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Order of the Disparity of Elements in Full Transformation Semigroup

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Abstract: ***Let*** α ***be a transformation from*** $X_n \rightarrow X_n$ ***where*** $X_n = \{1, 2, 3 \ldots n\}$***, a finite set, also let*** D_sT_n***,*** D_sO_n***,*** D_sD_n***and*** D_sC_n ***be the classes of disparity of elements of full transformation, order preserving, order decreasing, order preserving and decreasing full transformation semigroup respectively. The disparity of elements in full transformation semigroup is defined as*** $|W^+(\alpha) - W^-(\alpha)| = p$ ***where*** $p = \{1, 2\}$ ***,*** $W^+(\alpha) = max(Im(\alpha))$ ***and*** $W^-(\alpha) = min(Im(\alpha))$***.***

This research investigates the order of elements in D_sT_n ***and it subsemigroups,*** D_sO_n***,*** D_sD_n***and*** D_sC_n***.The elements in each semigroup were listed, arranged and tables were formed. The general nth term of the sequences were also derived. This research work has an important application in computational algebra and theoretical computer science.***

Keywords: *Disparity, Order Preserving, Order Decreasing, Order Preserving and Decreasing Full Transformation Semigroup.*

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I. INTRODUCTION

Semigroup theory is a generalization of group theory which has been popular topic in algebra for many years. The study of various semigroups of transformation which has aroused much interest in recent years have made significant contributions in the study of semigroup theory. However, full, partial and partial one-one transformation semigroups among other subsemigroups of transformation have been studied and have produced relevant results over the years.

Transformation semigroups are one of the most fundamental mathematical objects, they occur in theoretical computer science, where properties of language depend on algebraic properties of various transformation semigroups related to them. The finite semigroup of full transformation was reproduced by stull (1944). The early 1940s saw the publication of three highly influential papers (Rees (1940), Clifford (1941) and Dubreuil (1941)) which have shaped the subsequent development in the theory of semigroup. Up to 1941, most of the results obtained for semigroups had been analogues of results for groups or rings. The theory of semigroups went from strength in 1940s with an increasing number of papers appearing. As the decade progressed, the theory gradually gained momentum, culminating in the publication of highly influential papers.

In recent years, many authors have also published interesting research in semigroup transformation for example, Howie (1995), Laradji and Umar (2004a, 2004b, 2007), Adeniji and Makanjuola (2008,2010), Mogbonju, Ojo, Ogunleke (2012), Mogbonju, Azeez (2018), Mogbonju (2015), Adeniji (2012), Zubairu M.M and Bashir Ali (2018), Ibrahim, Iman, Adesola and Bakare (2019),

In this research, we introduce a new class of semigroup, disparity of elements in full transformation semigroup D_sT_n, order preserving D_sO_n , order decreasing D_sD_n and order preserving and order decreasing D_sC_n full transformation semigroup. The disparity of elements in full transformation has to do with the difference in the image α is defined as $|W^+(\alpha) - W^-(\alpha)| = p$, where $p = \{1,2\}$, $W^+(\alpha) = \max(Im(\alpha))$ and $W^-(\alpha) = \min(Im(\alpha))$. we also obtained the order of the semigroups.

II. PRELIMINARIES

- *Some Basic Definitions*

- *Group*

A group is a set G together with a law of composition, $*$, which has the following properties: For a, b, c, e $\in$ G,

- ✓ $a * (b * c) = (a * b) * c$ (associativity)
- ✓ $e \in G$ (Identity), $a * e = a = e * a$
- ✓ If $a \in G$, then $a^{-1} \in G$ (inverse), $a * a^{-1} = e = a^{-1} * a = e$

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- *Semigroup*

A semigroup is a set S together with a binary operation $*$, such $S \times S \to S$ such that for all $a, b, c \in S$, the operation satisfies the associative law $(a * b) * c = a * (b * c)$.

- *Subsemigroup*

Let $(S, *)$ be a semigroup and T a non empty subset of S. Then, T is a subsemigroup of S if it is closed under the semigroup operation $*$, that is, $a * b = b * a$, for all $a, b \in T$.

- *Order of Semigroup*

The order of a semigroup S is the number of elements in S, denoted by $|S|$.

- *Transformation*

Let X and Y be two non empty sets such that there is some rule f which assigns to each element $x \in X$, a unique element $y \in Y$, then this rule is said to be a transformation or mapping or a function.

- *Full Transformation Semigroup T_n*

T_n on the set $X_n = \{1,2,3,\ldots n\}$, is the set of all maps $\alpha : X_n \longrightarrow X_n$ under the operation of composition of mappings.

Let $X_n = \{1,2,3,\ldots n\}$. A transformation $\alpha : \mathrm{Dom}(\alpha) \subseteq X_n \longrightarrow \mathrm{Im}(\alpha) \subseteq X_n$ is said to be full if the $\mathrm{Dom}(\alpha) = X_n$.

