ISSN No: - 2456 – 2165

“A Study On Oscillation Criteria for Linear and Non-Linear Neutral Delay Differential Equations"

B. Maruthupandi Msc., B.Ed - M Phill (STUDENT)
Prist University Thanjavur -Tamilnadu

ISSN No: - 2456 – 2165

CHAPTER - I

INTRODUCTION

This dissertation entitled **"A STUDY ON OSCILLATION CRITERIA FOR LINEAR AND NON-LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS''** contains four chapters it is meant to study the basic concept and various interesting result on oscillation of second order Emden-fowler nonlinear neutral delay differential equations.

The chapter-1 is meant to recall the basic definition and various well known results relevant to this dissertation.

In chapter-2, we study about the basic definitions, and the most important results are:

Employing Riccati techniques and the integral averaging method,we establish interval oscillation criteria for the second-order Emden-Fowler neutral delay differential equation

$$\left[\left|x'(t)\right|^{\gamma-1}x'(t)\right]' + q_1(t)|y(t-\sigma)|^{\alpha-1}y(t-\sigma) \; + q_2(t)y(t-\sigma)=0$$

where $t \geq t_0$ and $x(t) = y(t) + p(t)y(t-\tau)$.

Consider the second-order Emden-Fowler neutral delay differential equation

$$\left[\left|x'(t)\right|^{\gamma-1}x'(t)\right]' + q_1(t)|y(t-\sigma)|^{\alpha-1}y(t-\sigma) + q_2(t)|y(t-\sigma)|^{\beta-1}y(t-\sigma) = 0 \qquad (1.1)$$

where $t \geq t_0$ and $x(t) = y(t) + p(t)y(t-\tau)$.

We assume that

(A1) τ and σ are nonnegative constants, α,β and γ are positive constants with $0 < \alpha < \gamma < \beta$

(A2) $q_1, q_2 \in C([t_0, \infty), R^+)$, $R^+ = (0, \infty)$

(A3) $p \in C([t_0, \infty), R)$, and $-1 < p_0 \leq p(t) \leq 1$, p_0 is a constant.

For any $\varphi \in C([t_0-\theta, t_0],R)$, $\theta = \max\{\tau, \sigma\}$, (1.1) has a solution y(t) extendable on $[t_0,\infty)$ satisfying the initial condition $y(t) \equiv \varphi(t)$ for $[t_0-\theta, t_0]$

$$y''(t) + q(t)|y(t)|^{\gamma-1}y(t) = 0, \;\; q \in C([t_0,\infty),R) and \;\; \gamma > 0 \qquad (1.2)$$

have been extended to the second order neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f\big((y-\sigma)\big) = 0 \qquad (1.3)$$

Under the assumption that the nonlinear function f satisfies The sub linear condition

$0 < \int_{0^+}^{\epsilon} \frac{du}{f(u)}, \int_{0^-}^{-\epsilon} \frac{du}{f(u)} < \infty \; forall \;\; \epsilon > 0$, as well as the super linear condition

$$0 < \int_{\epsilon}^{\infty} \frac{du}{f(u)}, \;\; \int_{-\epsilon}^{-\infty} \frac{du}{f(u)} < \infty, \qquad for\ all \;\; \epsilon > 0.$$

Also it will be of great interest to find some oscillation criteria for special case for (1.3), even for the Emden-Fowler neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)|y(t-\sigma)|^{\gamma-1}y(t-\sigma) = 0, \quad \gamma > 0. \tag{1.4}$$

The first beautiful interval criteria in this direction for some interval criteria for the oscillation of the second order linear ordinary differentia equation

$$(r(t)y'(t))'(t) + q(t)y(t) = 0 \tag{1.5}$$

Recently, Kong-type interval criteria to certain neutral differential equations.

In chapter-3, we study about the oscillation criteria for second order nonlinear neutral delay differential equations.

The oscillation of second order neutral differential equation

$$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0, \; t \geq t_0, \tag{1.6}$$

Where $p,q \in C([t_0,\infty),R)$, $f \in C(R,R)$:

Throughout this dissertation, we assume that $0 \leq p(t) \leq p_0 < +\infty$, $q(t) \geq 0$, and $q(t)$ is not identically zero on any ray of the form $[t_*,\infty)$ for any $t_* \geq t_0$, where p_0 is a constant, The oscillation of the second-order linear ordinary differential equation

$$x''(t) + p(t)x(t) = 0, \tag{1.7}$$

and used the class of functions as follows: Suppose there exist continuous functions

$$H, h : D \equiv \{(t, s) : t \geq s \geq t_0\} \rightarrow R \text{ such that } H(t, t) = 0, \; t \geq t_0,$$

$H(t, s) > 0$, $t > s \geq t_0$, and H has a continuous and non positive partial derivative on D with respect to the second variable.

then every solution of Eq. (1.7) oscillates. The oscillation criteria for second order linear equations for nonlinear analysis, we have

$$(r(t)x'(t))' + p(t)x(t) = 0, \tag{1.8}$$

and used the generalized Riccati substitution and established some new sufficient conditions for oscillation. If there exists a positive function $\in C^1([t_0,\infty),R^+)$ such that

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} [a(s)r(s)h(t,s)]ds < \infty,$$

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)[H(t,s)\psi(s) - \frac{1}{4}r(s)h^2(t,s)]ds = \infty,$$

Where $a(s) = \exp\{-\int_0^s g(u)du\}$ and $\psi(s) = \{p(s)+r(s)g^2(s)-(r(s)g(s))'\}$, then every solution of (1.8) oscillates. Riccati technique to obtain necessary and sufficient conditions for non oscillation of (1.8).Every solution of the equation

$(\frac{1}{t}x'(t))' + \frac{1}{t^3}x(t)=0$ is oscillatory.

