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Orthogonal Polynomials in Vibration Analysis of Nonhomogeneous Triangular Plate

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Abstract: In his paper, free transverse vibration behavior of nonhomogeneous equilateral triangular plate of variable thickness using 2-D boundary characteristic orthogonal polynomials in the Rayleigh-Ritz method is presented. The material's elastic properties and thickness are functions of in-plane coordinates. The first three natural frequencies for four different combinations of clamped, simply supported and free edges have been computed. Effects of various plate parameters on frequencies has been studied. Transverse displacements in the form of 3-D graphs for all the four boundary conditions have been presented. A comparison of results with those available in the literature has been presented.

Keywords: *Nonhomogeneous, Orthotropic, Equilateral triangular, Non-uniform, Rayleigh-Ritz.*

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I. INTRODUCTION

A large number of papers dealing with dynamic behavior of composite plates are available in the literature. Leissa [1-7] presents the review of work up to 1985 on linear vibration of plates. Confining our discussion on triangular isotropic/orthotropic plates, for later work up to 1998, the notable contributions are reported by Singh and Hassan [8] and Bambill et al. [9]. A survey of works concerning the vibration of triangular plates is reported in references [10-23]. Sakiyama and Huang [10] studied free vibration of right triangular plates of variable thickness using Green's function. Huang et al. [11] used Green's function to study free transverse vibration of anisotropic right triangular plates. Sakiyama et al. [12] conducted free vibration analysis of orthotropic right cantilever triangular plates. Haldar et al. [13] used finite element method for bending analysis of thick/thin triangular plate. Nallim et al. [14] studied vibration of general composite triangular plates with elastically restrained edges using Rayleigh-Ritz method. Zhou and Zheng [15] used moving least square Ritz method for vibration analysis of rectangular and triangular plates. Various other methods, such as finite element method [16, 20], superposition method [17], triangular differential quaderature method [18], Rayleigh-Ritz method [19, 21], Ritz method [22], series solution [23] have been used in the investigation of vibration of triangular plates.

Orthotropic plates are often nonhomogeneous. The references [24-27] are concerned with nonhomogeneous materials. However, rising temperature in aerospace technology and missile technology may also cause nonhomogeneity. The present paper studies transverse vibration of thin nonhomogeneous orthotropic equilateral non-uniform triangular plate. The approximate solution is obtained using Rayleigh-Ritz method for four different combinations of clamped (C), simply supported (S) and free (F) edge conditions. The Young's moduli, shear modulus and density of the plate material are dependent on in-plane coordinates. Mode shapes have been shown for the specified plate.

II. FORMULATION AND SOLUTION OF THE PROBLEM

A thin nonhomogeneous orthotropic equilateral triangular plate of varying thickness $h(x, y)$ and density $\rho(x, y)$ bounded by the sides $y = 0, y = c(x-a)/(b-a)$ and $y = cx/b$ specified by domain in $xy-$ plane is considered. The geometry of the plate is shown in Fig. 1.

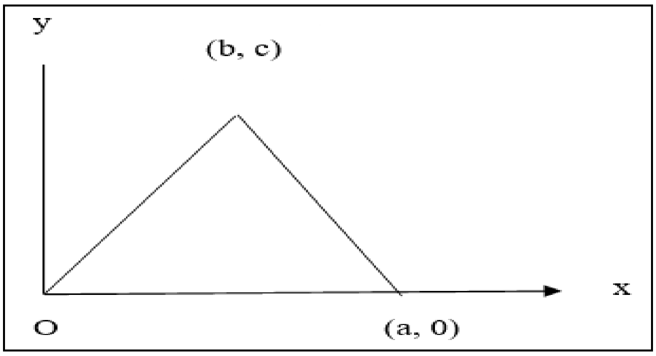

Fig 1 Geometry of the Equilateral Triangular Plate

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The strain energy of the plate is given by

$$V = \frac{1}{2}\iint_{R_1}[D_x w_{xx}^{\ 2} + 2\upsilon_x D_y w_{xx} w_{yy} + D_y w_{yy}^{\ 2} + 4D_{xy} w_{xy}^{\ 2}]\,dx\,dy \tag{1}$$

The kinetic energy of the plate is given by

$$T = \frac{1}{2}\iint_{R_1}\rho h\left(\frac{\partial w}{\partial t}\right)^2 dx\,dy\,, \tag{2}$$

Where $D_x = \frac{E_x h^3}{12(1-\upsilon_x\upsilon_y)}, D_y = \frac{E_y h^3}{12(1-\upsilon_x\upsilon_y)}, D_{xy} = \frac{G_{xy}h^3}{12}, D_x\upsilon_y = D_y\upsilon_x$, $w(x,y,t)$ is the deflection, t is the time, and E_x , E_y , υ_x,υ_y , G_{xy} have usual meanings.

The deflection function $w(x,y,t)$ is assumed to be,

$$w(x,y,t) = \overline{W}(x,y)\sin(\omega t). \tag{3}$$

With the help of relation (3), expressions (1) and (2) become

$$V_{\max} = \frac{1}{2}\iint_{R_1}[D_x \overline{W}_{xx}^{\ 2} + 2\upsilon_x D_y \overline{W}_{xx}\overline{W}_{yy} + D_y \overline{W}_{yy}^{\ 2} + 4D_{xy}\overline{W}_{xy}^{\ 2}]\,dx\,dy \tag{4}$$

$$T_{\max} = \frac{\omega^2}{2}\iint_{R_1}\rho h \overline{W}^2\,dx\,dy\,. \tag{5}$$

The Rayleigh quotient is given by

$$\omega^2 = \frac{V_{\max}}{(T_{\max}/\omega^2)}\,. \tag{6}$$

Using the transformation $(x,y) = (aX + bY, cY)$, the domain gets converted to right angle triangle.

The following variation [28] of Young's moduli, shear modulus and density of the plate material

$$E_x = E_1(1+\alpha X), E_y = E_2(1+\beta Y)\,,\ \ G_{xy} = G_0(1+\gamma_1 X + \gamma_2 Y), \rho = \rho_0(1+\delta_1 X + \delta_2 Y) \tag{7}$$

Where $X = x/a, Y = y/b, W = \overline{W}/a$ is considered.

