ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

Linear – Nonlinear Acoustic Wave Propagation Boundary in Air

Eghuanoye Ikata[1]; Godspower O. Ashaka[2*]

[1]Department of Physics Ignatius Ajuru University of Education Rumuolumeni, Port Harcourt, Nigeria
[2]Department of Physics/Instrumentation and Control Technology Federal Polytechnic of Oil and Gas Bonny, Nigeria

Corresponding Author: Godspower O. Ashaka[2*]

Publication Date: 2026/08/01

Abstract: In certain discussions of acoustic wave propagation, assumptions are introduced that lead to a linear approximation of the governing equations. Implicit in such assumptions is the notion of smallness of a parameter, the size of which is rarely given. We here give a value for the parameter, using a numerical procedure. This study models acoustic wave propagation in air, using a nonlinear acoustic wave equation in one space dimension, by a Riemann invariant based method of characteristics. We simulate propagation of a particle velocity wavelet along a straight pipe, closed at one end and open at the other. The initial amplitude of the wavelet lies between 0.1 and 200 m/s, assuming free space propagation. Data from the computation is analysed by exploiting the known '*steepening*' of a part of the wavelet in finite-amplitude wave propagation. We have calculated the slope of that part of the wavelet, at various times, for wavelets of different initial amplitudes. The results show clearly a demarcation between linear- and nonlinear-acoustic wave propagation (that is, a demarcation between finite-amplitude and small-amplitude wave propagation.) This demarcation is controlled by the opposing influences of energy loss and nonlinear effects. For an acoustic wavelet, the initial amplitude 50 m/s in terms of particle velocity (or 0.02 m in terms of particle displacement, or 0.15 in terms of acoustic Mach number) marks the upper limit beyond which nonlinear effects cannot be ignored. That is how small the size of the initial disturbance should be, for the assumptions made while deriving a linear wave equation to be valid.

Keywords: *Finite-Amplitude Wave; Method of Characteristics; Nonlinear Acoustic Wave Equation.*

How to Cite: Eghuanoye Ikata; Godspower O. Ashaka (2026) Linear – Nonlinear Acoustic Wave Propagation Boundary in Air. *International Journal of Innovative Science and Research Technology*, 11(7), 2245-2252. https://doi.org/10.38124/ijisrt/26jul835

I. INTRODUCTION

Acoustic wave is a disturbance in pressure (or density or velocity) that propagates in a compressible medium. In a fluid, the restoring force responsible for acoustic wave propagation is due to change in pressure. Acoustic wave is an example of a scalar field, as it can be represented using a single field function. Describing the wave by the field approach [1] the relevant field functions are the density ρ(r, *t*), particle velocity u(r, *t*) and pressure p(r, *t*), where r is position vector and *t* is time.

Mathematically, wave phenomenon is represented by a wave equation (supplemented with the appropriate boundary and initial conditions). Often, the acoustic wave equation is written as a second-order partial differential equation for the field function (chosen to denote the wave) though, it may be written as a system of coupled first-order partial differential equations [2, 3].

The acoustic wave equation is derived using the equation of continuity, equation of motion and equation of state. In general, these equations are nonlinear. Occasionally, their linear approximations are used. There are various methods [4 – 6] for obtaining a linear approximation of these equations but all rely on the notion of smallness of a parameter.

The assumptions made to obtain a linear approximation of the governing equations, and a linear acoustic wave equation, constrain one to deal with infinitesimal (or small) amplitude waves. If the wave amplitude is of any significant size, then one has to use the nonlinear equations. Using the nonlinear acoustic wave equation permits the treatment of finite-amplitude waves and shock waves [7, 8].

Knowledge of the boundary (in terms of wave amplitude size) separating linear (or small-amplitude) waves from nonlinear (or finite-amplitude) waves is of both practical and theoretical interest. In this study we try to identify the boundary region between linear and nonlinear acoustic wave propagation.

Here, we solve (numerically) the equations that govern nonlinear acoustic wave propagation in one space dimension and time. We vary the wave amplitude over a range and

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

observe how the amplitude influences the wave propagation. This enables us to locate an upper limit in wave amplitude size which separates linear from nonlinear acoustic wave propagation.

II. EQUATIONS OF CONTINUITY, MOTION AND STATE

The relationship between motion of a fluid particle and its compressions or dilations is given by the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) = 0 . \tag{1}$$

Ignoring the temperature dependence of the viscosities in a Newtonian fluid, the general equation of motion is Navier-Stokes equation, which can be written as

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\bullet\nabla)\mathbf{u}\right] = \rho\mathbf{b} - \nabla p + \left(\eta_B + \frac{4}{3}\eta\right)\nabla(\nabla\bullet\mathbf{u}) - \eta\nabla\times\nabla\times\mathbf{u} \tag{2}$$

Where η is shear viscosity, η_B is bulk viscosity and b is body force.

