ISSN No: - 2456 – 2165

Vertex Coloring with Basic Bound on Chromatic Number

G.Sangeetha
Msc., M Phill (Student)
Prist University-Thanjavur

CHAPTER-I Basic Definitions

CHAPTER-II Vertex Coloring with Basic Bound on Chromatic Number Related Theorems

INTRODUCTION

This thesis investigates problems in a number of deterrent areas of graph theory. These problems are related in the sense that they mostly concern the coloring or structure of the underlying graph.

The first problem we consider is in Ramsey Theory, a branch of graph theory stemming from the eponymous theorem which, in its simplest form, states that any esuriently large graph will contain a clique or anti-clique of a spiced size. The problem of ending the minimum size of underlying graph which will guarantee such a clique or anti-clique is an interesting problem in its own right, which has received much interest over the last eighty years but which is notoriously intractable. We consider a generalization of this problem. Rather than edges being present or not present in the underlying graph, each is assigned one of three possible colors and, rather than considering cliques, we consider cycles. Combining regularity and stability methods, we prove an exact result for a triple of long cycles.

We then move on to consider removal lemmas. The classic Removal Lemma states that, for n succinctly large, any graph on n vertices containing o(n3) triangles can be made triangle-free by the removal of o(n2) edges. Utilizing a colored hyper graph generalization of this result, we prove removal lemmas for two classes of multinomial. Next, we consider a problem in fractional coloring. Since ending the chromatic number of a given graph can be viewed as an integer programming problem, it is natural to consider the solution to the corresponding linear programming problem. The solution to this LP-relaxation is called the fractional chromatic number. By a probabilistic method, we improve on the best previously known bound for the fractional chromatic number of a triangle-free graph with maximum degree at most three.

Finally, we prove a weak version of Viking's Theorem for hyper graphs. We prove that, if H is an intersecting 3-uniform hyper graph with maximum degree _ and maximum multiplicity _, then H has at most 2_+_ edges. Furthermore, we prove that the unique structure achieving this maximum is _ copies of the Fanon Plane.

This thesis considers a number of problems in graph theory. A graph is an abstract mathematical structure formed by a set of vertices and edges joining pairs of those vertices. Graphs can be used to model the connections between objects; for instance, a computer network can be modeled as a graph with each server represented by a vertex and the connections between those servers represented by edges. Many problems in graph theory involve some sort of coloring, that is, assignment of labels or `colors' to the edges or vertices of a graph. Such problems fall broadly into two categories: The type of problem concerns the possibility of assigning colors to a graph while respecting some set of rules; the second concerns the existence of colored Structures in a graph whose coloring we do not control.

The graph coloring traces its origins to 1852, when Francis Guthrie observed that a map of the counties of England can be colored using four colors in such a way that adjacent counties receive different colors. The question of whether this is the case for any such map became known as the Four Color Problem and is, without doubt, the most well-known problem from the category above. This problem received much attention over the following century (see, for instance, [Wil03]) before, being answered in the a_rmative by Appeal and Hakes [AH77, AHK77] in 1976. The archetypal problem of the second type can also be phrased in a non-abstract form as follows: Suppose you were to invite multiple guests to a dinner-party and that those guests have not necessarily met each other previously. How many guests would you need to invite in order to guarantee that there will be three mutual acquaintances or three mutual strangers at the dinner table? Upon reading, it is less than obvious that such a question should have a answer | perhaps, for any size of party, there is a possible list of acquaintances and strangers without such a triad. In fact, it can easily be shown that the answer is six and, as we will see later, that no matter how large a collection of mutual acquaintances or collection of mutual strangers we require, there is size of gathering that will guarantee the existence of one or the other. However, the exact answer to this general problem is notoriously an interesting feature of many problems in Graph Theory (including the two problems above) is the contrast between the ease with which they may be stated and the apparent difficulty of their solution. This contrast is also apparent in most of the problems Considered in this thesis.