Thickness of the plates is taken as follows

$$h(X,Y) = ah_0(1+\psi_1 X)(1+\psi_2 Y)\,, \tag{8}$$

Where $E_1, E_2, \rho_0, G_0, h_0$ are material constants and thickness at origin. The displacement function is assumed as

$$W(X,Y) = \sum_{k=1}^{N} d_k \hat{\phi}_k(X,Y), \tag{9}$$

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Where N is the maximum number of terms taken in approximation, d_k's are unknowns and $\hat{\phi}_k$ are orthonormal polynomials.

Orthogonal polynomials ϕ_k over the region $0 \le X \le 1,\ 0 \le Y + X \le 1$ have been generated using the set of functions $S_k = s\, s_k, k = 1,2,3, \ldots \ldots$with

$$s = X^p Y^q (1 - X - Y)^r, s_k = \{1, X, Y, X^2, XY, Y^2, \ldots \ldots \ldots\}, \tag{10}$$

Where $p = 0,1\ or 2$ as the edge $X = 0$ is free, simply supported or clamped. Same justification can be given to q and r for the edges $Y = 0$ and $X + Y = 1$.

$$\emptyset_1 = S_1, \emptyset_k = S_k - \sum_{j=1}^{k-1} \beta_{kj} \emptyset_j,$$

$$\beta_{kj} = \frac{< S_k, \emptyset_j >}{< \emptyset_j, \emptyset_j >}, \quad j = 1,2,3,, \ldots \ldots (k-1), k = 2,3,4, \ldots \ldots, N. \tag{11}$$

The inner product $< \emptyset_1, \emptyset_2 >$ is defined as

$$< \phi_1, \phi_2 >= \int_0^1 \int_0^{1-X} (1 + \delta_1 X + \delta_2 Y)(1 + \psi_1 X)\ (1 + \psi_2 Y) \phi_1(X,Y)\ \phi_2(X,Y)\, dX\, dY, \tag{12}$$

And the norm of the function ϕ_1 is given by

$$\| \phi_1 \| = < \phi_1, \phi_1 >^{1/2} = \left[\int_0^1 \int_0^{1-X} (1 + \delta_1 X + \delta_2 Y)(1 + \psi_1 X)(1 + \psi_2 Y) \phi_1^{\,2}(X,Y)\, dX\, dY \right]^{1/2} \tag{13}$$

Orthonormal function $\widehat{\emptyset}_k$ is obtained as $\hat{\phi}_k = \dfrac{\phi_k}{\| \phi_k \|}$.

The eigenvalue problem is given by

$$\sum_{k=1}^{N} (a_{jk} - \Omega^2 \delta_{jk}) d_k = 0, j = 1,2,3, \ldots \ldots, N, \tag{14}$$

Where $H^* = E_2^{\,*} \upsilon_x + 2G^*$, $\Omega^2 = \dfrac{\rho_0 h_0 a^4 \omega^2}{H^*}$ and $\delta_{jk} = \begin{cases} 1, & if \quad j = k \\ 0, & if \quad j \ne k \end{cases}$

The integrals involved in Eq. (15) have been evaluated using the formula [8]

$$\int_0^1 \int_0^{1-X} X^p Y^q (1 - X - Y)^r\, dX dY = \frac{p!\, q!\, r!}{(p + q + r + 2)!}.$$

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III. BOUNDARY CONDITIONS AND RESULTS

The four boundary conditions namely CCC, CSC, CFC and SSS have been considered. The Eq. (14) is solved to get frequency parameter Ω. The values of different parameters are taken as follows.

$\alpha,\beta,\gamma_1,\gamma_2,\delta_1,\delta_2=-0.5(0.2)\ 0.5$; $\psi_1,\psi_2=-0.5(0.2)\ 0.5$; N=47, $b/c=1/\sqrt{3}, c/a=\sqrt{3}/2$.

$E_1^*/H^*=13.9,\ E_2^*/H^*=0.79, \upsilon_x=0.28$ from [28].

Table 1 shows the convergence of frequency parameter Ω. A comparison of frequency parameter Ω for isotropic and homogeneous triangular plate of uniform thickness is presented in Table 2.

Table 1 Convergence of Frequency Parameter Ω of Orthotropic Equilateral Triangular Plate for $\alpha=\beta=\gamma_1=\gamma_2=\delta_1=\delta_2=\psi_1=\psi_2=0.5$

		CCC				**CSC**	
		Mode				**Mode**	
N	**I**	**II**	**III**	**N**	**I**	**II**	**III**
10	240.355	405.801	535.880	10	214.630	407.809	495.632
30	238.471	381.225	516.060	30	207.850	345.220	469.454
46	238.468	381.028	515.794	46	207.834	344.898	468.598
47	238.468	381.028	515.794	47	207.834	344.898	468.598
		CFC				**SSS**	
		Mode				**Mode**	
N	**I**	**II**	**III**	**N**	**I**	**II**	**III**
10	137.479	277.854	352.381	10	127.561	283.562	371.837
30	135.740	243.555	346.482	30	119.989	228.791	333.764
46	135.718	243.405	346.106	46	119.981	228.530	333.056
47	135.718	243.405	346.106	47	119.981	228.530	333.056

Table 2 Comparison of Frequency Parameter Ω of Isotropic $(E_1^*/H^*=E_2^*/H^*=1, \upsilon_x=0.3)$ Homogeneous $(\alpha=\beta=\gamma_1=\gamma_2=\delta_1=\delta_2=0)$ uniform $(\psi_1=\psi_2=0)$ Equilateral Triangular Plate

Boundary Conditions	**Ref.**		**Mode**	
		I	**II**	**III**
CCC	14	99.0203	189.0076	189.0143
	29	99.020	189.02	189.22
	30	99.02	189.0	189.0
	31	99.0167	-	-
	Present	**99.019**	**189.006**	**189.006**
SSS	14	52.6381	122.8274	123.8374
	29	52.638	122.85	123.95
	30	52.64	122.8	122.8
	31	52.6368	-	-
	Present	**52.638**	**122.822**	**122.822**

The results are presented in Figs. 2-5. The frequency parameter Ω is observed decreasing in the order of boundary conditions CCC > CSC> CFC> SSS for the same set of values of plate parameters. The effect of nonhomogeneity parameter α on the frequency parameter Ω for $\beta=\pm0.5$, $\gamma_1=0.5$, $\gamma_2=0.5$, $\delta_1=\pm0.5$, $\delta_2=0.5$, $\psi_1=0.5$, $\psi_2=0.5$ is shown in Fig. 2. The frequency parameter Ω is observed increasing with increasing values of α for all four boundary conditions. The frequency parameter Ω is observed increasing with increasing values of β while decreasing with the increasing value of δ_1.