An important tool in the study of oscillation is the integral averaging technique. Say a function H = H(t, s) belongs to a function class H, denoted by

$$H \in H, \text{ if } H \in C(D, R_+ \cup \{0\}),$$

Where $D = \{(t, s) : t_0 \leq s \leq t < \infty$ and $R_+ = (0,\infty)$, which satisfies

$$H(t, t) = 0,\ H(t, s) > 0, \text{ for } t > s,$$

and has partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)} \text{ and } \frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}$$

Sun defined another type of function class X and considered the oscillation of the second-order nonlinear damped differential equation

$$(r(t)y'(t))' + p(t)y'(t) + q(t)f(y(t)) = 0, \tag{1.9}$$

The oscillation of the second-order neutral delay differential equation

$$[r(t)(y(t) + p(t)y(\sigma(t)))']' + \sum_{i=1}^{n} q_i(t) f_i(y(\tau_i(t))) = 0 \tag{1.10}$$

$$\omega(t) = r(t)\frac{z'(t)}{z(t)}, z(t) = y(t) + p(t)y(\sigma(t)),$$

The authors established some oscillation criteria for Eq. (1.10).

ISSN No: - 2456 – 2165

CHAPTER-II

PRELIMINARIES

Definition: 2.1

An equation involving derivatives or differential of one or more dependent variables with respect to one or more independent variables is called differential equation.

Examples:

1. $\frac{dy}{dx} = (x + sinx)$
2. $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + [\frac{dx}{dt}]^5 = e^t$
3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Definition:2.2

A differential equation involving derivatives with respect to single independent variable is called an Ordinary Differential Equation.

Equation 1 & 2 are Ordinary Differential Equations.

Definition: 2.3

A partial differential equation is an equation which equation contains one or more partial derivatives. Equation (3) is second order partial differential equation.

Definition: 2.4

A non trivial solution x(t) is said to be oscillatory, if it has arbitrarily large zeros for $t \geq t_0$,

That is, there exist a sequence of zeros $\{t_n\}(x(t_n)=0)$ of x(t) such that $\lim_{n\to\infty} s\, t_n = +\infty$

Definition: 2.5

A non trivial solution x(t) is said to be non oscillatory, if there exist a t_1, such that $x(t) \neq 0$ for all $t \geq t_1$

Definition: 2.6

Assume that $\Phi(t, s, l) \in X$: The operator $T_n[g, l, t]$ is

ISSN No: - 2456 – 2165

defined by

$$T_n[g, l, t] = \int_l^t \Phi^n(t, s, l) g(s) ds,$$

for $n \geq 1$, $t \geq s \geq l \geq t_0$ and $g \in C([t_0,\infty),R)$.s

Definition: 2.7

The function $\varphi = \varphi(t, s, l)$ is defined by

$$\frac{\partial \Phi(t, s, l)}{\partial s} = \varphi(t, s, l)\Phi(t, s, l)$$

It is easy to verify that $T_n[\cdot; l, t]$ is a linear operator and that it satisfies

$$T_n[g', l, t] = -nT_n[g\varphi; l, t], \text{ for } g \in C^1([t_0,\infty),R).$$

ISSN No: - 2456 – 2165

CHAPTER-III

OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

In this chapter, we give some new oscillation results for (1.6).

We start with the following oscillation results.

Theorem:3.1

Assume that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$: Further, suppose that there exists a function g Є C^1 ($[t_0, \infty)$, R)such that for some $\beta \geq 1$ and some H Є H, one has

$$\lim_{t\to\infty} sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\psi(s) - \left(1 + \frac{p_0}{\tau_0}\right) \frac{\beta}{4\sigma'(s)} u(s) h^2(t,s) \right] ds = \infty \tag{3.1}$$

where $\psi(t) := u(t)\{k\, Q(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\}$,

$Q(t):= \min \{q(t), q(\tau(t))\}$, $\quad u(t) := \exp \{2\int_{t_0}^{t} \sigma'(s) g(s) ds\}$.

Then every solution of

$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0$, $t \geq t_0$, is oscillatory.

Proof:

Let x be a non oscillatory solution of

$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0$, $t \geq t_0$,.

Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that

$x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, for all $t \geq t_1$: Define

$z(t) = x(t) + p(t)x(\tau(t))$ for $t \geq t_0$, then $z(t) > 0$ for $t \geq t_1$.

From (1.6), we have

$$z''(t)+q(t)f(x(\sigma(t)))=0, \; t\geq t_1$$

then by (b), we have

$$z''(t)=-q(t)f(x(\sigma(t)))$$

$$z''(t) \leq -kq(t)x(\sigma(t)) \leq 0, \; t \geq t_1 \tag{3.2}$$

It is obvious that $z''(t) \leq 0$ and $z(t) > 0$ for $t \geq t_1$

Implies $z'(t) > 0$ for $t \geq t_1$.