In adapting Navier-Stokes equation for acoustic wave phenomenon, it is appropriate to admit compressibility and discard circulation. These concepts are represented by the divergence and curl terms (of the velocity), respectively, in equation (2). Also, for acoustic wave in source-free medium, the term representing the body force will not apply. Though gravitational force is always present, acoustic wave with wavelength λ much smaller than a_o^2/g (where a_o is sound speed in undisturbed state and g is gravitational acceleration) is negligibly affected by gravity [9]. Finally, assuming a lossless medium the viscosity is zero and the equation of motion in a homogeneous inviscid Newtonian fluid reduces to Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\bullet\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p . \tag{3}$$

The equation of state gives the relationship between the internal restoring force and the corresponding deformation in a medium. The compression of a fluid causes a rise in its temperature and its dilation is accompanied by a temperature decrease, unless allowance is given for heat exchange. The compressions and dilations that compose acoustic wave succeed each other so rapidly that no heat flow is possible. In ordinary circumstances, the wavelength is relatively large and the thermal conductivity too small for any appreciable amount of heat to flow. Thus acoustic wave propagation is an adiabatic process.

The adiabatic equation of state in a fluid may be written, using Taylor expansion, as

$$p = f(\rho) = p_o + \left(\frac{\partial p}{\partial \rho}\right)_{\rho_o}(\rho - \rho_o) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)_{\rho_o}(\rho - \rho_o)^2 + \ldots.. \tag{4}$$

Where p_o is constant equilibrium pressure and ρ_o constant equilibrium density, in the fluid.

By definition the sound speed a_o is given through

$$a_o^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\rho_o} , \tag{5}$$

And the condensation s (that is, fractional change in density) is

$$s = \frac{(\rho - \rho_o)}{\rho_o} . \tag{6}$$

➢ *Nonlinear Terms and Assumptions in Linear Approximation*

Considering the equation of continuity (1), Euler equation (3) and equation of state (4), observe that nonlinearity is inherent in a general discussion of acoustic wave propagation. Nonlinearity enters such a discussion in various forms; namely:

- As a product of two acoustic variables, $\nabla\bullet(\rho\mathbf{u})$, in the equation of continuity;
- As a product of velocity and its derivative, $(\mathbf{u}\bullet\nabla)\mathbf{u}$, in the equation of motion;
- As a product of two nonlinear terms, $\frac{1}{\rho}\nabla p$, in the equation of motion;
- As a term proportional to ρ^n for n > 1, in the equation of state.

In linear acoustic wave propagation the equations of continuity, motion and state are made linear, by introducing the assumptions [5]:

- $\Delta\rho = (\rho - \rho_o) << \rho_o$, or that $\Delta p = (p - p_o) << p_o$;
- $(\mathbf{u}\bullet\nabla)\mathbf{u} << \partial\mathbf{u}/\partial t$, that is, $\mathbf{u}$ and $\nabla\bullet\mathbf{u}$ are very small;
- ρ_o is constant, or that, $\nabla\rho_o$ is very small.

Using the assumptions above, a linear acoustic wave equation written in terms of particle velocity u (or condensation s, or acoustic pressure Δp) can be derived [6]:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - a_o^2\nabla^2\mathbf{u} = 0 . \tag{7}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

Note that a theoretical discussion of acoustic wave assumes a fluid at rest in equilibrium. Hence, in equation (7) and subsequently u refers to particle velocity due to the acoustic disturbance.

Implicit in the above assumptions is the notion of smallness, either of the condensation s, or acoustic pressure Δp, or particle velocity u. One may thus want to ask: "*how small is small*?" Here, we attempt to answer the question numerically.

➢ *Features of Nonlinear Waves*

For acoustic wave propagation along x direction, in one space dimension, the continuity equation and Euler equation of motion reduce to.

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0 \tag{8a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{a^2}{\rho}\frac{\partial\rho}{\partial x}=0 \tag{8b}$$

Where the third term in equation (8b) follows from $\frac{\partial p}{\partial x}=\frac{\partial p}{\partial\rho}.\frac{\partial\rho}{\partial x}=a^2\frac{\partial\rho}{\partial x}$, using equation (5). In the nonlinear case, the sound speed *a* is, also, a function of space and time. The two coupled first-order partial differential equations (8) describe a nonlinear acoustic wave, in one space dimension.

Nonlinear wave equations have been studied both analytically and numerically [10, 11]. Analytical and numerical studies exist for the system of coupled equations (8) [12, 13]. From these studies it is observed that the basic features of a solution to a nonlinear wave equation are that:

- Local wave speed is not constant;
- Wave shape changes as wave propagates;
- A part of the solution may steepen as the wave propagates;
- A part of the solution may flatten as the wave propagates;
- The mathematical solution may be multi-valued at some later time, though the physical solution will develop a shock.