CHAPTER -I

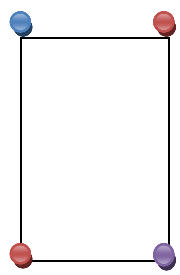
1.1 VERTEX COLOURING

Let G be a graph with vertex set V (G)

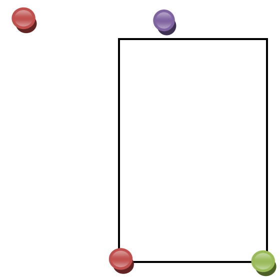
A (proper) vertex coloring of G is a labeling of the vertex set.

F: V (G) $\rightarrow \{1,2,3, \ldots \ldots k\}$

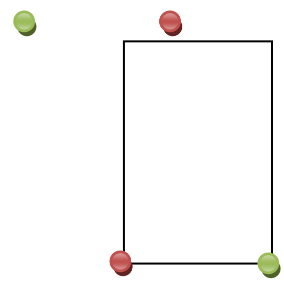
Example: 1



a proper coloring with 3 colors



not a proper coloring



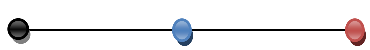
a proper coloring with 2 colors

1.2 K-COLOURING

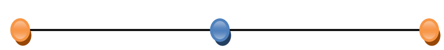
A k-coloring of a graph G is coloring of a using k-coloring

If G has a k-coloring then it is k- colorable

Example: 2



P_3 is 3 colorable



P_3 is 2 colorable

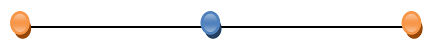
ISSN No: - 2456 – 2165

1.3 CHROMATIC NUMBER

The Chromatic Number of A Graph G, Denoted X(G) Is the Smallest K Such That K-Colourable

Example: 3

X (P_3) = 2



P_3 is 2 colorable

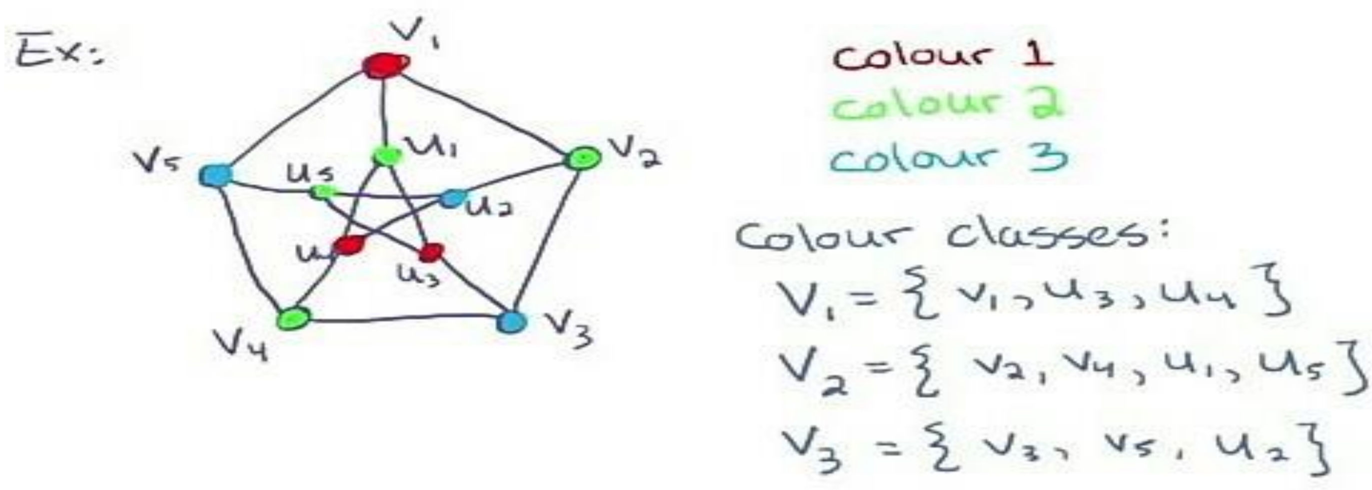
If X (G) = K we say that G is k- chromatic