Using (3.2) and the condition (b), there exists $t_2 \geq t_1$ such that for $t \geq t_2$,

We get

$$0 = z''(t) + q(t)f(x(\sigma(t)))$$

ISSN No: - 2456 – 2165

$$= z''(t) + q(t)f(x(\sigma(t))) + p_0[z''(\tau(t)) + q(\tau(t))f(x(\sigma(\tau(t))))]$$

$$=[z(t) + p_0 z(\tau(t))]'' + q(t)f(x(\sigma(t))) + p_0 q(\tau(t))f(x(\sigma(\tau(t))))$$

$$\geq[z(t) + \frac{p_0}{\tau_0} z(\tau(t))]'' + k[q(t)x(\sigma(t)) + p_0 q(\tau(t))x(\tau(\sigma(t)))]$$

$$\geq [z(t) + \frac{p_0}{\tau_0} z(\tau(t))]'' + kQ(t)[x(\sigma(t)) + p_0 x(\tau(\sigma(t)))]$$

$$\geq[z(t) + \frac{p_0}{\tau_0} z(\tau(t))]'' + kQ(t)z(\sigma(t)). \quad (3.3)$$

We introduce a generalized Riccati transformation

$$\omega(t) = u(t)\left[\frac{z'(t)}{z(\sigma(t))} + g(t)\right] \quad (3.4)$$

Note that $\sigma(t) \leq t$. Then we have $z''(t) \leq z'(\sigma(t))$ Differentiating (3.4), Thus, there exists $t_3 \geq t_1$ such that for all $t \geq t_3$,

$$\omega'(t)= u'(t)[\frac{z'(t)}{z(\sigma(t))} + g(t)]$$

$$+u(t)[\frac{z''(t)z(\sigma(t)) - z'(t)z'(\sigma(t))\sigma'(t)}{[z(\sigma(t))]^2} + g'(t)]$$

$$\leq u(t)\frac{z''(t)}{z(\sigma(t))} - u(t)\left[\sigma'(t)(\frac{z'(t)}{z(\sigma(t))})^2 + g'(t)\right] + u'(t)\frac{w(t)}{u(t)}$$

$$\omega'(t) \leq -2\sigma'(t)g(t)\omega(t) + u(t)\{\frac{z''(t)}{z(\sigma(t))} - \sigma'(t)\left[\frac{\omega(t)}{u(t)} - g(t)\right]^2 + g'(t)\}$$

$$= u(t)\frac{z''(t)}{z(\sigma(t))} + u(t)[-\sigma'(t)g^2(t) + g'(t)] - \sigma'(t)\frac{\omega^2(t)}{u(t)} \quad (3.5)$$

Similarly, we introduce another generalized Riccati transformation

$$\upsilon(t) = u(t)\left[\frac{z'(\tau(t))}{z(\sigma(t))} + g(t)\right] \quad (3.6)$$

Differentiating (3.6), note that $\sigma(t) \leq \tau(t)$, by (3.2) we have

$z'(\sigma(t)) \geq z'(\tau(t))$, then for all sufficiently large t, one has

$$v'(t) = u'(t)[\frac{z'(\tau(t))}{z(\sigma(t))} + g(t)] + u(t)[\frac{z''(\tau(t))\tau'(t)z(\sigma(t)) - z'(\sigma(t))z'(\tau(t))\sigma'(t)}{[z(\sigma(t))]^2} + g'(t)]$$

$$\upsilon'(t) \leq -2\sigma'(t)g(t)\upsilon(t) + u(t)\{\tau_0\frac{z''(\tau(t))}{z(\sigma(t))} - \sigma'(t)\left[\frac{\upsilon(t)}{u(t)} - g(t)\right]^2 + g'(t)\}$$

$$= \tau_0 u(t)\frac{z''(\tau(t))}{z(\sigma(t))} + u(t)[-\sigma'(t)g^2(t) + g'(t)] - \sigma'(t)\frac{\upsilon^2(t)}{u(t)} \quad (3.7)$$

From (3.5) and (3.7), we have

$$\left[\omega(t) + \frac{p_0}{\tau_0}\upsilon(t)\right]' \leq \frac{u(t)}{z(\sigma(t))}[z(t) + \frac{p_0}{\tau_0}z(\tau(t))]''$$

$$+(1 + \frac{p_0}{\tau_0})u(t)[-\sigma'(t)g^2(t) + g'(t)] - \frac{\sigma'(t)\omega^2(t)}{u(t)} - \frac{p_0}{\tau_0}\frac{\sigma'(t)\upsilon^2(t)}{u(t)}$$

By (3.3) and the above inequality, we obtain

$$\left[\omega(t)+\frac{p_0}{\tau_0}\upsilon(t)\right]' \leq -\psi(t)-\frac{\sigma'(t)\omega^2(t)}{u(t)}-\frac{p_0}{\tau_0}\frac{\sigma'(t)\upsilon^2(t)}{u(t)} \tag{3.8}$$

