ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jun1729

On Closed-Ness and Commutativity Properties of N-Posinormal and M-Hyponormal Class of Operators in Hilbert Spaces

Philemon Murimi[1]; Sammy Musundi[2]; Dr. Peter Muiruri[3]

[2]Professor, PhD

[1,2,3]Chuka University - Kenya

Publication Date: 2026/07/14

Abstract: The study of operators in Hilbert spaces is critical concept due to its wide applications in areas like quantum physics, computer programming and optimization of system per-formance in control engineering. This paper focuses on a class of n-posinormal and m-hyponormal operators in Hilbert spaces. Let H be a complex Hilbert space and B(H) be a bounded linear operator acting on H, then an operator T in B(H) is n-posinormal if there exist a positive operator $P \geq 0$ such that $T^nT^* = T^*PT^n$, where P is known as the n^{th} interrupter for the operator T. This paper studied the closed-ness aspect and commutativ-ity properties of this operator and the related class of m-hyponormal operator in Hilbert space. The results showed that if T is n-posinormal operator then, T^* is also n-posinormal operator.The results also were extended to the class of m-hyponormal operators. Similarly, the results showed that, if T is n-posinormal operator in Hilbert space H, then any operator S, unitary equivalent to T, is also n-posinormal operator. Furthermore, the results showed that the class of n-posinormal and m-hyponormal operators is closed in the operator norm topology. To achieve these results, the properties of linear operators, normal operators and posinormal operators were extended to the higher order n-posinormal and m-hyponormal operators.

Keywords: *Normal Operators, Hyponormal Operators, Adjoints, N Power Normal Opera-Tors, M-Hyponormal Operators and N-Posinormal Operators.*

How to Cite: Philemon Murimi; Sammy Musundi; Dr. Peter Muiruri (2026) On Closed-Ness and Commutativity Properties of N-Posinormal and M-Hyponormal Class of Operators in Hilbert Spaces. *International Journal of Innovative Science and Research Technology*, 11(6), 3320-3323. https://doi.org/10.38124/ijisrt/26jun1729

I. INTRODUCTION

Throughout this paper B(H) denotes the algebra of all bounded operator on Hilbert space H. A linear operator T on Hilbert space H is said to be bounded if there exist a constant $C \geq 0$ such that $\|T_x\| \leq C\|T_x\|$ for all $x \in H$. T is self adjoint if $T = T^*$, invertible with inverse S if there exist $S \in B(H)$ such that $ST = I = TS$, where I is the identity operator. An operator $T \in B(H)$ is said to be normal if it commutes with its adjoint, that is $TT^* = T^*T$, equivalently $TT^* - T^*T = 0$. An operator T is said to be positive if $T^* = T$ and $\langle Tx, x\rangle \geq 0$ for all $x \in H$. An operator T is said to be a posinormal if there exist an operator $P \geq 0$ such that $TT^* = T^*PT$ where P is called an interrupter for T. An operator T is called an n-posinormal operator if $T^nT^* = T^*PT^n$, where P is the n^th interrupter for T. An operator is hyponormal if $TT^* \leq T^*T$ and T is m-hyponormal operator if $T^mT^{*m} \leq T^{*m}T^m$ for a finite integer m.

II. METHODOLOGY

Adnan (2008), characterized normal operators and proved that an operator is n-power nor-mal if it commutes with T^n, that is $T^nT^* = T^*T^n$. He also explored on the closed-ness properties under scalar multiplication and unitary equivalence on n power normal opera-tors. He proved that n power normal operators were closed under scalar multiplication and under unitary equivalent.

Adnan's work was then extended to hyponormal operators by Messaoud G (2013). Messaoud G, proved that any operator S which is unitary equivalent to hyponormal oper-ator T is also an hyponormal operator. Then Kubrusly (2016),established that powers of normal operators remain normal, however the powers of hyponormal operators need not be hyponormal general.

Elmoctor (2021) described an operator T to be an n-posinormal operator if $R(T^n) \subseteq R(T^*)$ for any positive integer n. It was proved that n-posinormal operator contains posi-

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jun1729

normal operators as a subclass and is invariant under positive integer n.

➢ *Lemma 2.1*

Let $Q \in PN(H)$ be having a bounded Q_{MPI}. Then the Ran(Q) is closed.