1.4 COLOUR CLASSES

Let G $\neq$(V, E) A K-coloring of G partitions the vertex set v in to k sets $v_1, v_2, \ldots..v_n$ v_i is an independent set this means v =($v_1 \cup v_2 \cup_{,\ldots\ldots\ldots} \cup v_{k)}$

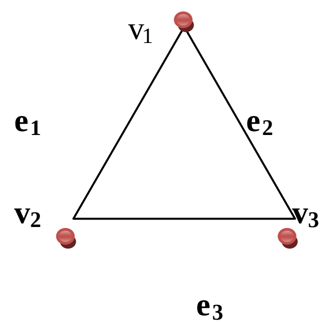
$v_i \cap v_j = \varphi$ for all i $\propto j$

The independent set $v_1 v_2 \ldots..v_k$ are called color classes.



1.5 GRAPH

G = consists of a set of adjusts V= (V_1,V_2,V_3) is called vertices and E = (e_1,e_2,e_3,….) is called edges such that each each e_k is defined with an unordered pair (v_i,v_j) associated with edge e_k are called the end of vertices of e_k the most common representation of a graph is by means of a diagram



1.6 ADJACENT VERTICES

Two vertices which are incident with a common edge are adjacent vertices

1.7 ADJACENT EDGES

Two edges which are incident with common vertex are adjacent

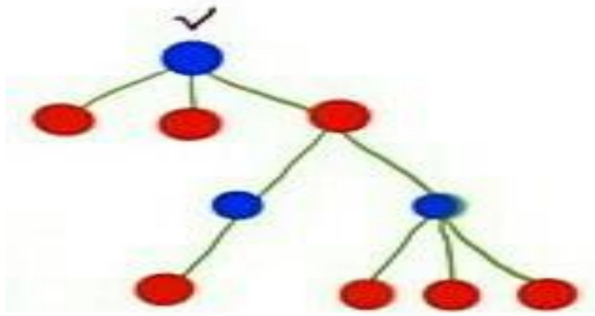
Example: E_2, e_6, e_9 incident with vertex v4

E_2, e_7 are adjacent with v4

V_4 and v_5 are adjacent with e7

1.8 TREE

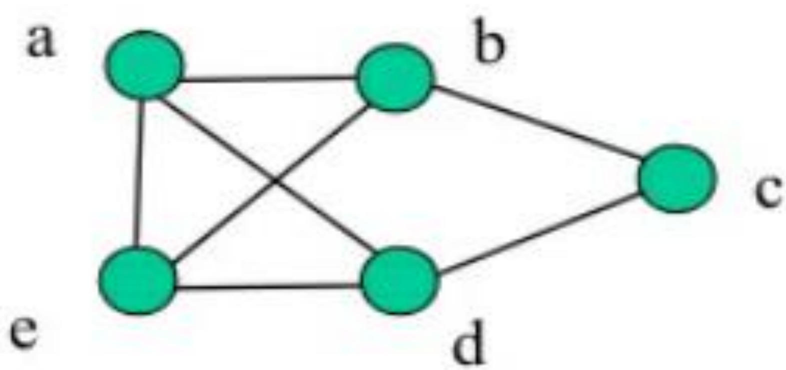
A tree is connected graph without any circuits immediately a tree has to be a single graph that is having neither neighed a self loaf nor parallel edges.