Multiplying (3.8) by H(t, s) and integrating from T to t, we have, for any $\beta \geq 1$ and for all $t \geq T \geq t_3$,

$$\int_T^t H(t,s)\psi(s)ds \leq -\int_T^t H(t,s)\omega'(s)ds-\int_T^t H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}ds$$

$$-\frac{p_0}{\tau_0}\int_T^t H(t,s)\upsilon'(s)ds-\frac{p_0}{\tau_0}\int_T^t H(t,s)\frac{\sigma'(s)\upsilon^2(s)}{u(s)}ds$$

Using integration by parts we get,

$$\int_T^t H(t,s)\psi(s)ds = -H(t,s)\omega(s)|_T^t-\int_T^t\left[-\frac{\partial H(t,s)}{\partial(s)}\omega(s)+H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}\right]ds$$

$$-\frac{p_0}{\tau_0}H(t,s)\upsilon(s)|_T^t-\frac{p_0}{\tau_0}\int_T^t[-\frac{\partial H(t,s)}{\partial(s)}\upsilon(s)+H(t,s)\frac{\sigma'(s)\upsilon^2(s)}{u(s)}]ds$$

Since H(t,t)=0

$$=H(t,T)\omega(T)-\int_T^t[h(t,s)\sqrt{H(t,s)}\omega(s)+H(t,s)\frac{\sigma'(s)\omega^2(s)}{u(s)}]ds$$

$$+\frac{p_0}{\tau_0}H(t,T)\upsilon(T)\frac{p_0}{\tau_0}\int_T^t[h(t,s)\sqrt{H(t,s)}\upsilon(s)+H(t,s)\frac{\sigma'(s)\upsilon^2(s)}{u(s)}]ds$$

$$=H(t,T)\omega(T)-\int_T^t[\sqrt{\frac{H(t,s)\sigma'(s)}{\beta u(s)}}\omega(s)+\sqrt{\frac{\beta u(s)}{4\sigma'(s)}}h(t,s)]^2ds$$

$$+\int_T^t\frac{\beta u(s)}{4\sigma'(s)}h^2(t,s)ds-\int_T^t\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\omega^2(s)ds$$

$$+\frac{p_0}{\tau_0}H(t,T)\upsilon(T)-\frac{p_0}{\tau_0}\int_T^t[\sqrt{\frac{H(t,s)\sigma'(s)}{\beta u(s)}}\upsilon(s)+\sqrt{\frac{\beta u(s)}{4\sigma'(s)}}h(t,s)]^2ds$$

$$+\frac{p_0}{\tau_0}\int_T^t\frac{\beta u(s)}{4\sigma'(s)}h^2(t,s)ds-\frac{p_0}{\tau_0}\int_T^t\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\upsilon^2(s)ds \tag{3.9}$$

From the above inequality and using monotonicity of H, for all $t \geq t_3$, we obtain

$$\int_{t_3}^t[H(t,s)\psi(s)-(1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds \leq H(t,t_0)|\omega(t_3)|+\frac{p_0}{\tau_0}H(t,t_0)$$

and, for all $t \geq t_3$,

ISSN No: - 2456 – 2165

$$\int_{t_0}^{t}[H(t,s)\psi(s)-(1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds$$

$$\le H(t,t_0)[\int_{t_0}^{t_3}|\psi(s)|ds+|\omega(t_3)|+\frac{p_0}{\tau_0}|\upsilon(t_3)|] \tag{3.10}$$

By (3.10),

$$\lim_{t\to\infty} sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}[H(t,s)\psi(s)-(1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds$$

$\le \int_{t_0}^{t_3}|\psi(s)|ds+|\omega(t_3)|+\frac{p_0}{\tau_0}|\upsilon(t_3)|< \infty$. This contradicts to

$$\lim_{t\to\infty} sup\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\psi(s)-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)\right]ds=\infty,$$

This completes the proof.

Corollary: 3.1

Suppose that $\sigma(t) \le \tau(t)$ for $t \ge t_0$.Furthermore, assume that there exists a function g Є $C^1([t_0,\infty),R)$ such that for some integer n >2 and some $\beta \ge 1$,

$$\lim_{t\to\infty} sup\, t^{1-n}\int_{t_0}^{t}(t-s)^{n-3}[(t-s)^2\psi(s)$$

$$-(1+\frac{p_0}{\tau_0})\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)]ds=\infty,$$

where, $\psi(t) := u(t)\{kQ(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\}$,

$$Q(t):= \min\{q(t), q(\tau(t))\},$$

$$u(t) :=\exp\{2\int_{t_0}^{t}\sigma'(s)g(s)ds\}.$$

Then every solution of

$$(x(t) + p(t)x(\tau(t)))''+ q(t)f(x(\sigma(t))) = 0, t \ge t_0,$$

Isoscillatory.