➢ *Preposition 2.2*

(C.S Kubrusly *et al*, 2007):

Take an arbitrary operator $T \in B(H)$

- T is posinormal if any of the following equivalent assertions are fulfilled:

✓ (a_1) $TT^* = T^*PT$ for some $P \geq 0$.
✓ (a_2) $TT^* \leq T^*PT$ for some $P \geq 0$.
✓ (a_3) $T = T^*L$ for some $L \in B(H)$.
✓ (a_4) $R(T) \subseteq R(T^*)$
✓ (a_5) $TT^* \leq \alpha^2 T^* T$ for some $\alpha \geq 0$.
✓ (a_6) $\|T^*x\| \leq \alpha \|Tx\|$ for some $\alpha \geq 0$ and every $x \in H$.

- T is coposinormal if T^* is posinormal.

➢ *Lemma 2.3*

$T \in B(H)$ is unitarily equivalent to a unitary operator if and only if it is a unitary operator.

III. RESULTS AND DISCUSSION

➢ *Theorem 2.4*

If T is an n-posinormal operator, then its adjoint T^* is an n-posinormal operator Proof From the definition an operator T is n-coposinormal if $T^*T^n = T^nRT^*$, for $R \geq 0$. Suppose that T is n-posinormal operator then there exist $P \geq 0$ such that $T^nT^* = T^*PT^n$,

Taking adjoints on both sides we get,

$(T^nT^*)^* = (T^*PT^n)^*$ $T^{**}(T^*)^n = (T^n)^*P(T^*)^*$ $T(T^*)^n = (T^*)^nPT$,

This implies that n-posinormal operator is closed under taking adjoints.

This theorem was extended to the related class of m-hyponormal operator. The results were presented in theorem 2.5

Theorem 2.5

Let $T \in B(H)$ be an m-hyponormal operator in Hilbert space H, then the adjoint T^* is not m-hyponormal in general.

- *Proof.*

Recall the definition of m-hyponormal operator, $(T^*T)^m \geq (TT^*)^m$, which simplifies to $T^{*m}T^m \geq T^mT^{*m}$, or equivalently, $T^{*m}T^m - T^mT^{*m} \geq 0$.

We then suppose T to be hyponormal operator and consider it's adjoint We proceed and replace T with it's adjoint, T^*, that is.

$(T^*)^{*m}(T^*)^m \geq (T^*)^m(T^*)^{*m}$, we then obtain, $T^mT^{*m} \geq T^{*m}T^m$

This is a contradiction to the definition of m-hyponormal operator, indeed the inequality sign reverses

➢ *Remark 2.6*

The validity of the closure of m-hyponormal operator under taking adjoints is guaranteed if and only if T^m is normal, that is $T^mT^{*m} = T^{*m}T^m$.

The next result on closed-ness was presented in the form of a theorem 2.7

➢ *Theorem 2.7*

If T is n- posinormal operator in Hilbert space, then any operator S unitary equivalent to T is also n-posinormal.

- *Proof*

Two operators T and S are unitary equivalence if there exists a unitary operator U such $S = UTU^*$.

We consider an operator T, that is n-posinormal and unitary equivalent to operator S. We rewrite the operator S in the form of $T(T^*)^n = (T^*)^nPT$.

thus $S^nS^* = (UTU^*)^n(UTU^*)^*$

since U is unitary then $UU^* = I$ $(UTU^*)^n = UT^nU^*$, $(UTU^*)^*) = UT^*U^*$.

$S^nS^* = (UT^nU^*)(UT^*U^*)$, substituting in the n-posinormality condition we get.

$UT^nT^*U^* = U(T^*QT^n)U^*$

$U(T^*QT^n) = (UT^*U^*)(UQU^*)(UT^nU^*)$

UQU^* is positive since unitary conjugate preserves positivity. Therefore, $S^nS^* = S^*(UQU^*)S^n$.

If T is n-posinormal, then any unitary equivalent operator S is also n-posinormal.

The results on unitary equivalence on n-posinormal operators were extended to the related class of m-hyponormal operators.

The findings were presented in the preposition 2.8

➢ *Preposition 2.8*

Let T be m-hyponormal operator on H, and let U be a unitary operator on H. Then the operator $S = U^*TU$ is also m-hyponormal operator.

- *Proof*

We define $S = U^*TU$, and then proceed to show that S is

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jun1729

also m-hyponormal operator. We compute;

$S^{*m}S^m - S^mS^{*m} = U^*(T^{*m}T^m - T^mT^{*m})U.$

Since $T^{*m}T^m - T^mT^{*m} \geq 0$, by assumption and U is unitary, then the lemma above implies

$U^*(T^{*m}T^m - T^mT^{*m})U \geq 0$. Therefore,

$S^{*m}S^m - S^mS^{*m} \geq 0$ and S is m-hyponormal operator. These results indicates that m-hyponormal is invariant under unitary equivalence.

