

Fractional Navier–Stokes Models for Nonlocal Turbulent Flows: Theory, Numerical Methods

Nilam Ghadage¹; Soni Pathak²

^{1,2}Department of Basic Sciences Chhatrapati Shivaji Maharaj University Panvel, Maharashtra, India

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Abstract: The classical Navier-Stokes equations have been astronomically used to describe fluid movement in engineering and scientific operations. still, their capability to model complex marvels similar as turbulent overflows, anomalous prolixity, and multiphase transport is limited because they calculate on original relations and memoryless hypotheticals. Fractional math provides a important fine frame for incorporating temporal memory and spatial nonlocality into fluid- inflow models. This review presents a comprehensive overview of fractional Navier – Stokes equations, including their fine foundations, governing phrasings, numerical result ways, and recent operations. Particular attention is given to fractional turbulence models, anomalous transport processes, and multiphase flux systems. Numerical approaches analogous as the Variational Iteration transform Method (VITM), Hybrid Finite Difference – Finite Element Method (FDM – FEM), Residual Power Series Method(RPSM), and spectral styles are critically examined. likewise, the advantages, limitations, and unborn exploration directions of fractional fluid dynamics are banded. The review highlights the eventuality of fractional models to ameliorate the vaticination of complex inflow gesteby landing long- range relations, memory goods, andnon-Gaussian transport mechanisms that are n't adequately represented by classical fluid models.

Keywords: Fractional Navier–Stokes Equations; Fractional Calculus; Turbulence Modeling; Fractional Laplacian; Anomalous Diffusion; Multiphase Flow; Nonlocal Transport; Numerical Methods.

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I. INTRODUCTION

Many scientific and engineering disciplines depend critically on fluid dynamics, including aerodynamics, energy systems, environmental simulations, and the development of artificial processes. The Navier-Stokes equations have historically guided the motion of fluids, determined by initial conditions and the fluid medium's immediate reaction. While these equations have been relatively effective in prognosticating laminar and relatively complex overflows, their operation is limited when dealing with systems characterized by significant nonlocal relations, multiscale dynamics, and memory-dependent geste .

Trials in numerous factual scripts, similar as turbulent overflows, pervious media transport, and miscellaneous accoutrements, show large disagreement from conventional prospects. These deviations are commonly manifested as anomalous diffusion, long-range correlations, and non-Gaussian statistical distributions. Similar marvels suggest that the underpinning physical processes are controlled by factors that go beyond the original frame anticipated in

classical fluid models. To address these limitations, fractional phrasings of fluid inflow equations have been introduced, furnishing a further generalized fine description that incorporates both temporal memory and spatial nonlocality. Importantly, these fractional models reduce to the classical Navier – Stokes equations when the fractional orders approach continuity, thereby conserving thickness with established proposition [1].

The objectification of fractional derivations into governing equations, still, introduces significant logical and computational challenges due to their nonlocal nature. To handle fractional Navier- Stokes systems well, numerous numerical and semi-analytical approaches have been created. With the variational replication transfigure approach [2], you can get close answers without using redundant tools like Adomian polynomials. This system is flexible and works well with computers[2], you can get close answers without using redundant tools like Adomian polynomials. This system is flexible and works well with computers.[2]. Contemporaneously, cold-blooded numerical ways that combine finite element styles for spatial approximation with

finite difference schemes for temporal discretization have shown excellent confluence gets and high stability rates, especially for nonlinear time- fractional problems [3]. also, standard fractional fluid inflow issues have been successfully answered using logical approximation ways like the residual power series system [4].

Fractional math has made major progress in turbulence modeling possible in addition to result methodologies. Two exemplifications of classical turbulence models that primarily calculate on empirical check hypotheticals are large circle simulation (LES) and Reynolds- equalled Navier- Stokes (RANS). These models constantly fail to capture the constitutionally nonlocal and multiscale nature of turbulence. Fractional check models have been proposed to address this problem and meliorate prophecy delicacy in a range of flux topologies, analogous as channel, pipe, and Couette flows, by explicitly incorporating spatial nonlocality into the governing equations [5].

Likewise, in order to more capture intermittent and non-Gaussian turbulence characteristics, fractional drivers deduced from Lévy-stable processes have been incorporated into LES fabrics to describe sub- grid- scale stresses [6].