Proof:

By theorem (3.1), we have

$$\lim_{t\to\infty} sup\frac{1}{(t-t_0)^{n-1}}\int_{t_0}^{t}[(t-s)^{n-1}\psi(s)$$

$$-\left(1+\frac{p_0}{\tau_0}\right)\frac{\beta}{4\sigma'(s)}(n-1)^2(t-s)^{n-3}u(s)]ds,$$

ISSN No: - 2456 – 2165

$$= \limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} (t-s)^{n-3}[(t-s)^2\psi(s) - (1+\frac{p_0}{\tau_0})\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)]d = \infty$$

Example: 3.2

Consider the second-order neutral differential equation

$$[x(t) + (3 + \sin t)x(t - \tau)]'' + \frac{\gamma}{t^2}x(t-\sigma) = 0, t \geq 1, \quad (3.11)$$

Where $\sigma \geq \tau$, $\gamma > 0$

Let $p(t) = 3 + \sin t$, $q(t) = \frac{\gamma}{t^2}$, $f(x) = x$, $g(t) = \frac{-1}{2t}$

Thenu(t) $= \exp\{-2\int_{t_0}^{t} g(t)dt\}$

$$= \exp\{-2\int_{t_0}^{t} \frac{-1}{2t}dt\}$$

$$= \exp(\log t)$$

$$= t,$$

$$\psi(t) = u(t)\left\{kQ(t) + \left(1+\frac{p_0}{\tau_0}\right)[\sigma'(t)g^2(t) - g'(t)]\right\}$$

$$= t\left\{k\frac{\gamma}{t^2} + \left(1+\frac{p_0}{\tau_0}\right)\left[(1)\frac{1}{4t^2} - \frac{1}{2t^2}\right]\right\}$$

Taking k=1, p=4 and $\tau_0 = \tau'(t) = 1$ we have,

$$\Psi(t) = (\gamma - 5/4)/t.$$

Applying Corollary 3.1 with n = 3, for any $\beta \geq 1$,

$$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-3}[(t-s)^2\psi(s)$$

$$-(1+\frac{p_0}{\tau_0})\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)]ds$$

$$= \limsup_{t\to\infty} t^{-2}\int_{1}^{t}(t-s)^0[(t-s)^2\frac{(\gamma-\frac{5}{4})}{s} - \beta s]ds$$

$$= \limsup_{t\to\infty} t^{-2}\int_{1}^{t}[(t^2-2ts+s^2)\frac{(\gamma-\frac{5}{4})}{s} - \beta s]ds$$

$$= \limsup_{t\to\infty} t^{-2}[(t^2\log s - 2ts + \frac{s^2}{2})\left(\gamma - \frac{5}{4}\right) - \beta\frac{s^2}{2}]_1^t$$

$$= \limsup_{t\to\infty} t^{-2}\left[\left(t^2\log t - 2t^2 + \frac{t^2}{2}\right)\left(\gamma-\frac{5}{4}\right) - \beta\frac{t^2}{2}\right] - \left[\left(t^2\log 1 - 2t + \frac{1}{2}\right)\left(\gamma - \frac{5}{4}\right) - \beta\frac{1}{2}\right]$$

= ∞, for $\gamma>5/4$.

Hence,

$$[x(t) + (3 + \sin t)x(t - \tau)]'' + \frac{\gamma}{t^2}x(t-\sigma) = 0, \ t \geq 1, \text{ is oscillatory for } \gamma>5/4.$$

Corollary 3.1 can be applied to the second-order

Euler Differential Equation

$$x''(t) + \frac{\gamma}{t^2}x(t) = 0, \ t \geq 1, \tag{3.12}$$

where $\gamma>0$. Let $p(t) = 0$, $q(t) = \gamma/t^2$, $f(x) = x$, $g(t) = -1/(2t)$

Then $u(t) = t$, $\Psi(t) =(\gamma -1/4)/t$. Take $k =1$, $p_0 = 0$.

Applying Corollary 4.1 with n = 3, for any $\beta \geq 1$,

$$\limsup_{t\to\infty} t^{1-n}\int_{t_0}^{t}(t-s)^{n-3}[(t-s)^2\psi(s) - \frac{\beta(n-1)^2(1+p_0)}{4}u(s)]ds$$

$$= \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t\left[\left(\gamma-\frac{1}{4}\right)\frac{(t-s)^2}{s} - \beta s\right]ds = \infty \ \text{ For } \gamma > 1/4.$$

Hence, $x''(t) + \frac{\gamma}{t^2}x(t) = 0$, $t \geq 1$is oscillatory for $\gamma > 1/4$.

It may happen that assumption (3.1) is not satisfied, or it is not easy to verify, consequently, Theorem 3.1 does not apply or is difficult to apply. The following results provide some essentially new oscillation criteria for equation

$$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0, \ t \geq t_0.$$

Theorem 3.2

Assume that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$, and for some H Є H,

$$0 < \inf_{s\geq t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}] \leq \infty. \tag{3.13}$$

Further, suppose that there exist functions g Є $C^1([t_0,\infty),R)$ and m Є $C([t_0,\infty),R)$ such that for all $T \geq t_0$and for some $\beta >1$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t[H(t,s)\psi(s) - (1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds \geq m(T), \tag{3.14}$$

Where $\psi(t) = u(t)\{kQ(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\}$,

$$Q(t)= \min\{q(t), q(\tau(t))\},$$

$$u(t) = \exp\{2\int_{t_0}^{t} \sigma'(s)g(s)ds\}.$$

Suppose further that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\sigma'(s)m_+^2(s)}{u(s)}ds = \infty, \tag{3.15}$$

where $m_+(t) := \max\{m(t), 0\}$.Then every solution of

$$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \text{ Isoscillatory.}$$

Proof:

Assuming, without loss of generality, that there exists a solution x of

$$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \text{ such that } x(t) > 0,$$

$$x(\tau(t)) > 0,\ x(\sigma(t)) > 0, \text{ for all } t \geq t_1 .$$

We define the functions

$$\omega(t) = u(t)\left[\frac{z'(t)}{z(\sigma(t))} + g(t)\right] \text{and}$$

$$\upsilon(t) = u(t)\left[\frac{z'(\tau(t))}{z(\sigma(t))} + g(t)\right]\cdot$$

we arrive at the inequality (3.9), which yields for $t > T \geq t_1$, sufficiently large

$$\frac{1}{H(t,T)}\int_{T}^{t}[H(t,s)\psi(s) - (1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds$$

$$\leq \omega(t) - \frac{1}{H(t,T)}\int_{T}^{t}\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\omega^2(s)ds + \frac{p_0}{\tau_0}\upsilon(t) - \frac{p_0}{\tau_0}\frac{1}{H(t,T)}\int_{T}^{t}\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}\upsilon^2(s)ds$$

Therefore, for $t > T \geq t_1$, sufficiently large

$$\lim_{t\to\infty} sup\frac{1}{H(t,T)}\int_{T}^{t}[H(t,s)\psi(s) - (1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds$$

$$\leq \omega(t) + \frac{p_0}{\tau_0}\upsilon(t) - \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}(\omega^2(s)$$

$$+ \frac{p_0}{\tau_0}\upsilon^2(s))ds$$

It follows from (3.14) that

$$\omega(t)+\frac{p_0}{\tau_0}\upsilon(t)\ \geq\ m(T)+\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds$$

for all $T \geq t_1$ and for any $\beta > 1$.Consequently, for all $T \geq t_1$, we obtain

$$\omega(T)+\frac{p_0}{\tau_0}\upsilon(T) \geq m(T), \quad \text{and} \tag{3.16}$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^t\frac{\sigma'(s)H(t,s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds$$

$$\leq \frac{\beta}{\beta-1}\left(\omega(t_1)+\frac{p_0}{\tau_0}\upsilon(t_1)-m(t_1)\right)<\infty \tag{3.17}$$

In order to prove that

$$\int_{t_1}^{\infty}\frac{\sigma'(s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds<\infty \tag{3.18}$$

suppose the contrary, that is,

$$\int_{t_1}^{\infty}\frac{\sigma'(s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds=\infty \tag{3.19}$$

Assumption

$$0<\inf_{s\geq t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}]\leq\infty$$

Existence of a $\rho>0$ such that

$$\inf_{s\geq t_0}[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}]>\rho \tag{3.20}$$

By (3.20), we have

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}>\rho>0,$$

and there exists a $T_2 \geq T_1$ such that $H(t, T_1)/H(t, t_0) \geq \rho$, for all $t \geq T_2$.

On the other hand, by virtue of (3.19), for any positive number κ, there exists a $T_1 \geq t_1$ such that, for all $t \geq T_1$,

$$\int_{t_1}^{t}\frac{\sigma'(s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds\geq\frac{\kappa}{\rho}.$$

Using integration by parts, we conclude that, for all $t \geq T_1$, $\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}\upsilon^2(s))ds$

$$=\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right][\int_{t_1}^{s}\frac{\sigma'(\upsilon)}{u(\upsilon)}(\omega^2(\upsilon)+\frac{p_0}{\tau_0}\upsilon^2(\upsilon))d\upsilon]ds$$

$$\geq \frac{\kappa}{\rho}\frac{1}{H(t,t_1)}\int_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]ds = \frac{\kappa}{\rho}\frac{H(t,T_1)}{H(t,t_1)} \tag{3.21}$$

It follows from (3.21) that, for all $t \geq T_2$,

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{H(t,s)\sigma'(s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}v^2(s))ds \geq \kappa,$$

Since κis an arbitrary positive constant, we get

$$\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\frac{\sigma'(s)H(t,s)}{u(s)}(\omega^2(s)+\frac{p_0}{\tau_0}v^2(s))ds = \infty,$$

which contradicts (3.17). Consequently, (3.18) holds, so

$$\int_{t_1}^{\infty}\frac{\sigma'(s)}{u(s)}\omega^2(s)ds < \infty, \qquad \int_{t_1}^{\infty}\frac{\sigma'(s)}{u(s)}v^2(s)ds < \infty$$

and, by virtue of (3.16)

$$\int_{t_1}^{\infty}\frac{\sigma'(s)m_+^2(s)}{u(s)}ds \leq \int_{t_1}^{\infty}\frac{\sigma'(s)\omega^2(s)+\left(\frac{p_0}{\tau_0}\right)^2\sigma'(s)v^2(s)+\frac{2p_0}{\tau_0}\sigma'(s)\omega(s)v(s)}{u(s)}ds$$

$$\leq \int_{t_1}^{\infty}\frac{\sigma'(s)\omega^2(s)+\left(\frac{p_0}{\tau_0}\right)^2\sigma'(s)v^2(s)+\frac{p_0}{\tau_0}\sigma'(s)[\omega^2(s)+v^2(s)]}{u(s)}ds < \infty,$$

which contradicts (3.15). This completes the proof.

Choosing H as in Corollary 3.1, it is easy to verify that condition (3.13) is satisfied because, for any $s \geq t_0$,

$$\lim_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty}\frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1.$$

Consequently, we have the following result.