In this study we exploit the aforementioned steepening (of a part of the wave shape) as the wave propagates. Given this steepening, the gradient (that is, slope) of that part of the profile which steepens will change, as the wave propagates. We have calculated this gradient at various time steps, for a wavelet of a certain initial amplitude. Then, we have compared the trend in the variation of the gradient (as the wave propagates) in wavelets of different initial amplitudes.

➢ *Acoustic Wave Equation in Polytropic Fluid*

The equation of state of a polytropic fluid is of the form [8]:

$$p=k\rho^{\gamma}, \tag{9}$$

Where k and γ are a constants. The pressure – density relation (9) is compatible with the ideal gas equation of state [12]

$$\left(\frac{p}{p_o}\right)=\left(\frac{\rho}{\rho_o}\right)^{\gamma}, \tag{10}$$

Where γ = 1.4 is ratio of specific heats, and the Tait equation of state for water [14]

$$\left(\frac{p}{p_o}\right)=(A+1)\left(\frac{\rho}{\rho_o}\right)^{n}-A, \tag{11}$$

With A = 3000, n = 7, p_o = 1 atm and ρ_o = 998 kg/m^3 at 20 °C.

Using equation (9)

$$a^2=\frac{dp}{d\rho}=\gamma k\rho^{\gamma-1}. \tag{12}$$

Recalling that the local sound speed *a* is a function of space and time, that is, $a=f(x,t)$, it follows that

$$\frac{\partial\rho}{\partial t}=\frac{2\rho}{(\gamma-1)a}\frac{\partial a}{\partial t}\quad\text{and}\quad\frac{\partial\rho}{\partial x}=\frac{2\rho}{(\gamma-1)a}\frac{\partial a}{\partial x} \tag{13}$$

We can rewrite equation (8) in terms of *a* and *u*, by eliminating ∂ρ/∂x and ∂ρ/∂t using equation (13), to obtain

$$\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}+\left(\frac{\gamma-1}{2}\right)a\frac{\partial u}{\partial x}=0, \tag{14a}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\left(\frac{2}{\gamma-1}\right)a\frac{\partial a}{\partial x}=0. \tag{14b}$$

Then, we multiply equation (14a) by 2/(γ – 1) to obtain

$$\frac{2}{\gamma-1}\left(\frac{\partial a}{\partial t}+u\frac{\partial a}{\partial x}\right)+a\frac{\partial u}{\partial x}=0. \tag{14c}$$

Finally, we add equations (14b) to (14c), and subtract equation (14c) from (14b) to obtain:

$$\left[\frac{\partial}{\partial t}+(u+a)\frac{\partial}{\partial x}\right]\left(u+\frac{2a}{\gamma-1}\right)=0, \tag{15}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

$$\left[\frac{\partial}{\partial t}+(u-a)\frac{\partial}{\partial x}\right]\left(u-\frac{2a}{\gamma-1}\right)=0\,. \tag{16}$$

The quantities $R^{\pm} = u \pm 2a/(\gamma - 1)$ are called Riemann invariants. Following the theory of characteristics for first-order quasi-linear partial differential equations [15] $R^{\pm}$ are constants on curves (called characteristics) in the x-t plane that have slopes $dx/dt = u \pm a$.

III. RIEMANN INVARIANT BASED METHOD OF CHARACTERISTICS

The method of characteristics, in its various adaptations [16 – 18], is a popular technique for numerically solving nonlinear wave equations. The coupled system of first-order partial differential equations (15) and (16) are solved, numerically, for the field functions $a(x, t)$ and $u(x, t)$. The numerical method [12] makes use of the fact that Riemann invariants $R^{\pm}$ are constants along characteristics in the x-t plane.

The space-time continuum of the problem is replaced with a space-time lattice, Figure 1, and the field values are specified at nodes of the computation lattice (a procedure called discretization). On Figure 1, the indices i and j refer to space and time axes, respectively, with $0 \le i \le M$ and $0 \le j \le N$. Thus we write $u(x, t) \equiv u(i\Delta x, j\Delta t) \equiv u_{i,j}$, where Δx is space increment and Δt is time increment.