Fractional fluid models are useful for further than just turbulence. In the case of anomalous transport, fractional discrimination equations give a strong frame for modeling Sub-diffusive and super-diffusive processes seen in complex systems like subterranean transport and miscellaneous accoutrements [7]. These models have been expanded to include a wider range of physical marvels where classical prolixity equations do not directly describe what happens [8]. In multiphase inflow systems, fractional phrasings help make check less complicated and better represent interphase relations, especially in gas – flyspeck and fluid – fluid systems [9]. Advanced computational fabrics predicated on finite element styles have also shown better performance in pretending compressible multiphase flows with heat transfer, giving better predictive delicacy than traditional finite volume approaches [10]. likewise, fractional prolixity models have been effectively employed in turbulent conduit flows, performing in enhanced prognostications of haste fields and flow structures across colourful Reynolds figures [11]. Experimental examinations further validate the necessity of fractional modeling approaches. Nonlinear correlations between inflow rate and pressure grade have been linked in studies of pervious medium overflows, pressing the limitations of traditional Darcy- grounded models and emphasizing the significance of considering nonlocal factors [12]. also, flyspeck- laden turbulent overflows have been modelled using mongrel stochastic – deterministic fabrics, successfully landing multiscale dynamics and stochastic relations that are else delicate to characterize with solely deterministic models [13].

II. RESEARCH GAP AND CONTRIBUTION

Despite the rapid-fire- fire development of fractional fluid models, several challenges remain, including high computational cost, difficulty in parameter estimation, and

limited experimental evidence for complex systems. also, a unified frame that integrates theoretical foundations, numerical styles, and advanced operations of fractional Navier – Stokes equations is still lacking in the literature. thus, the ideal of this review is to give a comprehensive and critical analysis of fractional Navier – Stokes equations, including their fine expression, numerical result ways, and operations in turbulence modeling, anomalous transport, and multiphase systems. Special emphasis is placed on assessing the effectiveness of different numerical styles and relating unborn exploration directions for advancing fractional fluid dynamics

III. FRACTIONAL GOVERNING EQUATIONS

➤ Fractional Continuity Equation

$$\frac{\partial^\alpha \rho(x,t)}{\partial t^\alpha} + \nabla^\beta \cdot (\rho u) = 0 \quad (1)$$

This equation introduces:

Temporal memory (α) which is depend on past states of the fluid flow

Spatial nonlocality (β) means dependence of the field variable at a given location on values across the entire spatial domain, rather than only on its immediate neighbourhood.

If $\alpha, \beta = 1$ then fractional continuity equation reduces to classical continuity equation.[1]

➤ Fractional Momentum Equation

Fractional Momentum Equation is written as:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (u \cdot \nabla^\beta)u = -\frac{1}{\rho} \nabla^\beta p + \nu(-\Delta)^\beta u \quad (2)$$

It extends the classical Navier Stokes equation by using fractional derivations in time and space. This formulation is better suited for modeling complicated and anomalous fluid flows because it captures memory effects through the time-fractional term and nonlocal relation through the spatial operators [1]

➤ Physical Interpretation

Fractional acceleration, the way a fluid moves depends on how it has moved in the past. This means that the fluid's current speed is affected by its history. Fractional convection means patches in the inflow interact over long distances, not just with near patches. Fractional prolixity represents non-standard spreading of instigation, where transport can be briskly or pokily than normal prolixity.

These features enable modeling of turbulence intermittency and energy cascade processes [6].

IV. MATHEMATICAL FOUNDATIONS

➤ *Caputo Fractional Derivative*

Caputofractional derivative is extensively used in modeling physical systems with memory and heritable properties. For a sufficiently smooth function $f(t)$, the Caputo derivative of order $\alpha \in (0,1)$ is defined as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau \tag{3}$$

Where,

- $f'(\tau)$ which tells how the function at past time τ .
- $(t - \tau)^{-\alpha}$: gives more importance to recent past and less to old past.

Caputo Fractional Derivatives is Suitable for physical initial conditions and it is widely used in fluid modeling [2], [3]

➤ *Fractional Laplacian*

The fractional Laplacian operator is an important fine tool used to model nonlocal and anomalous actions in turbulent overflows. In classical models, the standard Laplacian $(-\Delta)$ describes local diffusion. However, in numerous real- world turbulent systems, relations are n't purely original and involve long range goods. To describe this, we introduce the fractional Laplacian $(-\Delta)^\alpha$ with $0 < \alpha \leq 1$.