1.9 PATH

An open walk in which no vertex appears more than once is called a path

Example: (a, d, c, b, e)

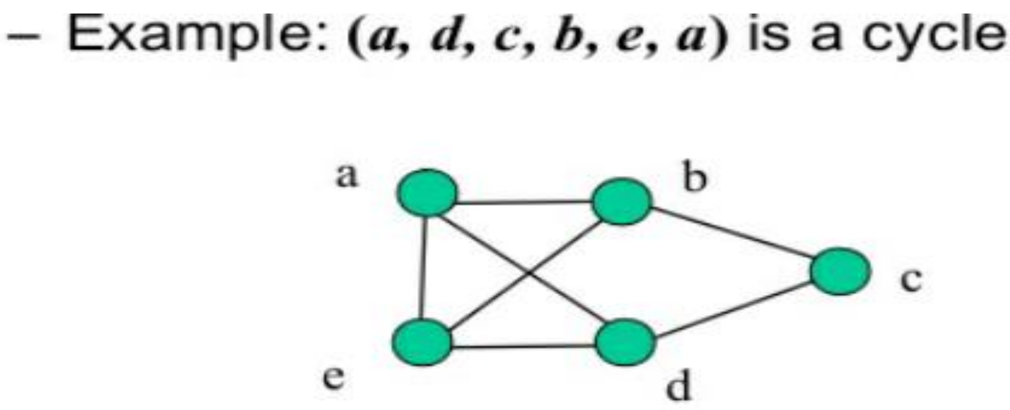


1.10 CIRCUIT

A closed walk in which no vertex except the initial and the final vertex appear more than once is called a circuit

1.11 CYCLE

A closed walk in which each vertex is of degree 2 is called a cycle



1.12 TRAIL

A walk of a graph G is called a trail if all it is edges are distinct

1.13 ACYCLIC

A graph is acyclic if it has no cycles

1.14 BIPARTITE GRAPH

V= (v_1,v_2,v_3,…..) X_1 =(v_1,v_3) X_z = =(v_2,v_4,v_5)

A graph G is called bipartite graph if its vertex set v is partitioned in to two non empty subsets x_1, x_2 in such that each edge is incident with one vertex form x_1 and other from x_2

1.15 COMPLETE BIPARTITE GRAPH

A complete bipartite graph is a graph. is a simple graph set of vertices can be partitioned in to two sets X and Y every edge is between a vertex in X and Y

1.16 TRIVIAL GRAPH

A graph with a single vertex no edges is called a trivial graph

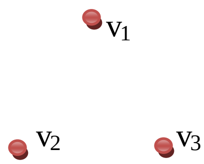
Example:

 v_1

1.17 NON TRIVIAL GRAPH

A grapg G = (v,e) where e = 0 is said to be no trivial graph

Example:



PROPERTIES

- If G has n vertices then $X(G) \leq n$
- $X(G)=1$ and G has no edges
- $X(C_{2n})=2$ and $x(C_{2n+1})=3$
- $X(K_n)=n$
- If H is a sub graph of G then $X(G)\geq X(H)$

This particularly useful when you find certain properties inside of a bigger graph for example if you find a triangle inside of your graph you know that you will need at least three colors to color your whole graph.

CHAPTER-II

VERTEX COLORING WITH BASIC BOUND ON CHROMATIC NUMBER

2.1 THEOREM

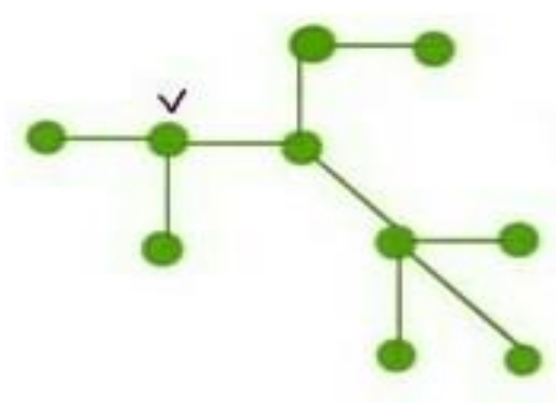
Every tree with at least 02 vertices is 02-chromatic graph

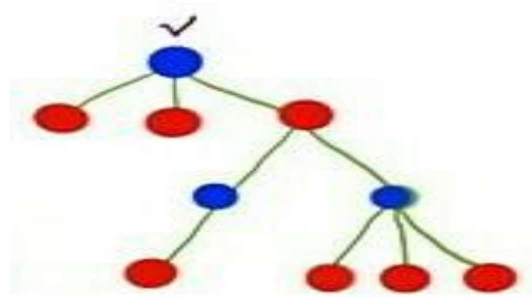
Proof.