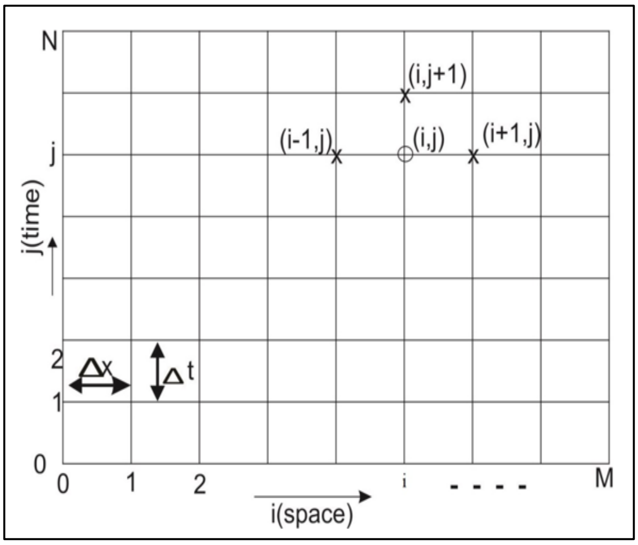


Fig 1 Computation Lattice

Suppose the initial values for a and u are given (or that a and u have already been computed at all nodes at time t_j) and the field values at an arbitrary node at time $t > 0$ are to be computed, say at node C($i, j + 1$) in Figure 2.

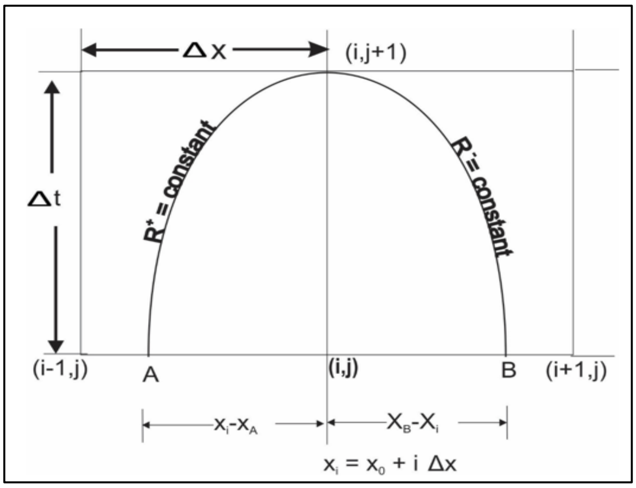


Fig 2 Characteristics on Rectangular Lattice

Through node C there are two characteristics (one of slope $u + a$ and the other of slope $u - a$) which intersect the x-axis at points A and B, respectively. Since a Riemann invariant is constant on a characteristic, $R_C^+ = R_A^+$ and $R_C^- = R_B^-$. That is,

$$u_C+\frac{2a_C}{\gamma-1}=u_A+\frac{2a_A}{\gamma-1}\,, \tag{17}$$

$$u_C-\frac{2a_C}{\gamma-1}=u_B-\frac{2a_B}{\gamma-1}\,. \tag{18}$$

Solving for a_C and u_C, by subtracting and adding equations (17) and (18), gives

$$a_C=\frac{\gamma-1}{4}(u_A-u_B)+\frac{1}{2}(a_A+a_B)\,, \tag{19}$$

$$u_C=\frac{1}{2}(u_A+u_B)+\frac{1}{\gamma-1}(a_A-a_B)\,. \tag{20}$$

However, since the points A and B, in Figure 2, are not nodes of the computation lattice, the field values at such points are not defined.

If the distance between A and B is small, the characteristics can be assumed to be straight lines with the approximate slopes $u_{i,j} + a_{i,j}$ and $u_{i,j} - a_{i,j}$. Noting that the slope has dimension of speed, from Figure 2, we can deduce locations of A and B as.

$$x_A=i\,\Delta x-(u_{i,j}+a_{i,j})\Delta t\,, \tag{21}$$

$$x_B=i\,\Delta x-(u_{i,j}-a_{i,j})\Delta t\,. \tag{22}$$

With the location of points A and B, the field values at those points are obtained by linear interpolation [19] using the

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

known field values at nodes on the same time level. Using the field at nodes (i – 1, j) and (i, j), a_A and u_A are given as.

$$a_A = a_{i,j} + \frac{\Delta t}{\Delta x}\left(u_{i,j} + a_{i,j}\right)\left(a_{i-1,j} - a_{i,j}\right), \tag{23}$$

$$u_A = u_{i,j} + \frac{\Delta t}{\Delta x}\left(u_{i,j} + a_{i,j}\right)\left(u_{i-1,j} - u_{i,j}\right). \tag{24}$$

Similarly, using field values at nodes (i, j) and (i + 1, j), a_B and u_B are given as

$$a_B = a_{i,j} - \frac{\Delta t}{\Delta x}\left(u_{i,j} - a_{i,j}\right)\left(a_{i+1,j} - a_{i,j}\right), \tag{25}$$

$$u_B = u_{i,j} - \frac{\Delta t}{\Delta x}\left(u_{i,j} - a_{i,j}\right)\left(u_{i+1,j} - u_{i,j}\right). \tag{26}$$

With a_A, u_A, a_B and u_B computed a_C and u_C (that is, $a_{i,\,j+1}$ and $u_{i,\,j+1}$) can be computed using equations (19) and (20). The values are first-order accurate, since we used linear interpolation. This low order of accuracy is not a limitation because we are interested in the trend in the variation of the gradient (and not in the individual values of the gradient).