This study also explored the closure of n-posinormal operator under the context of operator norm topology.

➢ *Preposition 2.9*

The class of n-posinormal operator is closed under the operator norm topology

- Proof

Let T_k be a sequence of n-posinormal operator converging in the norm of T. For each k, there exist a positive operator $P_k \geq 0$ satisfying :

$T^nT^* = T^*P_kT^n$, Since norm convergence implies convergence of products and adjoints

$k \quad k \qquad k$

(Norm convergence implies strong convergence of the operator powers). Thus,

$T^nT^* \longrightarrow T^*T^n$ and

$k \quad k \qquad k \quad k$

$T_kT^* \longrightarrow TT^*$ in operator norm.

Assume $\sup P_k = P < \infty$. Taking limits preserves the operator equality hence,

$T^nT^* = T^*PT^n$. Hence T is a posinormal.

Similarly the study established the closure property of m-hyponormal operator under the operator norm topology. Theorem 2.10 was derived

➢ *Theorem 2.10*

The class of m-hyponormal operator is closed in the operator norm topology

- *Proof*

Let T_n be a sequence of m-hyponormal operators in B(H) such that $T_n \longrightarrow T$ in the oper-ator norm. We proceed and and show that T is m-hyponormal operator.

Since the map $S \longrightarrow S^m$ is norm continuous on B(H), we have

$(T^*)^m \longrightarrow (T^*)^m$ in the operator norm, for each n, m-hyponormality of T_n gives, $(T^*)^mT^m$ - $T^m(T^*)^m \geq 0$

$n \qquad n \quad n \quad n$

Consider the operator

$A_n = (T^*)^mT^m$ - $T^m(T^*)^m$

$n \qquad n \quad n \quad n$

By continuity of multiplication in B(H), $A_n \longrightarrow A$ in the norm where

A= $(T^*)^mT^m$ - $T^m(T^*)^m$

$n \qquad n \quad n \quad n$

Since the cone of positive operators is closed in the operator norm topology, the limit of A is also positive. Hence,

$(T^*)^mT^m$ - $T^m(T^*)^m \geq 0$

Therefore T is m-hyponormal operator. This proves that class of m-hyponormal operator is norm closed.

IV. CONCLUSIONS

This research paper focused on the closed-ness properties of n-posinormal operator and the related class of m-hyponormal operators in Hilbert spaces. Several properties have been investigated including closed-ness under taking adjionts,under unitary equivalence and also on operator norm topology. Future research will be towards the spectral proper-ties of n-posinormal operators and the related class of m-hyponormal operators.

➢ *Disclaimer (Artificial Intelligence)*

Author(s) hereby declare that NO generative AI technologies such as Large Language Models(ChatGPT,COPILOT, etc) and text-to-image generators have been used during writ-ing or editing of manuscripts.

➢ *Competing Interests*

Authors have declared that no competing interests exist.

- Authors contribution: The work was carried out in collaboration among all the authors.
- All the authors read and approved the final manuscript.

REFERENCES

[1]. Adnan A. S. Jibril, On n-Power Normal Operators, The Arabian Journal for Sciences and Engineering, Vol. 33, No. 2A, 2008, 247 - 251.

[2]. Kubrusly C.S (2012), *Spectral Theory of Operators on Hilbert Spaces, Birkhauser/Springer, New York.*

[3]. Rhaly H, JR.(2014), A superclass of the posinormal operators, New York J. Math, 30 , 497–506.

[4]. Mahmoud, S. A. O. A., Saddi, A.,Gherairi, K. (2020). Some results on higher orders quasi-isometries. *Hacettepe Journal of Mathematics and Statistics,* 49(4), 1315-1333.

ISSN No:-2456-2165 https://doi.org/10.38124/ijisrt/26jun1729

[5]. Lee M.Y and S. H. Lee (2006), On powers of p -posinormal operators, Sci. Math. Jpn. 64 , 97–101.
[6]. Ould Beiba, E. M. (2021). n-Power-posinormal operators. Methods of Functional Analysis and Topology, 27(1), 18-24.
[7]. Bourdon P.S and D. Thompson (2022), Posinormal composition operators on H^2,J. Math. Anal.
[8]. S. H. Lee, W. Y. Lee, (1996). A spectral mapping theorem for the Weyl spectrum.,Glasgow Math. J. 38, 61–4.
[9]. Bouldin R (1973), The product of operators with closed range, Tohoku Math. J. 25 , 359–363.