Let T be a tree with V (T) $\geq$ 2

Let v$\epsilon V(T)$ and consider T to be **rooted** at v.

T rooter at V





Let v be colored with color 01

Let the neighbors of v be colored with colour 02

Let the neighbors of those vertices be colored 01

.

.

.

.

Continue.

In a tree there is a unique bath between any 02 vertices now along any path in T, the vertices alternate colors so no pair of adjacent vertices receive the same colour

Now this a 2 coloring of T and X(T) = 2 since I V(T) I ≥ 2

2.2 THEOREM

X(G)=2 G is bipartite (G is no odd cycle)

Proof:

Aim to show

X(G)=2 G is bipartite

Contrupositive G is not bipartite X(G)>2 then G has an odd cycle C_{2R+1} and

Hence X(G)≥ X (C_{2R+1})=3

Aim to show

G is bipartite X(G)=2

SUPPOSE G is bipartite then V(G)=X∪Y such that every edge of G is of the form e =xy where x∈X abd y∈Y THEN COLOUR of all vertices of Y colour o2 therefore X(G)=2

BOUNDS ON THE CHROMATIC NUMBER:

Assigning distinct colors to distinct vertices always yields a proper coloring, so 1≤X(G)≤n The only graphs that can be 1-colored are edgeless graphs.

A complete graph of *n* vertices requires colors. In an optimal coloring there must be at least one of the graph's *m*edges between every pair of color classes, so

$$X(G)(X(G)-1) \leq 2m$$

If *G* contains a clique of size *k*, then at least *k* colors are needed to color that clique In other words, the chromatic number is at least the clique number

$$X(G) \geq W(G)$$

For perfect graphs this bound is tight.

The 2-colorable graphs are exactly the bipartite graphs, including trees and forests. By the four color theorem, every planar graph can be 4-colored.

A greedy coloring shows that every graph can be colored with one more color than the maximum vertex degree

$$X(G)\leq \Delta (G) +1$$

Complete graphs have X(G) = n and

Δ (G) = n-1 and odd cycles have X(G) =3

And Δ(G) = 2, so for these graphs this bound is best possible. In all other cases, the bound can be slightly improved

2.3 THEOREM

For every graph G. $X(G)\leq \Delta (G)+1$

Proof.

by induction on n basis n=1 G=k X(G)=1 Δ (G) =0 assume the result hold for every graph with n-1 vertices (n≥1)

Let G be a graph on n vertices let $v\epsilon V(G)$

noe g-v is a graph on n-1 vertices

$X (G-V) \leq \Delta (G-V)+1$

so G-V can be colored

NOTE:

$Deg_{E}(v) \leq \Delta(\epsilon)$

The neibours of v use all $\leq \Delta$ (G)

Case: 01

$\Delta (G) = \Delta (G-V)$

Then there is at least one color of the Δ (G-V) +1 = Δ (G) + 1 colors. Not used by the neighbors of v so v can be colored with the color

ISSN No: - 2456 – 2165

Case: 02

Δ (G) $\neq \Delta$ (G-V)

Then Δ (G-V) $< \Delta$ (G)

Using a new colour for v we will have a Δ ((G-V) + 2) –colouring of G

Since

Δ ((G-V) + 2) $\leq \Delta$ (G) +1

We have X (G) $\leq \Delta$ (G) +1

Hence we proved X (G) $\leq \Delta$ (G) +1

2.4 THEOREM

If G is connected, X (G) $\leq \Delta$(G) unless G is complete or an odd cycle.

Proof.