The behavior of the frequency parameter Ω with nonhomogeneity parameter γ_1 for $\alpha=0.5$, $\beta=0.5$, $\gamma_2=\pm0.5$, $\delta_1=\pm0.5$, $\delta_2=0.5$, $\psi_1=0.5$, $\psi_2=0.5$ is shown in Fig. 3. The frequency parameter Ω is observed increasing with increasing value of γ_1. The frequency parameter Ω is found increasing with increasing value of

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γ_2. The value of Ω is observed decreasing with the increasing value of δ_1.

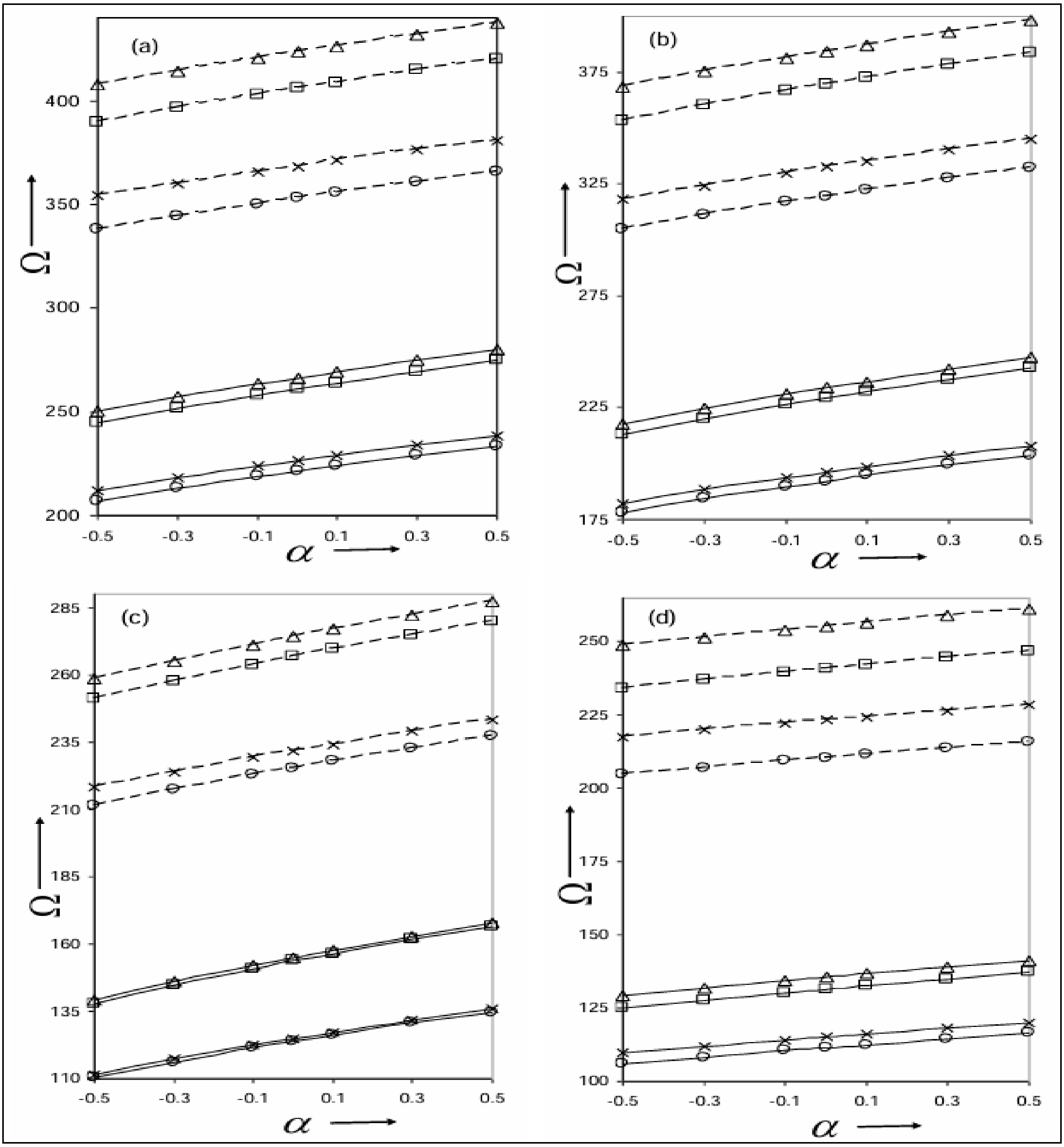

Fig 2 Graph of Ω for (a) CCC (b) CSC (c) CFC (d) SSS Plate for $\gamma_1 = \gamma_2 = \delta_2 = \psi_1 = \psi_2 = 0.5.$, Mode I ; Mode II; $\Delta, \beta = -0.5, \delta_1 = -0.5$; o, $\beta = -0.5, \delta_1 = 0.5$; $\Delta, \beta = 0.5, \delta_1 = -0.5$; ×, $\beta = 0.5, \delta_1 = 0.5$

The effect of density parameter δ_1 on the frequency parameter Ω for, $\beta = 0.5$, $\gamma_1 = 0.5$, $\gamma_2 = 0.5$, $\delta_2 = \pm 0.5$, $\psi_1 = \pm 0.5$, $\psi_2 = 0.5$ is shown in Fig. 4. The frequency parameter Ω is observed decreasing with increasing value of δ_1 for all the four boundary conditions. The value of Ω is observed decreasing with the increasing value of δ_2 and increasing with increasing value of ψ_1.

