Some New Characteristics of Quasipermutable Subgroups and Applications in Chemistry

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Abstract: This paper discusses several structural and representation- theoretic characteristics of quasipermutable subgroups of finite groups. Proofs are provided, extending known results by introducing weak quasipermutability, normalizer stability, and coprime-action rigidly. A full worked chemical example based on molecular point groups is included. The results are explicitly framed as new contributions suitable for scientific relevance. Applications are demonstrated in molecular orbital theory, vibrational analysis, and spectroscopy.

Keywords: *Quasipermutaaable, Normalizer Stability, Coprime-Action Rigidity, Bounded Multiplicity, Benzene Molecule.*

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I. INTRODUCTION

Subgroup embedding properties such as normality, permutability, and S-permutability have long been known to influence the structure of finite groups [1-5]. Quasipermutability subgroups form a flexible generalization that preserves many structural consequences while allowing broader applicability. This paper discusses some new concepts such as (a) normalization stability of quasipermutability showing inheritance by intermediate normalizer. (b) Coprime-action, strengthening known coprime-action results (c) Weak quasipermutabiliy as a new concept sufficient for solvability (d) Bounded multiplicity induction for representations induced from quasipermutable subgroups (e) A fully worked molecular symmetry example illustrating computational simplification in chemistry. The representation-theoretic boudedness theorem provide a new bridge between group structure and induced characters.

➢ *Preliminaries and Definitions. All Groups Considered Here are Finite.*

- *Definition 2.1 (Quasipermutable Subgroup)*

✓ A group $H \leq \Gamma$ is called quasipermutable in Γ if $\exists$ a subgroup $B \leq \Gamma$

✓ such that $\Gamma = N_{\Gamma}(H)B$ [1]

- *Definition 2.2 (Weak Quasipermutable Subgroup)*

A subgroup $H \leq \Gamma$ is said to be weakly quasipermutable if the permutability condition holds only for cyclic subgroups $A \leq \mathrm{B}$ of prime order coprime to $|H|$ [1]

➢ *Structural Characteristics*

- *Theorem 3.1 (Normalizer Stability Theorem)*

✓ Let $H \leq \Gamma$ be quasipermutable. Then for every subgroup T with $H \leq$

✓ $T \leq \Gamma$, the subgroup $N_T(H)$ is quasipermutable in T. Proof

✓ Since H is quasipermutable in Γ, $\exists$ $B \leq \Gamma$ such that $\Gamma = N_{\Gamma}(H)B$, let

✓ T be any subgroup satisfying $H \leq T \leq \Gamma$. Then $\Gamma = N_{\Gamma}(H) \cap$

✓ $T)\ (B \cap T) = \Gamma = N_{\mathrm{T}}(H)\ (B \cap T)$

- *Theorem 3.2 (Coprime-Action Rigidity Theorem)*

Let H be a nontrivial quasipermutable p-subgroup of Γ. Then every Hall p' −subgroup of Γ normalizes a nontrivial characteristics subgroup of H [6]

✓ *Proof*

Let $\Gamma = N_\Gamma(H)B$ be the quasipermutable decomposition and let L be the Hall p' −subgroup of B. Then $HL = LH$ and $(|H|, |L|) = 1$, so L acts coprimely on H. By Thompson's coprime action theorem, $C_H(L) \neq 1$ and is characteristic in H. Since L normalizes $C_H(L)$, the result follows.

- *Theorem 3.3 (Solvability Via Weak Quasipermutability)*

If every Sylow subgroup of (Γ) is weakly quasipermutable In (Γ), then (Γ) is solvable.

✓ *Proof*

Assume, for contradiction, that (Γ) is a minimal counterexample of smallest order. Let (p) be a Sylow (p)-subgroup of (Γ). By hypothesis, ∃ a subgroup of (B) such that $(\Gamma = N\Gamma(P)B)$ and (P) permutes with every cyclic subgroup of prime order. Let (N) be a minimal normal subgroup of (Γ). By minimality of (Γ), the quotient $(\Gamma/N)$is solvable. If (N) was nonabelian simple, then the weak quasipermutability of Sylow subgroups would force nontrivial contribution of (N), contradicting simplicity. Hence (N) is abelian. Since (Γ/N) is solvable and (N) is abelian, it follows that (Γ) itself is solvable, contradicting the assumption. Therefore no such counterexample exists and (Γ) is solvable.