We may assume $\Delta = \Delta$ (G) ≥ 3, since the result is easy otherwise. Our proof proceeds by induction on Δ, and, for each Δ, we will use induction on n. The induction starts at n = Δ+1, and the theorem is true in this case, since if IGI = n+1 and G $\neq$ Kn+1 we can colour G with Δ colours by using the same colour for some two non-adjacent vertices. Therefore, suppose n $\geq \Delta$+ 2.

Case 1.

There is a vertex v such that G−v is disconnected. Let the components of G−v Be C1,…,Ct. Consider the graphs induced by G on the vertex sets C1∪ $\{V\}$,….,Ct ∪ $\{V\}$. We may _-colour each of these graphs by induction (if one of the graphs is complete or An odd cycle, its maximum degree must be strictly less thanΔ). Switching colours within Some of these colorings if necessary, we may assume that v gets colour 1 in all t colorings, which we can therefore combine to get a Δ-colouring of G.

Case 2.

G−v is connected for all v, but there are two non-adjacent vertices v and w Such that G −v − w is disconnected. You will understand the following argument better if you draw some Figures to illustrate it.

Let A be a component of G $-$v $-$ w and let B = V (G) \ (V (A) $\cup \{V, W\}$). If there are no edges from v to A, then G$-$ w is disconnected, which we are assuming is not the case.
Therefore, there is at least one edge from v to A. Similarly, there is at least one edge from w to A, at least one edge from v to B, and at least one edge from w to B.

Write G1 for the graph obtained from G by deleting B, and G2 for the graph obtained from G by deleting A. It is tempting at this point to Δ-colour G1 and G2 by induction and then combine the colorings, but it may not be possible to combine the colorings (to see why, consider the case when G is an odd cycle). Instead, we note that, from the above observations, v and w have degree at most $\Delta - 1$ in both G1 and G2, so that we may Δ-colour G3 = G1+vw and G4 = G2+vw by induction, unless one of them is complete (if either of them is an odd cycle, we can Δ-colour it since $\Delta > 2$). Such colorings, if they exist, can be combined because v and w will be forced to have different colors in both of them: we can then switch colors if necessary to ensure that v and w are colored 1 and 2 respectively in both colorings. If G3 is a clique on Δ+ 1 vertices, then each of v and w must have degree 1 in G2 (since both have degree Δ in G3 and $\Delta - 1$ in G1). In G2, we can combine v and w into a single vertex, obtaining a graph G5, which can be Δ-coloured by induction. Therefore, there are Δ-colourings of both G1 and G2 in which both v and w get the same colour. These colorings can be combined to provide a Δ-colouring of G.

Case 3.

G $-$v $-$ w is connected for every pair of non-adjacent vertices v and w. Select a vertex u of maximum degree Δ. Since G $\neq$ Kn, some pair of neighbors v and w of u are not adjacent. We define v1 = v; v2 = w; vn = u and, working backwards from vn-1 to v3, we ensure that each vi has some neighbor among{vi 1, … … …. vn}: this is possible since G $-$v $-$ w is connected. Running the greedy algorithm with this ordering of the vertices, we see that v1 = v and v2 = w both get colour 1, and also that we never need to use colour Δ + 1 on v3…vn-1, since each such vi has only at most Δ - 1 neighbors among the already colored vertices. Finally, when we come to color vn, two of its Δ neighbours have received the same colour (1), so that one of the colours 1…Δ is available to color v_n this completes the induction step.