Corollary:3.2

Suppose that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$.Furthermore, assume that there exist functions g Є $C^1([t_0,\infty),R)$ and m Є $C([t_0,\infty),R)$ such that for all $T \geq t_0$, for some integer n >2

and some $\beta \geq 1$, $\limsup_{t\to\infty} t^{1-n}\int_{T}^{t}(t-s)^{n-3}[(t-s)^2\psi(s)$

$$-(1+\frac{p_0}{\tau_0})\frac{\beta(n-1)^2}{4\sigma'(s)}u(s)]ds \geq m(T)$$

where u and ψ are as in Theorem 3.1. Suppose further that (3.15) holds, where $m_+$is as in Theorem 3.2. Then every solution of (x(t) + p(t)x(τ(t)))"+ q(t)f(x(σ(t))) = 0,

t ≥ t_0, isoscillatory.

From Theorem 3.2, we have the following result.

Theorem: 3.3

Assume that σ(t) ≤τ(t) for t ≥ t_0.Further, suppose that H Є H, there exist functions

g Є $C^1([t_0,\infty),R)$ and m Є $C([t_0,\infty),R)$ such that for all T ≥ 0 and for some β >1,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)}\int_T^t [H(t,s)\psi(s) - (1+\frac{p_0}{\tau_0})\frac{\beta}{4\sigma'(s)}u(s)h^2(t,s)]ds \geq m(T), \quad (3.22)$$

Where

$$\psi(t) = u(t)\{kQ(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\},$$

$$Q(t)= \min\{q(t), q(\tau(t))\},$$

$$u(t) = \exp\{2\int_{t_0}^t \sigma'(s)g(s)ds\}.$$

Suppose further that (3.15) holds, where m_+ is as in Theorem 3.2.

Then every solution of (x(t) + p(t)x(τ(t)))"+ q(t)f(x(σ(t))) = 0, t ≥ t_0 ,isoscillatory.

Theorem:3.4

Assume that σ(t) ≤ τ(t) for t ≥ t_0.Further, assume that there exists a function Φ Є X, such that for each l ≥ t_0, for some n ≥ 1,

$$\limsup_{t\to\infty} T_n[\psi(s) - \frac{n^2}{4}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\sigma'(s)}; l, t] > 0, \quad (3.23)$$

where ψ, u are defined as in Theorem 3.1, the operator T_n is defined by

$T_n[g,l,t] = \int_l^t \Phi^n(t,s,l)g(s)ds$, and

φ = φ(t, s, l) is defined by $\frac{\partial\Phi(t,s,l)}{\partial s} = \varphi(t,s,l)\Phi(t,s,l)$

Then every solution of

(x(t) + p(t)x(τ(t)))"+ q(t)f(x(σ(t))) = 0, t ≥ t_0,isoscillatory.

Proof:

ISSN No: - 2456 – 2165

Assuming, without loss of generality, that there exists a solution x of

$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0$, $t \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\sigma(t)) > 0$, for all $t \geq t_1$.

We define the functions

$$\omega(t) = u(t)\left[\frac{z'(t)}{z(\sigma(t))} + g(t)\right] \quad \text{and}$$

$$\upsilon(t) = u(t)\left[\frac{z'(\tau(t))}{z(\sigma(t))} + g(t)\right].$$

we arrive at the inequality (3.8). Applying $T_n[\cdot; l, t]$ to (3.8), we get

$$T_n\left[\left[\omega(s) + \frac{p_0}{\tau_0}\upsilon(s)\right]'; l, t\right] \leq T_n\left[-\psi(s) - \frac{\sigma'(s)\omega^2(s)}{u(s)} - \frac{p_0}{\tau_0}\frac{\sigma'(s)\upsilon^2(s)}{u(s)};\ l, t\right].$$

$$T_n[\psi(s); l, t] \quad \leq \ T_n\left[n\varphi\omega(s) - \frac{\sigma'(s)\omega^2(s)}{u(s)} + n\frac{p_0}{\tau_0}\varphi\upsilon(s) - \frac{p_0}{\tau_0}\frac{\sigma'(s)\upsilon^2(s)}{u(s)};\ l, t\right] \qquad (3.24)$$

Hence, from (3.24) we have

$$T_n\left[\left[\omega(s) + \frac{p_0}{\tau_0}\upsilon(s)\right]'; l, t\right] \leq T_n\left[\frac{n^2}{4}\left(1 + \frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\sigma'(s)}; l, t\right]$$

that is,

$$T_n\left[\psi(s) - \frac{n^2}{4}\left(1 + \frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\sigma'(s)}; l, t\right] \leq 0,$$

Taking the super limit in the above inequality, we get

$$\limsup_{t\to\infty} T_n\left[\psi(s) - \frac{n^2}{4}\left(1 + \frac{p_0}{\tau_0}\right)\frac{u(s)\varphi^2(s)}{\sigma'(s)}; l, t\right] \leq 0,$$

which contradicts (3.23). This completes the proof. If we choose

$$\Phi(t, s, l) = \rho(s)(t - s)^{\alpha}(s - l)^{\beta} \qquad (3.25)$$

for $\alpha, \beta > ½$, and $\rho \in C^1([t_0,\infty), (0,\infty))$, then we have

$$\varphi(t, s, l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}. \qquad (3.26)$$

Thus by Theorem 3.4, we have the following oscillation result.