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The effect of thickness parameter ψ_1 on the frequency parameter Ω for $\alpha=\pm0.5$, $\beta=0.5$, $\gamma_1=0.5$, $\gamma_2=0.5$, $\delta_1=0.5$, $\delta_2=0.5$, $\psi_2=\pm0.5$ is shown in Fig. 5. The frequency parameter Ω is observed increasing with increasing value of ψ_1 for all the four boundary conditions. The value of Ω is observed increasing with the increasing value of α as well as ψ_2.

The behavior of frequency parameter Ω for third mode of vibration remains same for all the four boundary conditions. Transverse displacement in first three modes is shown in the form of 3-D graphs in Fig. 6 for all the four boundary conditions.

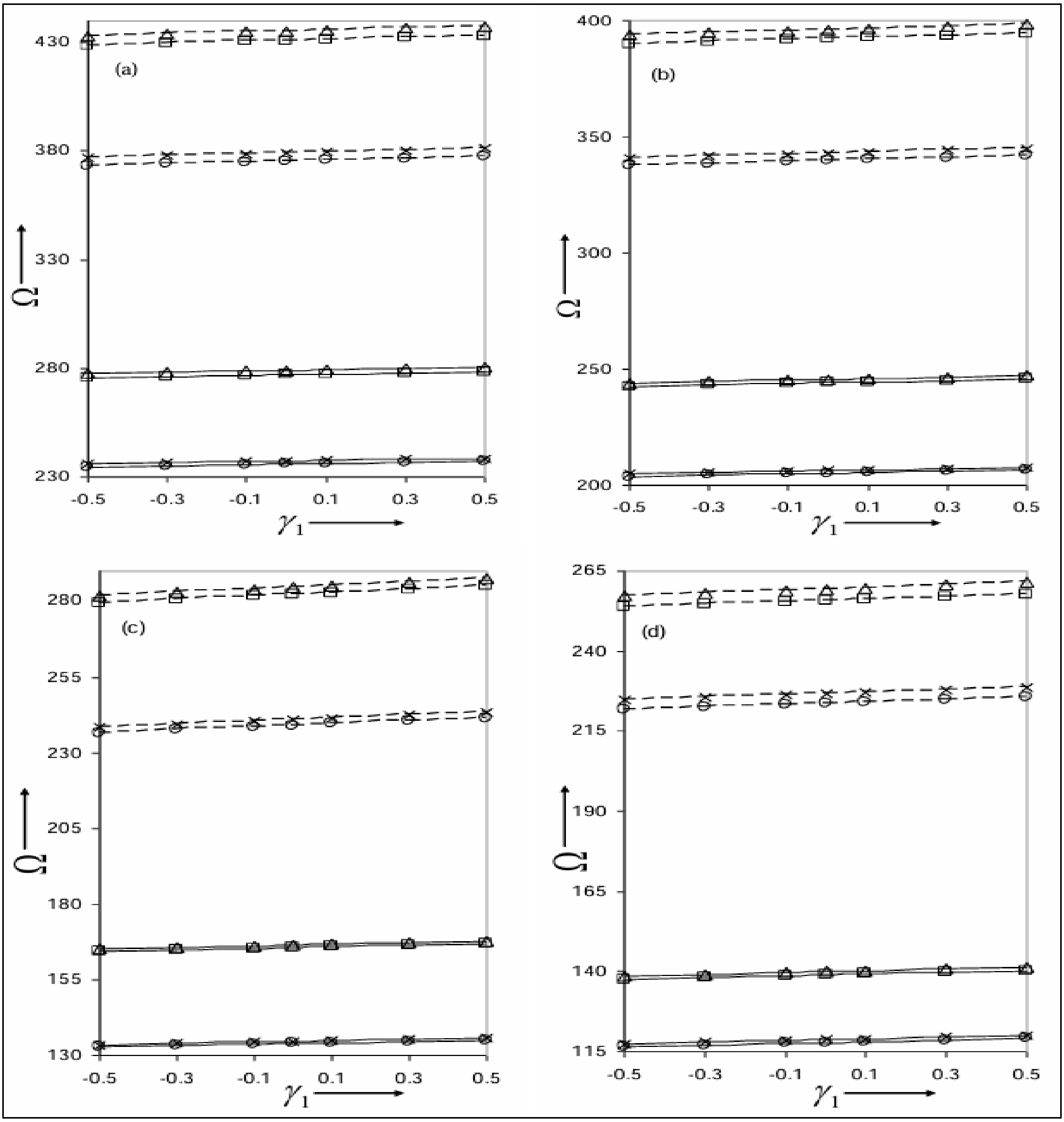

Fig 3 Graph of Ω for (a) CCC (b) CSC (c) CFC (d) SSS Plate for $\alpha=\beta=\delta_2=\psi_1=\psi_2=0.5.$ Mode I ; Mode II;
$\Delta, \gamma_2=-0.5, \delta_1=-0.5$; o, $\gamma_2=-0.5, \delta_1=0.5$; $\Delta, \gamma_2=0.5, \delta_1=-0.5$; $\times, \gamma_2=0.5, \delta_1=0.5$