REFERENCES

[1]. Arenboim, L.; Elkin, M. (2009), "Distributed (Δ + 1)-coloring in linear (in Δ) time", Proceedings of the 41st Symposium on Theory of Computing, pp. 111–120, doi:10.1145/1536414.1536432, ISBN 978-1-60558-506-2

[2]. Panconesi, A.; Srinivasan, A. (1996), "On the complexity of distributed network decomposition", Journal of Algorithms, 20

[3]. Schneider, J. (2010), "A new technique for distributed symmetry breaking" (PDF), Proceedings of the Symposium on Principles of Distributed Computing

[4]. Schneider, J. (2008), "A log-star distributed maximal independent set algorithm for growth-bounded graphs" (PDF), Proceedings of the Symposium on Principles of Distributed Computing

[5]. Beigel, R.; Eppstein, D. (2005), "3-coloring in time $O(1.3289^n)$", Journal of Algorithms, 54 (2)): 168–204, doi:10.1016/j.jalgor.2004.06.008

[6]. Björklund, A.; Husfeldt, T.; Koivisto, M. (2009), "Set partitioning via inclusion–exclusion", SIAM Journal on Computing, 39 (2): 546–563, doi:10.1137/070683933

[7]. Brélaz, D. (1979), "New methods to color the vertices of a graph", Communications of the ACM, 22 (4): 251–256, doi:10.1145/359094.359101

[8]. Brooks, R. L.; Tutte, W. T. (1941), "On colouring the nodes of a network", Proceedings of the Cambridge Philosophical Society, 37 (2): 194–197, doi:10.1017/S030500410002168X

[9]. de Bruijn, N. G.; Erdős, P. (1951), "A colour problem for infinite graphs and a problem in the theory of relations" (PDF), Nederl. Akad. Wetensch. Proc. Ser. A, 54: 371–373 (= Indag. Math. 13)

[10]. Byskov, J.M. (2004), "Enumerating maximal independent sets with applications to graph colouring", Operations Research Letters, 32 (6): 547–556, doi:10.1016/j.orl.2004.03.002

[11]. Chaitin, G. J. (1982), "Register allocation & spilling via graph colouring", Proc. 1982 SIGPLAN Symposium on Compiler Construction, pp. 98–105, doi:10.1145/800230.806984, ISBN 0-89791-074-5

[12]. Cole, R.; Vishkin, U. (1986), "Deterministic coin tossing with applications to optimal parallel list ranking", Information and Control, 70 (1): 32–53, doi:10.1016/S0019-9958(86)80023-7

[13]. Cormen, T. H.; Leiserson, C. E.; Rivest, R. L. (1990), Introduction to Algorithms (1st ed.), The MIT Press

[14]. Dailey, D. P. (1980), "Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete", Discrete Mathematics, 30 (3): 289–293, doi:10.1016/0012-365X(80)90236-8

[15]. Duffy, K.; O'Connell, N.; Sapozhnikov, A. (2008), "Complexity analysis of a decentralised graph colouring algorithm" (PDF), Information Processing Letters, 107 (2): 60–63, doi:10.1016/j.ipl.2008.01.002

[16]. Fawcett, B. W. (1978), "On infinite full colourings of graphs", Can. J. Math., 30: 455–457, doi:10.4153/cjm-1978-039-8

[17]. Fomin, F.V.; Gaspers, S.; Saurabh, S. (2007), "Improved Exact Algorithms for Counting 3- and 4-Colorings", Proc. 13th Annual International Conference, COCOON 2007, Lecture Notes in Computer Science, 4598, Springer, pp. 65–74, doi:10.1007/978-3-540-73545-8_9, ISBN 978-3-540-73544-1

[18]. Garey, M. R.; Johnson, D. S. (1979), Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, ISBN 0-7167-1045-5

[19]. Garey, M. R.; Johnson, D. S.; Stockmeyer, L. (1974), "Some simplified NP-complete problems", Proceedings of the Sixth Annual ACM Symposium on Theory of Computing, pp. 47–63, doi:10.1145/800119.803884

[20]. Goldberg, L. A.; Jerrum, M. (July 2008), "Inapproximability of the Tutte polynomial", Information and Computation, 206 (7): 908–929, doi:10.1016/j.ic.2008.04.003

[21]. Goldberg, A. V.; Plotkin, S. A.; Shannon, G. E. (1988), "Parallel symmetry-breaking in sparse graphs", SIAM Journal on Discrete Mathematics, 1 (4): 434–446, doi:10.1137/0401044