➤ *Definition (Fourier Form)*

The fractional Laplacian is defined using the Fourier transfigure as:

$$(-\Delta)^\alpha u(x) = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F}\{u\}(\xi)) \tag{4}$$

Where \mathcal{F} and \mathcal{F}^{-1} denote the Fourier and inverse Fourier transforms, respectively.

This shows that the driver modifies the frequence factors of the function, allowing the modeling of multi-scale and irregular marvels [6].

➤ *Integral Form*

The fractional Laplacian can also be written as:

$$(-\Delta)^\alpha u(x) = C_{d,\alpha} \int_{\mathbb{R}^d} \frac{u(x)-u(s)}{|x-s|^{2\alpha+d}} ds \tag{5}$$

Where $C_{d,\alpha}$ is a normalization constant.

This form easily shows that:

The value at a point x depends on all other points in the sphere

It captures long- range relations (nonlocal goods)

➤ *Physical Meaning in Turbulence*

In turbulent flux modeling, especially in Large Eddy Simulation(LES), the fractional Laplacian is used to represent sub grid scale(SGS) stresses. These stresses arise due to undetermined small- scale movements.

Turbulence frequently exhibits heavy- tailed behaviour, which can be described using Lévy α -stable distributions. At microscopic levels, particle motion follows random jumps (non-Gaussian behavior)When scaled up, this leads to a fractional Laplacian term in governing equations.

Therefore, the fractional Laplacian provides a rigorous way to model anomalous prolixity and energy transfer in turbulence[6].

➤ *Relation with Navier – Stokes Equations*

For incompressible inflow, the classical Navier – Stokes equations are

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \frac{1}{\rho} \nabla p - \nu \Delta v = 0, \nabla \cdot v = 0 \tag{6}$$

In turbulence modeling (RANS/LES), additional stress terms appear. These can be modeled using the fractional Laplacian as:

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \frac{1}{\rho} \nabla p - \nu \Delta v + \xi (-\Delta)^\alpha v = 0 \tag{7}$$

Where:

- ξ : turbulent diffusivity
- α : fractional order (often $\alpha = 1/5$ in turbulence studies)
- This expression replaces classical prolixity with fractional prolixity, which better captures turbulent transport and irregular inflow structures [11].

➤ *Anomalous Diffusion Law*

$$\langle x^2(t) \rangle \sim t^\alpha$$

$\alpha < 1$ then sub-diffusion

$\alpha > 1$ then super-diffusion

Observed in real systems [7], [8].

V. VARIATIONAL ITERATION TRANSFORM METHOD (VITM)

The iterative correction functional is given by:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\tau) [L(u_n) + N(u_n)g(x, t)] d\tau \tag{8}$$

Where

- L : linear operator
- N : nonlinear operator
- $\lambda(\tau)$: Lagrange multiplier

The fractional Navier–Stokes equations are nonlinear and involve memory terms, making classical analytical solutions difficult. VITM is used because:

- It avoids discretization, reducing numerical complexity
- It does not require Adomian or He’s polynomials, simplifying implementation
- It provides fast convergence for nonlinear fractional equations [2]

VI. HYBRID FINITE DIFFERENCE-FINITE ELEMENT METHOD (FDM–FEM)

➤ *Time Discretization (L1 Scheme):*

$$D_t^\alpha u^n = \frac{1}{\Delta t^\alpha} \sum_{k=0}^{n-1} b_k (u^{n-k} - u^{n-k-1}) \tag{9}$$

Where,

- $b_k = (k + 1)^{1-\alpha} - k^{1-\alpha}$
- $D_t^\alpha u^n$: Fractional (Caputo) derivative at time level n
- Δt : Time step size
- u^n : Numerical solution at current time t_n
- u^{n-k}, u^{n-k-1} : Solutions at previous time levels (past values)
- b_k : Weight coefficients that depend on α

➤ *Weak Form (FEM):*

$$\int_{\Omega} \frac{\partial^\alpha u}{\partial t^\alpha} v \, d\Omega + \int_{\Omega} \nabla u \cdot \nabla v \, d\Omega = \int_{\Omega} f v \, d\Omega \tag{10}$$

Where,

- Ω : Spatial domain (region of the problem)
- u : Unknown solution (e.g., velocity, temperature)
- v : Test function (used in FEM)
- $\frac{\partial^\alpha u}{\partial t^\alpha}$: Fractional time derivative (memory effect)
- ∇u : Gradient of u (spatial variation)
- $\nabla u \cdot \nabla v$: Represents diffusion or spatial interaction
- f : Source term (external force/input)
- $\int_{\Omega} (\cdot) \, d\Omega$: Integration over the domain

The finite difference system (L1 scheme) is used to directly capture the time memory goods, and the finite element system is good for handling complex shapes. Combining these two styles, the intercross approach is attained, which exploits the advantages of the two styles. This system is able of modeling anomalous prolixity in turbulent overflows and provides high delicacy and trustability for working nonlinear fractional Navier – Stokes equations [3].