- *Order-Preserving O_n*

A transformation $\alpha \in T_n$ is said to be order - preserving if for all $x, y \in \mathrm{Dom}(\alpha)$, $x \leq y \implies x\alpha \leq y\alpha$.

- *Order-Decreasing D_n*

A transformation $\alpha \in T_n$ is said to be order-decreasing if for all $x, y \in \mathrm{Dom}(\alpha)$, $x\alpha \leq x$.

- *Order Preserving and Decreasing C_n*

A transformation $\alpha \in T_n$ is said to be Order Preserving (Decreasing) if for all $x, y \in \mathrm{Dom}(\alpha), x\alpha \leq x$ ($x\alpha \leq y\alpha$).

III. METHODOLOGY

In this research, the elements in the semigroup D_sT_n were constructed by identifying the disparities of elements from the full transformation semigroup $T_n = n^n$ followed by the elements of the subsemigroups, order preserving D_sO_n, order decreasing D_sD_n and order preserving and decreasing D_sC_n full transformation semigroup , the order of arrangement were studied to derive the general n^{th} terms for the sequence formed. It is shown that the composition of mapping in the semigroup is associative.

- *Theorem 3.1:*

Let $\alpha \in S$ such that $S = \{D_sT_n, D_sO_n, D_sD_n, D_sC_n\}$ and let $X_n = \{1,2,3,\ldots.n\}$, then, $S = |W^+(\alpha) - W^-(\alpha)|$ for all $|W^+(\alpha) - W^-(\alpha)| = p$ and $p = 1,2$.

- *Proof:*

Since $\alpha \in S$ such that $S = \{D_sT_n, D_sO_n, D_sD_n, D_sC_n\}$, then, $\alpha \in D_sT_n$, D_sO_n , D_sD_n , D_sC_n for all $X_n = \{1,2,3,\ldots.n\}$,

Now,

$S = |W^+(\alpha) - W^-(\alpha)|$ for all $W^+(\alpha) = \max(Im(\alpha))$ and $W^-(\alpha) = \min(Im(\alpha))$

$$\text{Let } x\alpha = \max(Im(\alpha)) \text{ and } y\alpha = \min(Im(\alpha))$$

$$\implies S = |x\alpha - y\alpha| \leq 2 \text{ for all } x, y \in X_n$$

$$\text{This implies that } \mathrm{Im}(\alpha) = \{i, i+1\}, i = 1,2,\ldots,n-1.$$

$$\text{Hence, } S = |x\alpha - y\alpha| \leq 2.$$

- *Proposition 3.1:*

Let D_sT_n be disparity of elements of full transformation semigroup. Composition of mappings is associative if α, β and δ are partial mappings such that the composition $\delta(\beta\alpha)$ is defined if and only if the composition $(\delta\beta)\alpha$ is defined and both are equal.

- *Proof:*

Define the mappings $\alpha: W \to X, \beta: X \to Y$ and $\delta: Y \to Z$ which can be shown as

$$\alpha(w) = x, \beta(x) = y \text{ and } \delta(y) = z \text{ where } x \in X, y \in Y, \text{and } z \in Z$$

$$\implies (\alpha\beta) = y \text{ and } \beta(\delta) = z$$

$$\implies \alpha(\beta\delta) = (\beta\alpha)\delta$$

Hence, $\alpha(\beta\delta)$ is defined, since $(\beta\alpha)\delta$ is defined and are both equal. Associativity of composition naturally leads to the notation of semigroup. Thus, the disparity of elements in full transformation are semigroup with respect to the composition of full transformation.

- *For Example:*

$$\text{Let } S = D_sT_3$$

$$\text{Let} \quad a = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\right\} \in D_sT_3$$

$$b = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\right\} \in D_sT_3$$

$$c = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\right\} \in D_sT_3$$

$$a(bc) = (ab)c$$

$$a \circ b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$b \circ c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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$$a(bc) = \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$$

$$(ab)c = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$$

Therefore, $a(bc) = (ab)c$, hence, D_sT_n is associative.

- *Observations on D_sT_n*

Elements in D_sT_n were obtained from the relation $\alpha\colon \mathrm{Dom}\alpha \subseteq X_n \longmapsto \mathrm{Im}\alpha \subseteq X_n, X_n = \{1,2,3,\ldots n\}$ and $|W^{+}(\alpha) - W^{-}(\alpha)| = p$ where p is the set of $\{1,2\}$, $W^{+}(\alpha) = \max\left(Im(\alpha)\right)$ and $W^{-}(\alpha) = \min\left(Im(\alpha)\right)$.