The numerical scheme is stable provided it satisfies the Courant-Friedrichs-Lewy stability condition [20] which states that for convergence to occur the domain of dependence of a numerical scheme must include the domain of dependence of the partial differential equation determined by the characteristics. In this case, the stability condition requires that the time increment Δt satisfy [12].

$$\frac{\Delta t}{\Delta x}\left|u + a\right| \le 1 \text{ and } \frac{\Delta t}{\Delta x}\left|u - a\right| \le 1. \tag{27}$$

➢ *Numerical Experiment, Initial and Boundary Conditions*

The numerical experiment simulates propagation of acoustic wavelet in air, along a pipe of constant cross section with the left-end closed and the right-end open to the atmosphere, whose length is 3 m; that is, $0 \le x \le 3$. The pipe's length is divided into small intervals Δx = 0.02 m, resulting in 151 points along the x-axis; that is, $0 \le i \le 151 = M$.

At time t = 0 a particle velocity wavelet of initial amplitude u_m and wavelength 1 m propagating along positive x direction is introduced in the pipe, near its closed end. The initial shape of the wavelet is defined by

$$u_{i,o} = \begin{cases} \dfrac{i}{12}u_m & , \ 0 \le i \le 12 \\ \dfrac{25-i}{13}u_m & , \ 12 < i \le 38 \\ \dfrac{i-50}{12}u_m & , \ 38 < i \le 50 \\ 0 & , \ i > 50 \end{cases} \tag{28}$$

The initial condition for sound speed $a_{i,o}$ is obtained using [13]

$$a = a_o \pm \frac{\gamma - 1}{2}u \quad , \quad 0 \le i \le 151, \tag{29}$$

Choosing same sign as x, where a_o = 340 m/s is sound speed in the undisturbed state and γ = 1.4 is ratio of specific heats.

The boundary conditions are deduced by noting that at a closed end the particle velocity u is zero at all times, while at an end open to the atmosphere the density, pressure or sound speed is same as that of the undisturbed state. Hence.

$$\begin{aligned} a_{M,\,j+1} &= a_o \\ u_{o,\,j+1} &= 0 \end{aligned} . \tag{30}$$

Referring to Figure 2, at the closed left-end of the pipe, only one backward characteristic can be drawn through a node along which the Riemann invariant R^- is constant. This corresponds to a wave propagating in the negative x direction. Thus using (29) (with the negative sign) and equating its values at points i = 0 and i = B (in Figure 2).

$$a_{o,\,j+1} - \frac{\gamma - 1}{2}u_{o,\,j+1} = a_B - \frac{\gamma - 1}{2}u_B \tag{31a}$$

Or, since from equation (30) $u_{o,\,j+1}$ = 0, the boundary condition is

$$a_{o,\,j+1} = a_B - \frac{\gamma - 1}{2}u_B \tag{31b}$$

Where a_B and u_B are computed for i = 0 using equations (25) and (26).

Likewise, at the open right-end of the pipe, only one backward characteristic can be drawn through a node along which the Riemann invariant R^+ is constant. This corresponds to a wave propagating in the positive x direction. Following same procedure as above leads to the boundary condition.

$$u_{M,\,j+1} = u_A + \frac{2}{\gamma - 1}\left(a_A - a_o\right) \tag{31c}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

Where a_A and u_A are computed for $i = M$ using equations (23) and (24).

The numerical method presented in the preceding section together with the stated initial and boundary conditions were coded in Python programming language, and the program was executed for a value of initial amplitude. The program was run for up to 250 time increments, to test numerical stability. Program execution simulates propagation of the wavelet along the pipe.

For each execution of the program, we used a different value of initial amplitude. The values of initial amplitude u_m used were 200, 100, 50, 25, 12.5, 10, 6.25, 1 and 0.1, m/s.

During program execution, with given initial amplitude, we extracted data representing the wavelet (as it propagates along the pipe) at time intervals of 10Δt between 0 and 150Δt. The data, at any instant, denotes the wavelet shape and its position along the pipe. Figure 3 is a typical plot of such data for wavelet with initial amplitude 100 m/s, at 0Δt and at 80Δt (when the slope is a maximum).