VII. RESIDUAL POWER SERIES METHOD (RPSM)

The Residual Power Series Method (RPSM) is a semi-analytical method to solve fractional partial differential equations (FPDEs) especially nonlinear problems with memory effects. It constructs the result as a fractional power

series without taking discretization, making it computationally effective [4].

Consider the FPDE:

$$D_t^\beta u(\xi, t) = N(u) + R(u), 0 < \beta \leq 1 \tag{11}$$

With initial condition $u(\xi, 0) = f(\xi)$, where D_t^β is the Caputo derivative.

RPSM assumes a fractional series solution:

$$u(\xi, t) = \sum_{n=0}^{\infty} \frac{f_n(\xi) t^{n\beta}}{(1+n\beta)} \tag{12}$$

And its truncated form:

$$u_k(\xi, t) = f(\xi) + \sum_{n=1}^k \frac{f_n(\xi) t^{n\beta}}{(1+n\beta)} \tag{13}$$

A residual function is defined as:

$$\text{Res}_{u,k} = D_t^\beta u_k - N(u_k) - R(u_k) \tag{14}$$

And coefficients are obtained using:

$$D_t^{(k-1)\beta} \text{Res}_{u,k}(\xi, 0) = 0. \tag{15}$$

The method ensures convergence, as the error decreases with increasing terms, and the solution is unique if the sequence converges [4] RPSM has been effectively applied to nonlinear systems such as fractional Navier–Stokes equations, yielding highly accurate results.

VIII. THE FRACTIONAL FINITE DIFFERENCE METHOD

(FDM) is a numerical approach used to compare fractional operators, particularly the fractional Laplacian, in fractional discriminational equations. It generalizes the classical finite- difference setting by allowing nonlocal relations.

The fractional Laplacian can be discretized as

$$(-\Delta)^\beta u_i \approx \sum_j w_{ij} (u_i - u_j) \tag{16}$$

Where weights depend on the spatial distance between nodes, which reflects the inherent nonlocal behavior of fractional operators.

This is the most widely used approach due to its simplicity and applicability for structured grids. It differs from classical methods by taking into account long-range interactions and is thus particularly well adapted to the modeling of fractional diffusion and turbulence phenomena. Recent studies reveal that fractional FDM enhances the prediction of velocity in turbulent duct flows and captures the scaling behavior at different Reynolds

numbers. The results indicate an improved representation of turbulent diffusion over traditional approaches [11].

IX. SPECTRAL METHODS AND FRACTIONAL OPERATORS

Spectral methods provide a very accurate approximation framework for fractional operators, in particular the fractional Laplacian, through expansion in eigenfunctions.

The fractional Laplacian is expressed as:

$$(-\Delta)^\beta u = \sum_k \lambda_k^\beta \hat{u}_k \phi_k \tag{17}$$

Where λ_k and ϕ_k are eigenvalues and eigenfunctions of the Laplacian operator, and \hat{u}_k are the spectral coefficients. They are preferred due to their high accuracy for smooth problems and efficiency for nonlocal fractional operators. In particular, for turbulence modeling where the capture of small scale structures is important. It turns out that the presence of the fractional Laplacian improves the modeling of nonlocal diffusion and turbulence intermittency, which are often associated to Lévy-stable processes. This approach describes reasonably well the non-Gaussian behavior observed in turbulent flows [6].

X. FINAL CRITICAL COMPARISON

Table 1 Final Critical Comparison

Method	Why Used	Key Finding	Limitation
VITM	Avoid discretization	Fast convergence	Limited to simple problems
Hybrid FEM–FDM	Accuracy + geometry	Stable & accurate	High cost
RPSM	Analytical solution	High precision	Not scalable
FDM	Simplicity	Good for turbulence	Low accuracy
Spectral	High precision	Captures nonlocality	Expensive

XI. FRACTIONAL TURBULENCE MODELING

➤ *Motivation for Fractional Turbulence Models*

Traditional turbulence models such as Reynolds-Averaged Navier–Stokes (RANS) and Large Eddy Simulation (LES) are based on local interactions and Gaussian statistics. But turbulent flows exhibit nonlocal interactions, intermittency and anomalous transport due to multiscale vortex dynamics. Fractional calculus offers an efficient framework for the incorporation of memory effects and long-range interactions into turbulence models, and thus an improved description of turbulent transport processes.[6].