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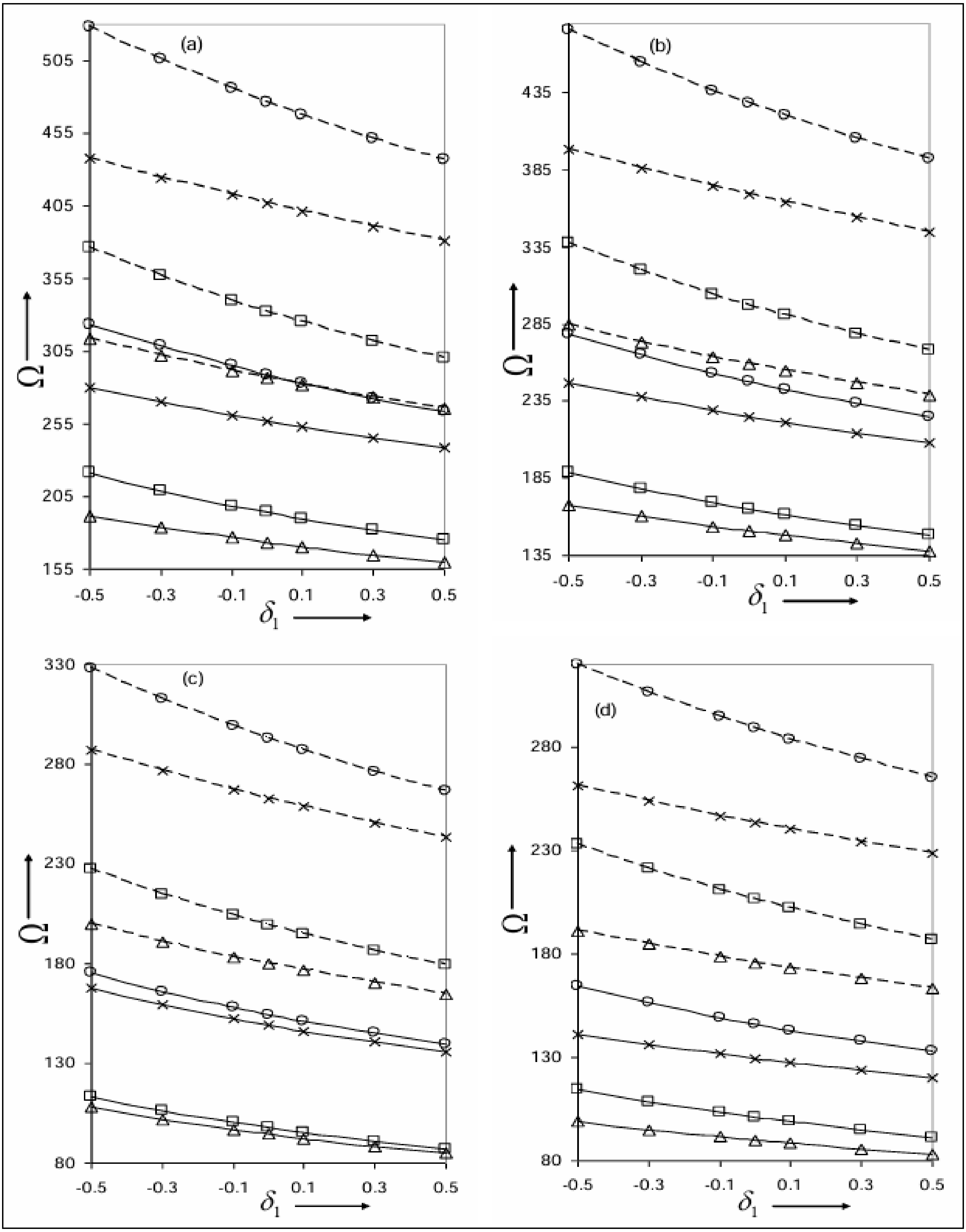

Fig 4 Graph of Ω for (a) CCC (b) CSC (c) CFC (d) SSS Plate for $\alpha = \beta = \gamma_1 = \gamma_2 = \psi_2 = 0.5.$, Mode I; Mode II;
Δ, $\delta_2 = -0.5, \psi_2 = -0.5$; o, $\delta_2 = -0.5, \psi_2 = 0.5$; Δ, $\delta_2 = 0.5, \psi_2 = -0.5$; ×, $\delta_2 = 0.5, \psi_2 = 0.5$

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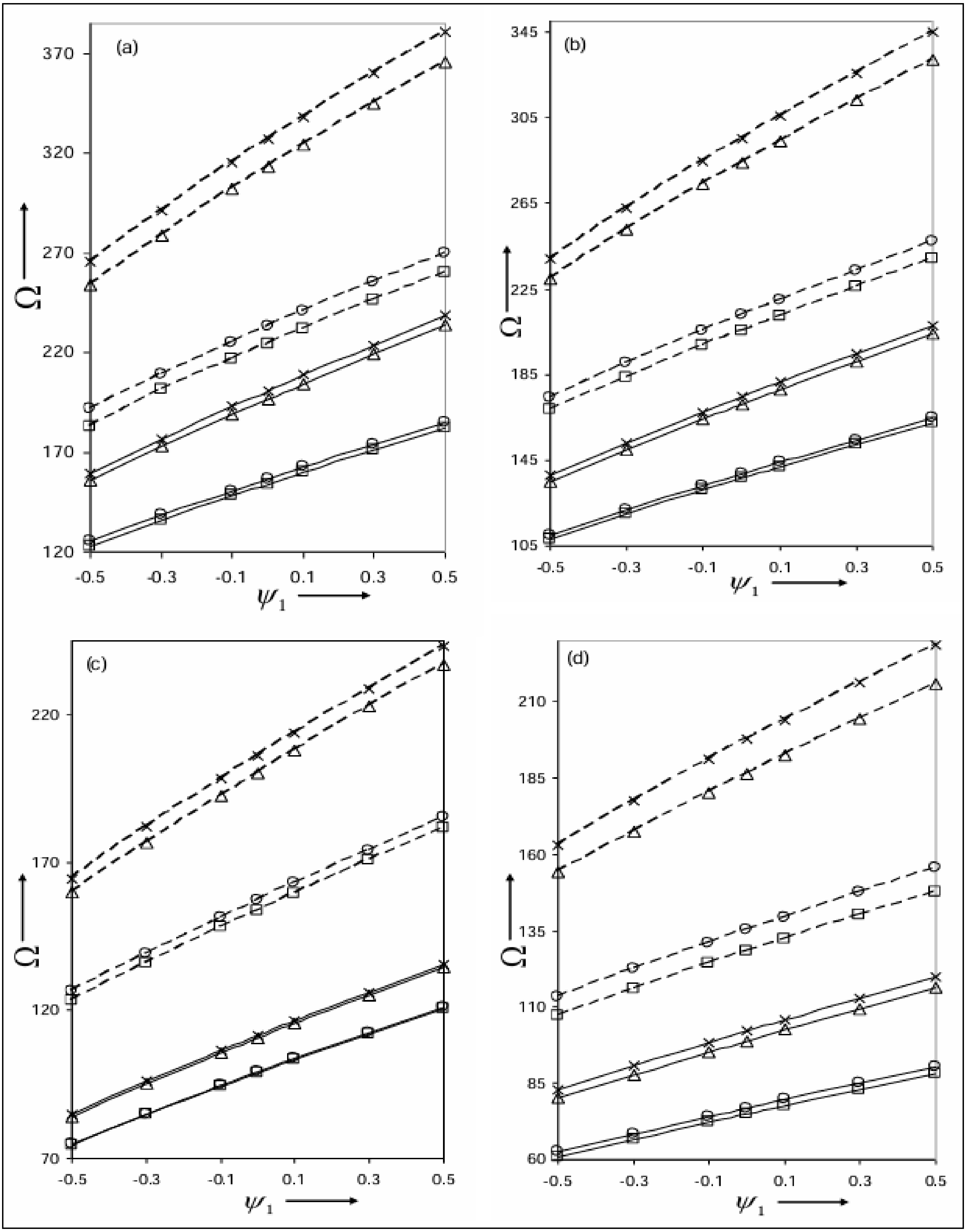