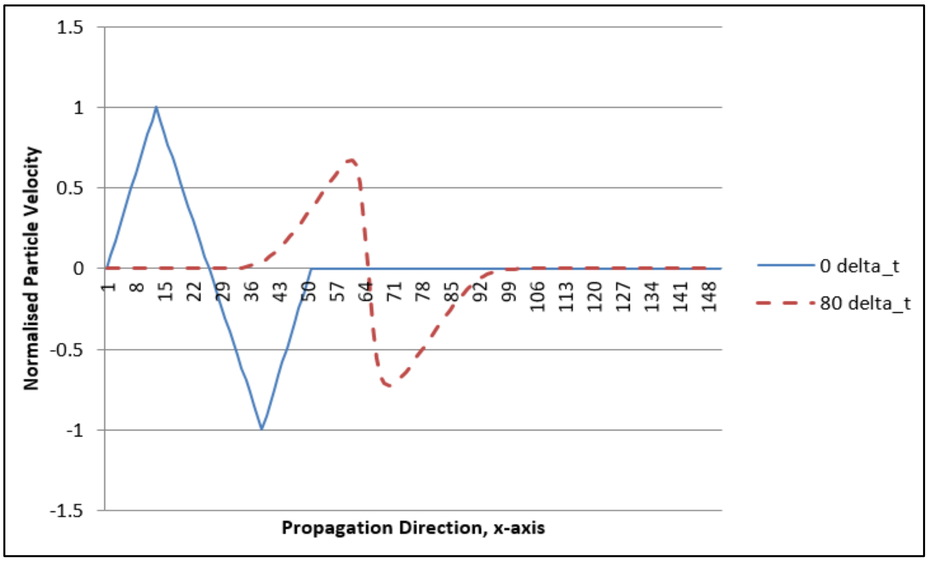
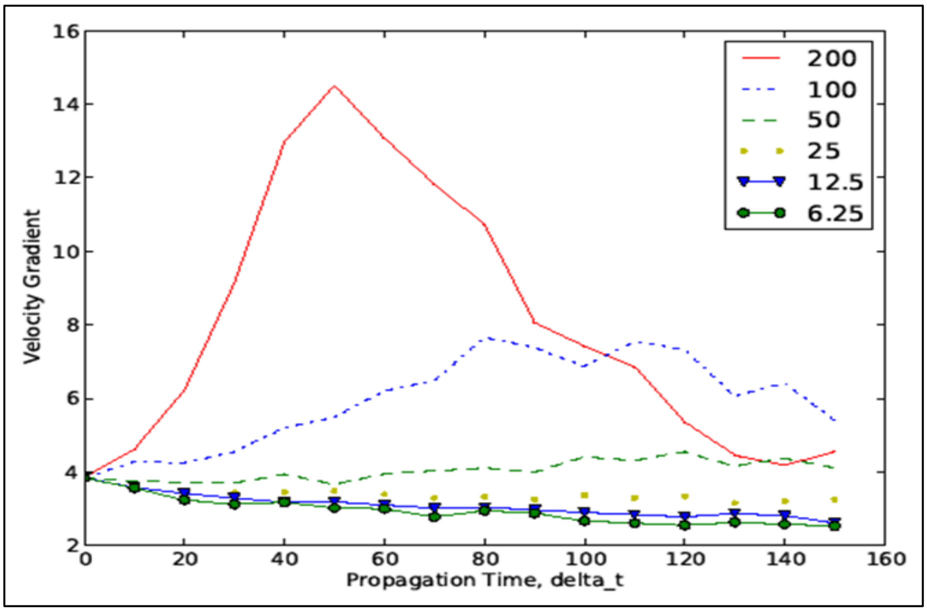


Fig 3 Wavelet Shape and Position, for u_m = 100 m/s

For each time, the data was plotted and the slope (that is, velocity gradient) of that part of the wavelet which steepens was calculated. Thus we obtained information on variation of velocity gradient (of that part of the wavelet which steepens) with time, as the wavelet propagates down the pipe. This information is presented graphically in the next section, for various initial amplitudes.

IV. RESULTS AND DISCUSSION

The results from our numerical experiment are shown in Figures 4. Each curve on Figure 4 is a plot of the slope against propagation time, for given initial amplitude. Initially, at time 0Δt, the slope is about 3.8 (for all initial amplitudes).



ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

Fig 4 Variation of Slope of Central Portion of Wave Profile with Propagation.

From these curves, it is evident that the slope is not constant with propagation time. Overall, we observe that the slope may decrease or increase. This implies that there are (at least) two opposing influences on the value of the slope. Earlier studies [10, 12] have shown that nonlinearity can manifest as increasing slope (what is often reported as "steepening"). A decrease in slope may imply energy loss. Energy loss manifests as decrease in amplitude of wavelet, as it propagates. This energy loss may be physical [21] or numerical [3, 16]. Though we assumed an inviscid fluid, it is shown in [21] that there may arise an induced relaxation viscosity opposing nonlinearity in finite-amplitude wave propagation. Also, in [22] it is suggested that a velocity gradient can induce viscous forces. Thus, once the particle velocity varies with position acoustic wave propagation involves energy loss [1].