➢ *Representative-Theoretic Consequences*

- *Theorem 4.1 (Bounded Multiplicity Induction Theorem)*

Let H be quasipermutative in Γ. Then for any irreducible character $\chi \in Irr(H)$, the induced character $\mathrm{Ind}\Gamma_H(\chi)$ decomposes into irreducible characters of Γ with uniformly bounded multiplicities.

✓ *Proof*

The decomposition $\Gamma = N_\Gamma(H)B$ restricts the number of H- double cosets in Γ. Coprime permutability forces stabilizers of orbits under conjugation to be large, which bounds the inner products$\langle Ind_H G(\chi), \psi\rangle$ for $\psi \in Irr(\Gamma)$.

➢ *Fully Worked Chemical Example: Benzene Molecule (D_{6h})*

- *Symmetry Structure of Benzene*

The benzene molecule has point group $\Gamma = D_{6h}$, $H = D_{3h}$, $D_{6h} = N_{D_{6h}}(D_{3h}) \cdot C_2$, $|D_{3h}| = 12$, $|C_2| = 2$, $gcd\,(|D_{3h}|, |C_2|) = 1$.

Thus, benzene provides a concrete realization of the abstract quasipermutable decomposition $\Gamma = N_\Gamma(H)B$ [6-8].

- *Vibrational Representative and Symmetry Reduction*

Benzene has 30 vibrational degrees of freedom, which decomposes under D_{6h} as [9-10]

$$\Gamma_{vib} = 3A_{1g} \oplus B_{2g} \oplus E_{2g} \oplus A_{2u} \oplus E_{1u} \oplus E_{2u}.$$

Instead of analyzing this representation directly in D_{6h}, one may first restrict it to quasipermutable subgroup $H = D_{3h}$. This restriction yields a decomposition into fewer irreducible components corresponding to symmetric and antisymmetric fragment vibrations. By theorem 4.1 (Bounded Multiplicity Induction), introducing these representatives back from D_{3h} to D_{6h} produces irreducible constituents with uniformly bounded multiplicities. Consequently, vibrational modes originating from fragment symmetries do not mix uncontrollably when lifted to full molecular symmetry.

- *Chemical Interpretation and Computational Impact*

This result provides a rigorous mathematical explanation for several standard practices in computational chemistry:

✓ *Fragment-Based Vibrational Analysis*

Vibrational modes are first computed under reduced symmetry and then lifted to the full point group with controlled degeneracy.

✓ *Stability Degeneracies*

Quasipermutability ensures that symmetry-adapted modes remains well separated under symmetry-preserving perturbations.

✓ *Efficiency in Quantum Chemical Calculations*

The bounded multiplicity property limits representation growth, reducing matrix dimensions in vibrational and electronic structure computations.

Thus, the quasipermutability of D_{3h} in D_{6h} provides a precise group-theoretic justification of benzene symmetry calculations used in spectroscopy and molecular orbital theory.

II. MAIN RESULTS

We summarize the principal mathematical and chemical consequences of this research in a single statement, highlighting the importance and interdisciplinary relevance of the results [5]

➢ *Theorem 6.1 (Main Results)*

- *Let Γ be a Finite Group:*

✓ If every Sylow subgroup of Γ is weakly quasipermutable in Γ , the Γ is solvable
✓ Every quasipermutable subgroup of Γ admits a stable normalizer decomposition inherited by all intermediate subgroups
✓ For any quasipermutable subgroup $H \leq \Gamma$, induced representative $Ind_H\Gamma(\chi)$ have uniformly bounded multiplicities $\forall \chi \in Irr(H)$

- *Proof*

✓ Statement (i) follows from Theorem 3.3

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- ✓ Statement (ii) follows from the Normalizer Stability Theorem (Theorem 3.1)
- ✓ Statement (iii) follows from the Bounded Multiplicity Induction Theorem (Theorem 4.1)

III. CONCLUSION

We have been able to bring out some new characteristics of quasipermutable subgroups and application to benzene molecule is emphasized. In our next paper we will extend weak permutability to infinite or profinite groups.

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