- *Some of the Elements in D_sT_n are:*

When $n = 1$

$$D_sT_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = 1 \text{ element},$$

When n = 2

$$D_sT_2 = \left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\right\} = 2 \text{ elements},$$

When n = 3

$$D_sT_3 =$$

$$\left\{\begin{matrix}\begin{pmatrix}1 & 2 & 3\\1 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 2 & 1\end{pmatrix}\\ \begin{pmatrix}1 & 2 & 3\\2 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 3 & 2\end{pmatrix}\\ \begin{pmatrix}1 & 2 & 3\\1 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}\\ \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 3 & 1\end{pmatrix}\end{matrix}\right\} = 24 \text{ elements}$$

- *Observations on D_sO_n*

These are the elements of D_sO_n for $h = 1,2$ and $n = 1,2,3,\ldots,n$

At n=1,

$$D_sO_1 = 0,$$

At n=2,

$$D_sO_2 = \left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\right\} = 1 \text{ element},$$

At n=3,

$$D_sO_3 = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 3\end{pmatrix}\right\} = 7 \text{ elements}$$

- *Observations on D_sD_n*

These are the elements of D_sD_n for $h = 1,2$ and $n = 1,2,3,\ldots,n$

At n=1,

$$D_sD_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}\right\} = 1 \text{ element},$$

At n=2,

$$D_sD_2 = \left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\right\} = 1 \text{ element},$$

At n=3,

$$D_sD_3 = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}\right\} = 5 \text{ elements}$$

- *Observations on D_sC_n*

These are the elements of D_sC_n for $h = 1,2$ and $n = 1,2,3,\ldots,n$

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At n=1, $D_sC_1 = 0$, $|\text{Im}(1)| = 0, |\text{Im}(2)| = 0$

At n=2,

$$D_sC_2 = \left\{\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}\right\} = \text{ 1 element,}$$

At n=3,

$$D_sC_3 = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\1 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\1 & 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}\right\} = \text{ 4 elements}$$

IV. MAIN RESULTS

➢ *Results in D_sT_n*

Table 1 The Table Displays the Sequence of Elements in D_sT_n

n	$\lvert S \rvert = \frac{19}{2}n^4 - \frac{165}{2}n^3 + 268n^2 - 368n + 174$
1	1
2	2
3	24
4	142
5	659

- *Theorem 4.1:* Let $S = D_sT_n$, then $|S| = \frac{19}{2}n^4 - \frac{165}{2}n^3 + 268n^2 - 368n + 174$

✓ *Proof:*

Let $\alpha \in S$ with $\lvert \alpha y - \alpha x \rvert \leq 2$ and considering the sequence on the table 1 for $|W^+(\alpha) - W^-(\alpha)| = 1$, the polynomial obtained for the order of disparity of elements in full transformation $\lvert S \rvert = \frac{19}{2}n^4 - \frac{165}{2}n^3 + 268n^2 - 368n + 174$, is chosen from the $\text{Im}(\alpha)$ such that $\text{Im}(\alpha) = \{i, i+1\}$ and since it is a full transformation, then $\text{Im}(\alpha)$ occur in i ways with $i = 1,2,3,\ldots n-1$ then the result follows from $n-1$ elements which is group in $\lvert S \rvert = \frac{19}{2}n^4 - \frac{165}{2}n^3 + 268n^2 - 368n + 174$

➢ *Results in D_sO_n*

Table 2 The Table Displays the Sequence of Elements in D_sO_n

N	$\lvert S \rvert = \frac{1}{2}n^3 - \frac{1}{2}n^2 - n + 1$
1	0
2	1
3	7
4	21
5	46

- Theorem 4.2: Let $\alpha \in D_sO_n$ then $\lvert S(\alpha) \rvert = \frac{1}{2}n^3 - \frac{1}{2}n^2 - n + 1$

✓ *Proof:*

The result follow from the elements in D_sO_n, it is known that $|W^+(\alpha) - W^-(\alpha)| = 1,2$ which implies for every $\alpha \in D_sO_n$, $\lvert S(\alpha) \rvert = \frac{1}{2}n^3 - \frac{1}{2}n^2 - n + 1$and the table above shows the elements of D_sO_n.

➢ *Results in D_sD_n*

Table 3 The Table Displays the Sequence of Elements in D_sD_n

N	$\lvert S \rvert = \frac{1}{2}n^4 - \frac{13}{3}n^3 + \frac{31}{2}n^2 - \frac{71}{3}n + 13$
1	1
2	1
3	5
4	17
5	53

➢ *Results in D_sC_n*

Table 4 The Table Displays the Sequence of Elements in D_sC_n

N	$\lvert S \rvert = -\frac{1}{6}n^3 + 2n^2 - \frac{23}{6}n + 2$
1	0
2	1
3	4
4	8
5	12

V. CONCLUSION

This research investigates the ordder of disparity of elements in full transformation semigroup D_sT_n and it subsemigroups D_sO_n, D_sD_n and D_sC_n. We also obtained the general n^{th} term for the sequences derived. The study found that the disparity of elements in the considerd semigroup is bounded by 2. These results have a wide range of applications in computational algebra, coding theory,automata theory,game theory, it will also be of importance in some branches of physics and theoretical computer science.

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