Specifically, the slope for u_m = 200 and 100 begins to increase and later decreases, with propagation time. The slope decrease commences much earlier (after about 50Δt) for wavelet with u_m = 200 than for wavelet with u_m = 100. Also, the wavelet with u_m = 200 attains a maximum (about 14.5) and much higher slope before the wavelet with u_m = 100. However, the wavelet with u_m = 50 seems to have an almost constant slope (about 4.0) as it propagates.

The slope variation over time for wavelets with u_m = 25, 12.5 and 6.25 tends to decrease with propagation time. This trend is reproduced for wavelets with u_m = 10, 1.0 and 0.1 (so the curves are omitted). Though the slope at 150Δt is about 2.3 for u_m = 1.0 compared with about 2.5 for u_m = 10.

By comparing the curves for wavelets with u_m = 100, 50 and 25 m/s, we can observe clearly the transition from linear to nonlinear wave propagation.

The curve for wavelet with initial amplitude u_m = 25 (or lower) shows a situation in which energy loss predominates (if any nonlinearity exists). Thus, such wavelets may propagate, essentially, as linear (that is, small-amplitude) waves.

The curve for wavelet with initial amplitude u_m = 50 seems to imply that the opposing influences of energy loss and nonlinearity are about balanced. The slope stays near 4.0, for this wavelet. In the neighbourhood of initial amplitude u_m = 50, these opposing influences start almost at the same time and, essentially, cancel each other during the propagation time.

The curve for wavelet with initial amplitude u_m = 100 (or higher) clearly imply nonlinearity, as slope starts-out increasing. Energy loss sets-in later and tries to oppose the nonlinear effect. After 150Δt, the opposing influences try to return the slope to about 4.0. As we did not analyse the data beyond 150Δt, we cannot say if the slope for such nonlinear (that is, finite-amplitude) waves will reach the low values attained by the linear waves. A careful inspection of the curve for u_m = 200 seems to suggest that the slope value will hover round 4.0, beyond 150Δt.

Thus, the results suggest that for an acoustic wavelet denoted by particle velocity, the initial amplitude of 50 m/s marks the boundary between linear and nonlinear wave propagation. A value of 50 m/s seems strange as, by intuition, one may have difficulty accepting the number 50 as *small*. However, assuming a plane wave, the amplitude (that is, maximum value) of condensation s_m, acoustic pressure Δp_m and particle velocity u_m are related by [5, p 107].

$$\Delta p_m = \rho_o a_o u_m \tag{32}$$

$$s_m = \frac{\Delta p_m}{\rho_o a_o^2} = \frac{u_m}{a_o} \equiv \text{Ma}\,, \tag{33}$$

Where the Mach number Ma is ratio of maximum particle velocity to local sound velocity. The third term in equation (33) is written using equation (32).Note from equation (33) that the maximum value of condensation is equivalent to the acoustic Mach number.

The particle velocity 50 m/s is equivalent to an acoustic Mach number Ma of about 0.15. In [7, p1-2] it is shown that the one-dimensional wave equation in terms of particle displacement η in a gas can be written as.

$$\left(1+\frac{\partial \eta}{\partial x}\right)^{\gamma+1}\frac{\partial^2 \eta}{\partial t^2} = a_o^2\frac{\partial^2 \eta}{\partial x^2}\ ;\ \left(\frac{\partial \eta}{\partial x}\right)_{\max} = \frac{u_m}{a_o} = M\text{a} \tag{34}$$

The Mach number enters equation (34) through the coefficient of the double time derivative. The significance of equation (34) is that a finite non-zero acoustic Mach number Ma will affect the wave shape and speed, as the wave propagates. Our results show that for a single wavelet, an acoustic Mach number of about 0.15 marks the boundary between linear and nonlinear wave propagation. However, the effect of the acoustic Mach number for a continuous wave (over many wavelengths) will be cumulative, and we expect an acoustic Mach number lower than 0.15 to mark the boundary between linear and nonlinear continuous acoustic wave propagation.

Furthermore, in [23] we have an illuminating illustration for a typical acoustic wave of moderate intensity with an approximate equation that links maximum particle velocity u_m, frequency f and maximum particle displacement η_m:

$$u_m = 2\pi\eta_m f = 2\pi\eta_m a_o/\lambda\,. \tag{35}$$

Thus, for an acoustic wavelet denoted by particle displacement, the maximum particle displacement.

$$\eta_m = \frac{u_m \lambda}{2\pi a_o}\,. \tag{36}$$

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jul835

Given a_o = 340 m/s, u_m = 50 m/s and λ = 1m, η_m is about 0.02 m. If one imagines that η_m (= 0.02 m) is the maximum horizontal displacement of a simple pendulum whose length is 1 m then calling 0.02 m "small" wont challenge intuition.

V. CONCLUSION

We have modelled acoustic wave propagation in air, considering a nonlinear wave equation in one space dimension, using a Riemann invariant based method of characteristics. The model simulates propagation of a particle velocity wavelet along a straight pipe, with one end closed and the other end open. The initial amplitude of the wavelet lies in the interval 0.1 m/s to 200 m/s, and propagation is allowed for sufficiently long time (before the on-set of reflection effects).