Corollary:3.3

Suppose that $\sigma(t) \leq \tau(t)$ for $t \geq t_0$ .Further, assume that for each $l \geq t_0$, there exists a function $\rho \in C^1([t_0,\infty), (0,\infty))$ and two constants $\alpha, \beta > ½$,such that for some

$n \geq 1$,

ISSN No: - 2456 – 2165

$$\limsup_{t\to\infty}\int_l^t \rho^n(s)\,(t-s)^{n\alpha}(s-l)^{n\beta}[\psi(s)-\frac{n^2}{4}$$

$$\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\sigma'(s)}(\frac{\rho'(s)}{\rho(s)}+\frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)})^2]ds>0.$$

Where $\psi(t) = u(t)\{kQ(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\}$,

$Q(t)= \min\{q(t), q(\tau(t))\}$,

$u(t) =\exp\{2\int_{t_0}^{t}\sigma'(s)g(s)ds\}$.

Then every solution of

$(x(t) + p(t)x(\tau(t)))''+ q(t)f(x(\sigma(t))) = 0$, $t \geq t_0$, isoscillatory.

If we choose

$$\Phi(t,s,l)=\sqrt{H_1(s,l)H_2(t,s)} \tag{3.27}$$

whereH_1, H_2 Є H, then we have

$$\varphi(t,s,l)=\frac{1}{2}(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}}-\frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}), \tag{3.28}$$

where$h_1^{(1)}(s,l), h_2^{(2)}(t,s)$ are defined as the following:

$$\frac{\partial H_1(s,l)}{\partial s}=h_1^{(1)}(s,l)\sqrt{H_1(s,l)}\text{and}$$

$$\frac{\partial H_2(t,s)}{\partial s}=-h_2^{(2)}(t,s)\sqrt{H_2(t,s)}, \tag{3.29}$$

According to Theorem 3.4, we have the following oscillation result.

Corollary: 3.4

Suppose that$\sigma(t) \leq \tau(t)$ for $t \geq t_0$.Further, assume that for each $l \geq t_0$, there exist two functions H_1,H_2ЄH such that for some $n \geq 1$

$$\limsup_{t\to\infty}\int_l^t (\sqrt{H_1(s,l)H_2(t,s)})^n\,[\psi(s)$$

$$-\frac{n^2}{16}\left(1+\frac{p_0}{\tau_0}\right)\frac{u(s)}{\sigma'(s)}\left(\frac{h_1^{(1)}(s,l)}{\sqrt{H_1(s,l)}}-\frac{h_2^{(2)}(t,s)}{\sqrt{H_2(t,s)}}\right)^2]ds>0,$$

Where $\psi(t) = u(t)\{kQ(t)+(1 + p_0/\tau_0)[\sigma'(t)g^2(t)-g'(t)]\}$,

$Q(t)= \min\{q(t), q(\tau(t))\}$,

$u(t) =\exp\{2\int_{t_0}^{t}\sigma'(s)g(s)ds\}$.

Then every solution of

$$(x(t) + p(t)x(\tau(t)))'' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \text{isoscillatory.}$$

In the following, we give some new oscillation results for equation

$$(x(t) + p(t)x(\tau (t)))'' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \text{when } \sigma(t) \geq \tau (t) \text{ for } t \geq t_0.$$

ISSN No: - 2456 – 2165

BIBLIOGRAPHY

[1]. R. P. Agarwal, S. R. Grace, Oscillation theorems for certain neutral functional differential equations, Comput. Math. Appl. 38 (11–12) (1999) 1–11.

[2]. J. Dˇzurina, I. P. Stavroulakis, Oscillation criteria for second-order delay differential equations,Appl. Math. Comput. 140 (2'‘3) 445–453.

[3]. S. R. Grace, B. S. Lalli, Oscillation of nonlinear second order neutral differential equations,Rat. Math. 3 (1987) 77–84.

[4]. M. K. Grammatikopoulos, G. Ladas, A. Meimaridou, Oscillation of second order neutral delaydifferential equations, Rat. Mat. 1 (1985) 267–274.

[5]. B. Karpuz, J. V. Manojlovi´c, ¨ O. ¨ Ocalan, Y. Shoukaku, Oscillation criteria for a class of second-order neutral delay differential equations, Appl. Math. Comput. 210 (2'‘9) 303–312.

[6]. H. J. Li, Oscillatory theorems for second-order neutral delay differential equations, NonlinearAnalysis. 26 (1996) 1397–1409.

[7]. H. J. Li, Oscillation of solutions of second-order neutral delay differential equations withintegrable coefficients, Mathl. Comput. Modelling 25 (1997) 69–79.

[8]. W. T. Li, R. P. Agarwal, Interval oscillation criteria related to integral averaging techniquefor certain nonlinear differential equations, J. Math. Anal. Appl. 245 (2'‘0) 171–188.

[9]. X. Y. Lin, X. H. Tang, Oscillation of solutions of neutral differential equations with a super linearneutral term, Appl. Math. Lett. 20 (2'‘7) 1016–1022.

[10]. L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, 1995.