➤ *Fractional RANS Closure Model*

To account for nonlocal turbulent interactions, the Reynolds stress closure can be expressed using a fractional Laplacian operator as

$$\nabla \cdot \tau = C(-\Delta)^\alpha u \tag{18}$$

Where,

- τ : Reynolds stress tensor
- α : fractional order
- C : model coefficient

Unlike classical RANS models that require empirical closure assumptions, the fractional formulation directly incorporates spatial nonlocality. Applications to channel, pipe and Couette flows have reported prediction errors less than 1%, demonstrating improved accuracy and robustness of turbulent flow simulations[5].

➤ *Fractional Subgrid-Scale (SGS) Stress Models*

For LES, the unresolved subgrid stresses can be modeled as

$$\tau_{sgs} = C(-\Delta)^\alpha u \tag{19}$$

Fractional SGS models are motivated by the non-Gaussian and intermittent nature of turbulence. These models, incorporating Lévy-stable processes and fractional operators, are more suitable for describing long-range correlations and energy transfer across scales. Results have shown an improved prediction of the velocity fluctuations and turbulence statistics in comparison with conventional SGS models.[6].

➤ *Fractional Diffusion Models*

Fractional diffusion equations are commonly written as

$$\frac{\partial u}{\partial t} = -(-\Delta)^\alpha u \tag{20}$$

Where,

- Δ : diffusion coefficient
- A : fractional order.

These models describe anomalous diffusion and nonlocal transport processes that are frequently observed in turbulent flows. Applications to channel and duct flows have demonstrated improved agreement with experimental observations compared with classical diffusion models [6].

➤ *Physical Significance*

Fractional turbulence models provide a generalized framework for representing:

- Long-range interactions between turbulent eddies,
- Memory effects in flow evolution,
- Non-Gaussian turbulence statistics,
- Anomalous transport and diffusion.

Therefore, fractional formulations offer a more realistic description of turbulent dynamics than conventional integer-order models, particularly in complex and multiscale flow systems [5,6].

XII. CHALLENGES AND FUTURE RESEARCH DIRECTIONS

Despite recent advances, several challenges remain in the development of fractional Navier–Stokes models.

➤ *Fractional-Order Selection*

The determination of appropriate fractional-order parameters remains a major challenge. Model predictions are highly sensitive to these parameters, and their physical interpretation requires further investigation.

➤ *Advanced Turbulence Modeling*

Fractional RANS and LES models have shown promising results for canonical turbulent flows. However, their application to complex three-dimensional and high-Reynolds-number flows requires further study.

➤ *Multiphase Flow Applications*

Although fractional formulations have been applied to multiphase systems, the development of reliable interphase force models and turbulence closures remains an open research problem.

➤ *Computational Challenges*

The nonlocal nature of fractional operators increases computational cost. Efficient numerical algorithms and parallel computing strategies are needed for large-scale simulations.

➤ *Machine Learning and PINNs*

The integration of fractional Navier–Stokes equations with machine learning and Physics-Informed Neural Networks (PINNs) is an emerging research area that may improve computational efficiency and predictive accuracy.

➤ *Experimental Validation*

Most studies are numerical, while experimental validation remains limited. Future work should focus on validating fractional models using laboratory and industrial flow data.

Overall, advances in adaptive fractional modeling, data-driven techniques, and experimental verification are expected to further enhance the applicability of fractional fluid dynamics in turbulence and multiphase flow simulations.

XIII. CONCLUSION

Fractional Navier–Stokes equations extend classical fluid models by incorporating memory effects and spatial nonlocality, making them suitable for describing turbulence, anomalous diffusion, and multiphase flows. This review has described the mathematical foundations, numerical solution methods and recent applications of fractional fluid mechanics.

The reviewed studies demonstrate that fractional models can better capture nonlocal transport, intermittency, and complex flow behavior compared with conventional approaches. However, the computationally challenging parameter selection, costs and experimental validation are pending. Future advances in numerical methods, machine learning, and adaptive fractional modelling are expected to further enhance the practical applicability of fractional fluid-flow models.

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