The analysis of the computed data exploits the known "steepening" of a part of the wavelet in finite-amplitude wave propagation. We have calculated the slope of this part of the wavelet and compared its variation with propagation time, for wavelets of different initial amplitudes.

The results show a clear demarcation between linear wave propagation and nonlinear wave propagation, as wave amplitude is varied. The demarcation is controlled by the opposing influences of energy loss and nonlinear effects, each of whose significance depends on the initial amplitude and propagation time.

For an acoustic wavelet, denoted by the particle velocity, an initial amplitude of 50 m/s seems to mark the upper limit beyond which nonlinear effects cannot be ignored. Interpreted in terms of particle displacement, this value (50 m/s) corresponds to an acoustic wave with initial particle displacement 0.02 m. If expressed in terms of acoustic Mach number the value corresponds to about 0.15.

We conclude that for wavelets with initial amplitude above 50 m/s the linear wave equation is inadequate, and a nonlinear wave equation should be used. That is how small the amplitude of the initial disturbance should be for propagation to be in the linear (or small-amplitude) domain or, put in mathematical terms, for the assumptions made while deriving a linear acoustic wave equation to be valid.

REFERENCES

[1]. Ikata, Eghuanoye 2008 *J. Nig. Assoc. Math. Phys*. 13 161

[2]. Ikata E and Tay G 1998 *IL Nuovo Cimento* 20D(12) 1779

[3]. Ikata, Eghuanoye 2010 *J. Nig. Assoc. Math. Phys*. 17 197

[4]. Chester C R 1971 *Techniques in Partial Differential Equations* (Tokyo: McGraw-Hill) p.49

[5]. Kinsler L E, Frey A R, Coppens A B and Sanders J V 1982 *Fundamentals of Acoustics, 3rd ed*. (New York: John Wiley) p. 100 – 102

[6]. Symon K R 1979 *Mechanics, 3rd ed*. (Reading, MSS: Addison-Wesley) p. 330 – 331

[7]. Beyer R T 1974 *Nonlinear Acoustics* (USA Department of Navy, Naval Sea Systems Command) Chapters 3 & 4

[8]. Debnath, Lokenath 2005 *Nonlinear Partial Differential Equations for Scientists and Engineers, 2nd ed*. (Boston, USA: Birkhauser) Chapters 6 & 7

[9]. Lighthill J 1978 *Waves in Fluids* (Cambridge: Cambridge University Press) p. 76 – 85

[10]. Olver P J 2014 *Introduction to Partial Differential Equations* (Heidelberg, Switzerland: Springer) Section 2.3

[11]. Schiesser W E and Silebi C A 1997 *Computational Transport Phenomena* (Cambridge: Cambridge University Press) Part 1, M3

[12]. Chow C- Y 1983 *An Introduction to Computational Fluid Mechanics, corrected ed*. (Boulder, CO: Seminole Publishers) Section 2.14

[13]. Coulson C A and Jeffrey A 1988 *Waves – A Mathematical Approach to the Common Types of Wave Motion, 2nd ed*. (Essex, England: Longman Scientific & Technical) Chapter 9

[14]. White F M 2006 *Viscous Fluid Flow, 3rd ed*. (New York: McGraw-Hill) p. 55

[15]. Lapidus L and Pinder G F 1999 *Numerical Solution of Partial Differential Equations in Science and Engineering* (New York: John Wiley) Section 1.2.3

[16]. Ferziger J H 1998 *Numerical Methods for Engineering Applications* (New York: John Wiley) Section 8.2

[17]. Bukiet B, Pelesko J, Li X L and Sachdev P L 1996 *Computers Math. Applic*. 31(7) 75-97

[18]. Nei Y, Fu K and Lv X 2018 *Hindawi Advances in Mathematical Physics* vol 2018, ID 3469534, 10pages

[19]. Chapra S C and Canale R P 2010 *Numerical Methods for Engineers, 6th ed*. (New York: McGraw-Hill) p. 489

[20]. Morton K W and Mayers D F 1994 *Numerical Solution of Partial Differential Equations* (Cambridge: Cambridge University Press) p.88

[21]. Borisov A A, Vakhgel't A F and Nakoryakov V E 1980 *Zhurnal Prikladnoi Mekhaniki I Teknicheskoi Fiziki* No 5 33-38 (English Trans.)

[22]. Bird R B, Stewart W E and Lightfoot E N 2005 *Transport Phenomena, 2nd ed*. (Singapore: John Wiley Asia) p. 11 – 37

[23]. Butkov, Eugene 1968 *Mathematical Physics* (Reading, MSS: Addison-Wesley) p. 320.