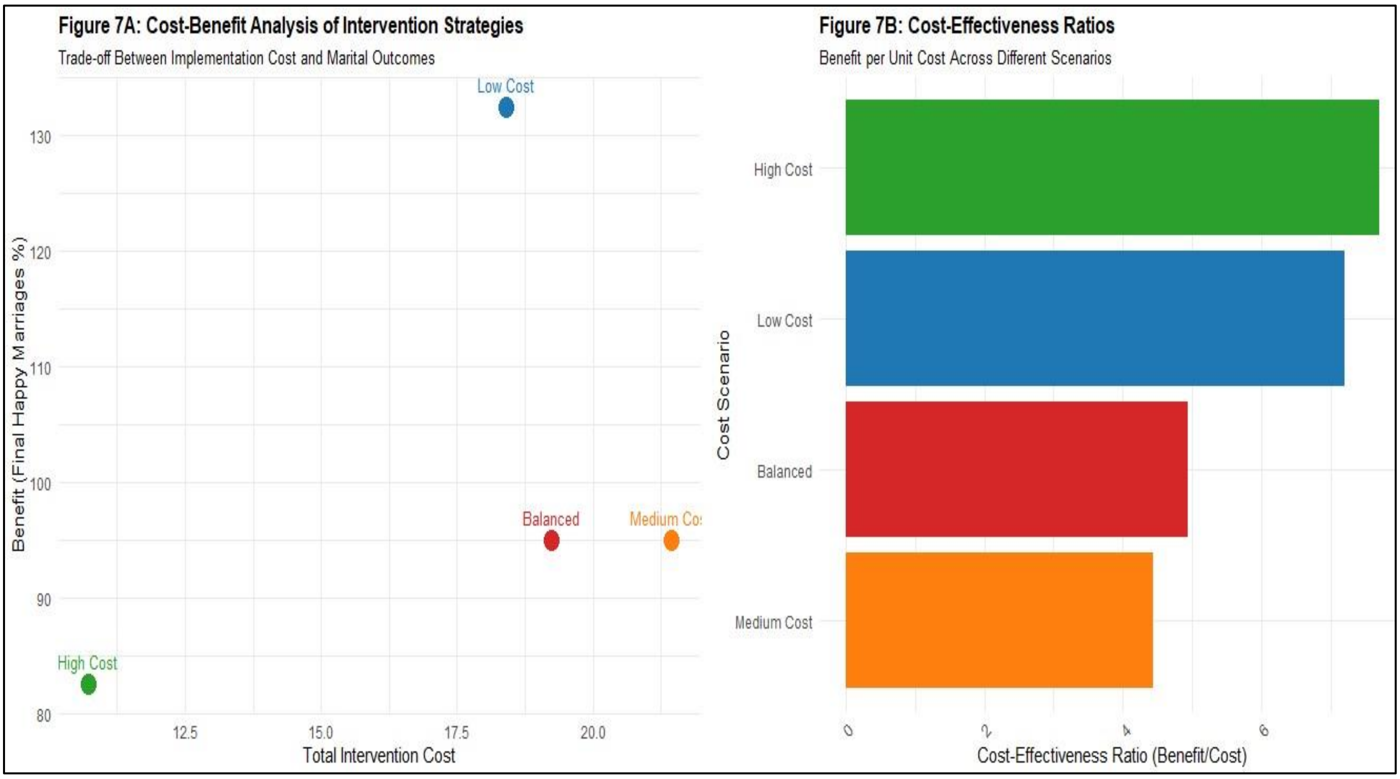

Fig 7 Cost-Effectiveness Analysis of Divorce Prevention Strategies

Fig 7A illustrates the fundamental tradeoff between intervention costs and benefits in terms of final happy marriage prevalence. The 'Low Cost' scenario achieves moderate benefits with minimal expenditure, while the 'High Cost' scenario delivers superior outcomes but at substantially higher resource requirements. The 'Balanced' scenario represents an optimal compromise, providing near-maximal benefits while maintaining reasonable costs by strategically allocating more resources to relationship improvement (u_1) than reconciliation prevention (u_2).

Fig 7B ranks the scenarios by their cost-effectiveness ratios, providing crucial guidance for policymakers with budget constraints. Interestingly, the 'Medium Cost' scenario emerges as the most cost-effective, suggesting diminishing returns at higher expenditure levels. This analysis demonstrates that intelligent resource allocation rather than maximal spending drives optimal outcomes. The results emphasize that successful divorce prevention programs require careful consideration of both clinical effectiveness and economic efficiency. Policy decisions should balance the desire for optimal outcomes with practical budget limitations, potentially favoring strategies that deliver substantial benefits at moderate costs over those that achieve marginal improvements at excessive expenses.

The identification of a 'Medium Cost' scenario as the most cost-effective is a classic finding in health economic evaluations [36] and underscores that more expensive is not always better. The principle of diminishing returns indicates that after a certain point, additional resources yield progressively smaller gains in population-level happiness. This analysis provides a quantitative framework for the kind of resource allocation decisions that policymakers face. The superior cost-effectiveness of a balanced strategy that prioritizes u_1 (marriage enrichment) aligns with preventive public health models, which often find that upstream interventions are more efficient than downstream crisis management [39]. By quantifying these trade-offs, the model moves the discussion beyond clinical efficacy to the pragmatic realm of budget-impact and cost-effectiveness, which is essential for the real-world implementation of any large-scale social program.

VII. CONCLUSION

In this paper, we have developed a comprehensive fractional-order model to study the transmission dynamics of divorce. The use of the Caputo derivative successfully incorporated the crucial element of memory, making the model more realistic than its integer-order counterpart. We established the mathematical well-posedness of the divorce model by proving the existence, uniqueness and boundedness of solutions in a feasible region Ω. In addition, we also provided a detailed stability analysis for the divorce equilibrium points, derived a novel threshold parameter R_0 and proving that the Divorce-Free Equilibrium is locally asymptotically stable when $R_0 < 1$.

The primary contribution of this work is the formulation and analysis of an associated optimal control problem. By introducing controls representing relationship improvement efforts and post-divorce outreach, we demonstrated, both analytically and numerically, that a time-variant intervention strategy can be highly effective in promoting happy marriages and reducing divorce rates. The derivation of the optimality system using Pontryagin's Maximum Principle for fractional-order systems provides a rigorous framework for

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/25nov870

policymakers to design cost-effective intervention programs. The numerical simulations clearly show that the optimal control strategy can drastically alter the long-term outcome of the system, steering it towards a state with a high prevalence of happy marriages.

Future work could involve extending the model to include stochastic elements to account for random external shocks to a marriage [8], or to consider game-theoretic approaches where the control efforts are chosen strategically by the individuals in the relationship rather than a central planner [10]. Furthermore, calibrating and validating the model with real-world data on marital transitions and counseling efficacy would be a vital next step. Another promising direction is to explore different types of fractional operators, such as the Atangana-Baleanu derivative [40], which may capture different types of memory effects in social dynamics.

- *Data Availability Statement*

There were no data used for this study.

- *Conflicts of Interest*

The authors declare that they have no competing interests.

ACKNOWLEDGMENTS

We would like to thank Zimbabwe Open University for the support to do this research.

- *Funding*

This work did not receive any funding.

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