Fig 5 Graph of Ω for (a) CCC (b) CSC (c) CFC (d) SSS Plate for $\alpha = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = 0.5$. Mode I; Mode II; $\Delta, \psi_2 = -0.5$, $\beta = -0.5$; o, $\psi_2 = -0.5$, $\beta = 0.5$; $\Delta, \psi_2 = 0.5$, $\beta = -0.5$; $\times, \psi_2 = 0.5$, $\beta = 0.5$

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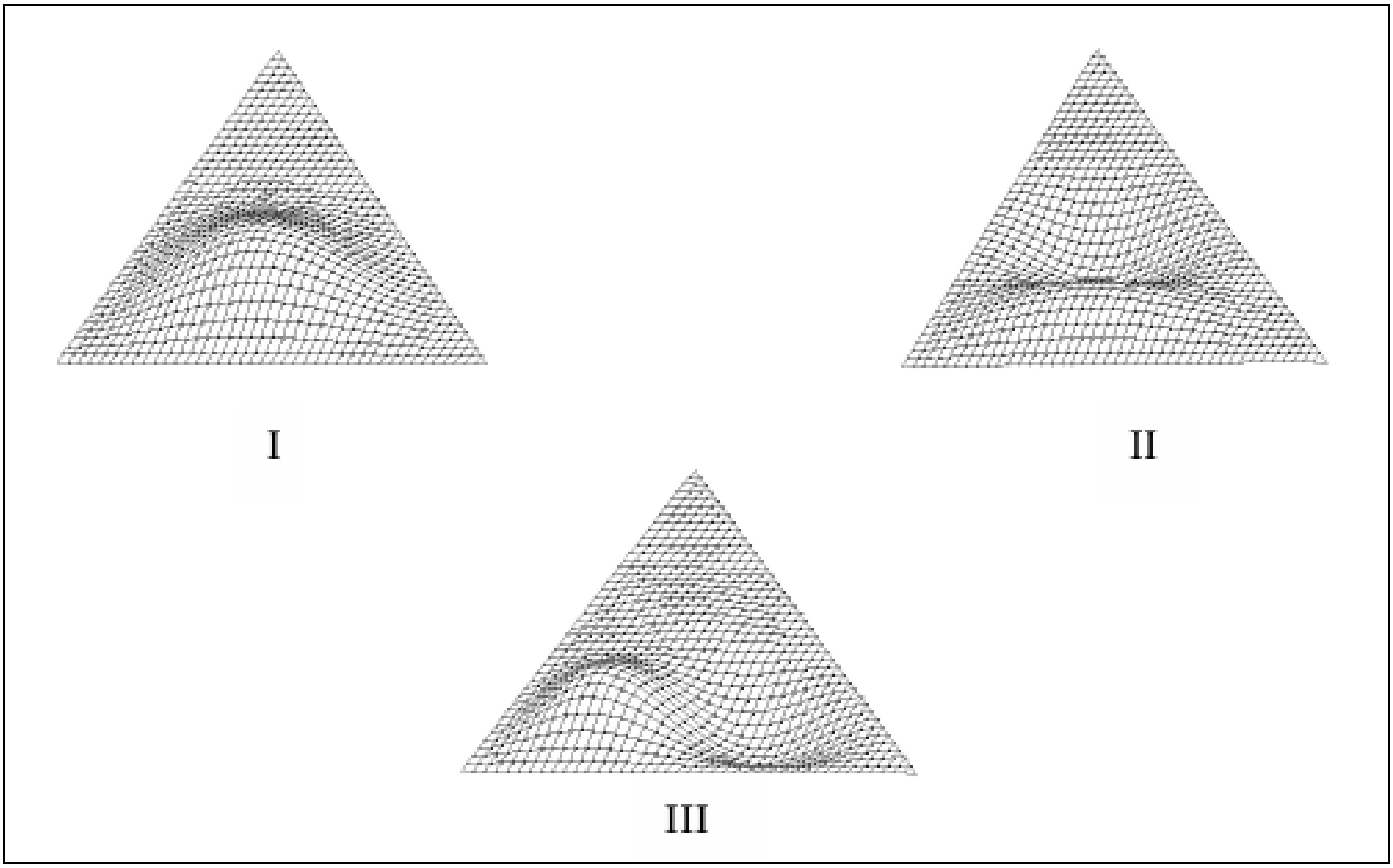
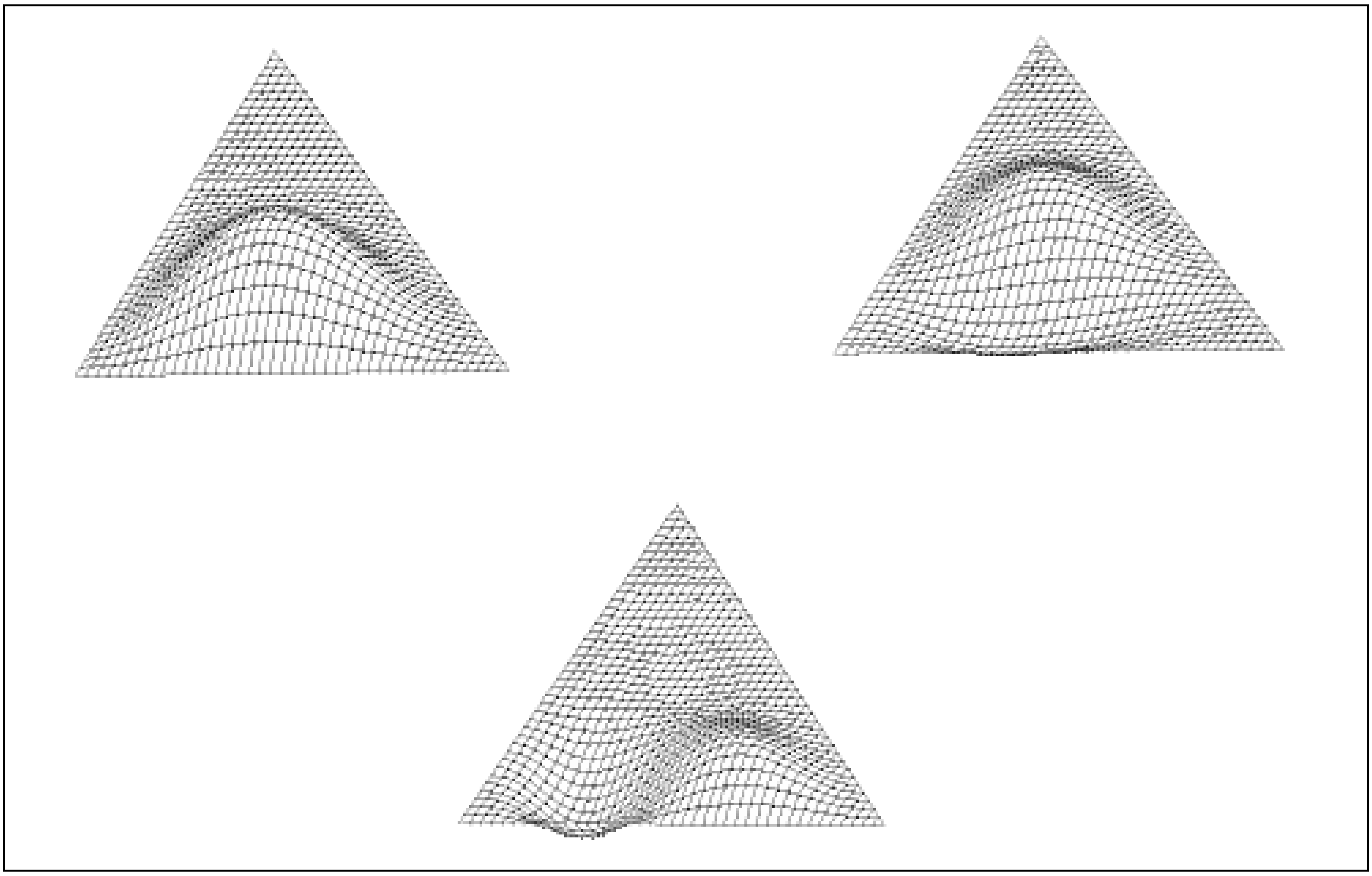
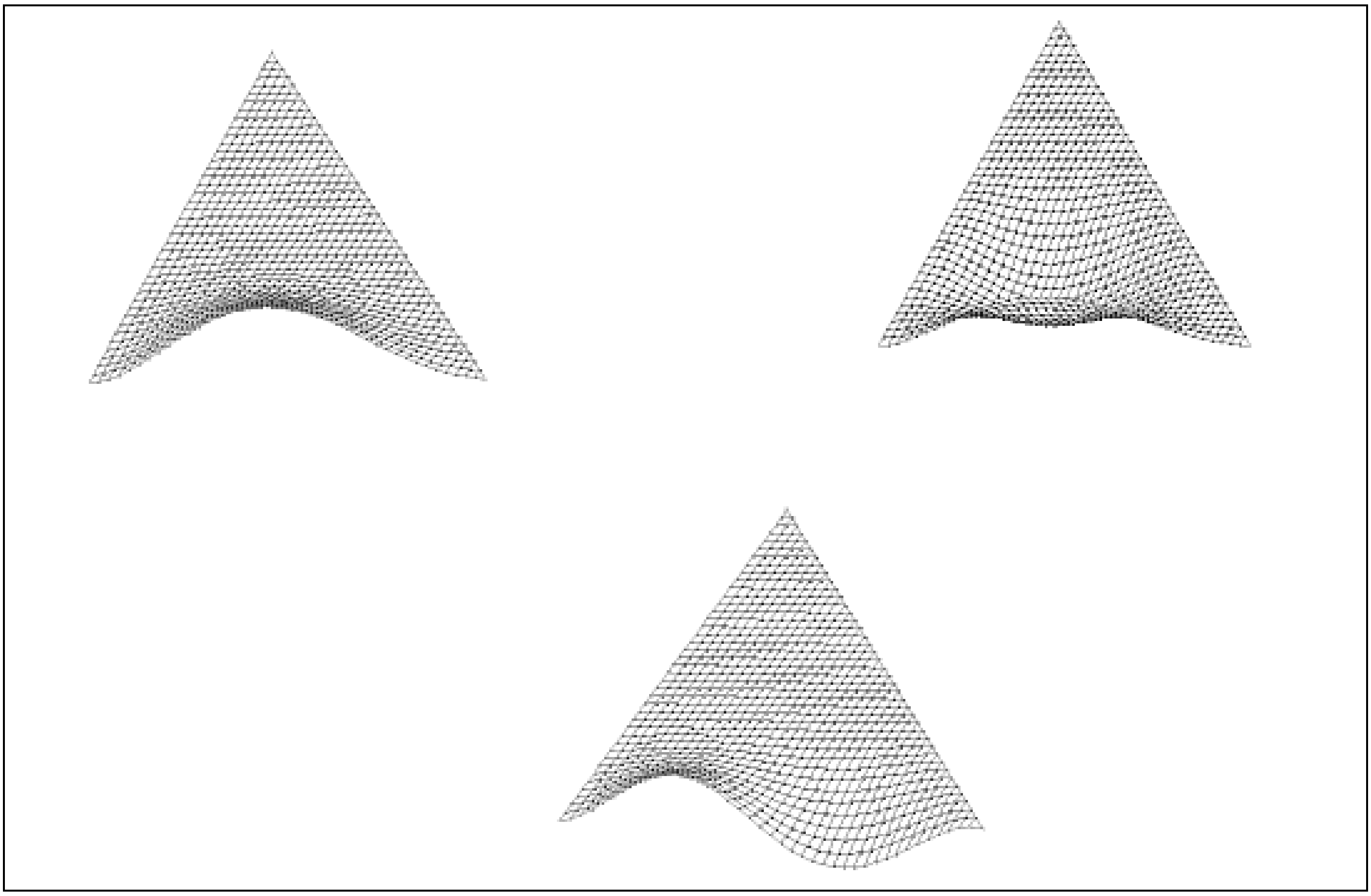


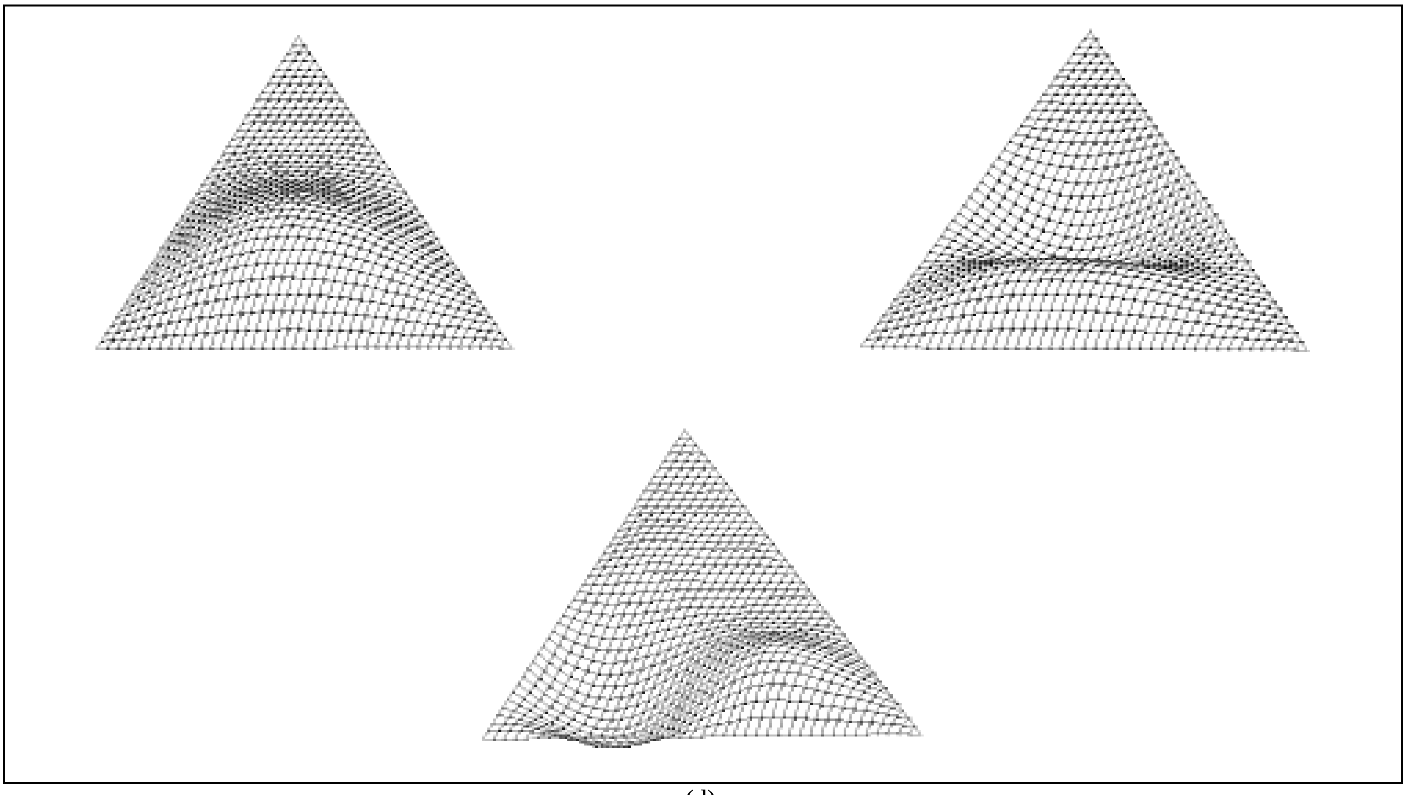
(a)



(b)



(c)



(d)

Fig 6 First Three Mode Shapes of (a) CCC (b) CSC (c) CFC (d) SSS Orthotropic Equilateral Triangular Plate for
$$\alpha = \beta = \gamma_1 = \gamma_2 = \delta_1 = \delta_2 = \psi_1 = \psi_2 = 0.5.$$

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IV. CONCLUSION

From the analysis one can observe that the frequency parameter Ω increases as the values of the the parameters $\alpha, \beta, \gamma_1, \gamma_2$ increase and effect is opposite with the increasing values of δ_1 and δ_2 . The frequency parameter also increases as the thickness of the plate increases towards the edge $X+Y=1$. For the first mode of vibration, frequency parameter Ω observes the percentage variations as -6.4 to 5.4, -7.1 to 6.0, -10.8 to 8.7 and -4.9 to 4.4 for CCC, CSC, CFC and SSS boundary conditions, respectively for corresponding change in α from -0.5 to 0.5. For the first mode of vibration, frequency parameter Ω observes the percentage variations as -0.6 to 0.6, -0.8 to 0.8, -1.1 to 1.0 and -1.2 to 1.2 for CCC, CSC, CFC and SSS boundary conditions, respectively for corresponding change in γ_1 from -0.5 to 0.5. The corresponding variations are -10.5 to 8.1, -11.2 to 8.5, -13.3 to 9.6 and -10.7 to 8.2 when δ_1 changes from -0.5 to 0.5.

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