Study on Fuzzy Topological Space

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CHAPTER - 1

INTRODUCATION

This dissertation entitled " A STUDY ON FUZZY TOPOLOGICAL SPACE" contains four chapters. It is meant to study the basic concept and carious interesting result on Fuzzy Topological Space.

The chapter – 1 is meant to recall the basic definition and various well known results relevant to this dissertation.

In chapter – 2 we study about the basic definition and the most important theorems such as Fuzzy topological space, Basis and Subbasis for FTS, Closure and interior of Fuzzy sets, Neighbourhood, Fuzzy Continuous maps, Sequences of Fuzzy Sets.

In chapter – 3 we study about the Compact fuzzy topological space, Fuzzy regular space, Fuzzy normal space, Other Separation axioms.

CHAPTER - 2

FUZZY TOPOLOGICAL SPACE

2.1. Fuzzy Topological Space

Definition 2.1.1. A family $\tau \subset I^X$ of fuzzy sets is called a fuzzy topology for X if it satisfies the following three axioms:

(1) $\overline{0}, \overline{1} \in \tau$.

(2) $\forall$ A,B $\in \tau \Rightarrow A \wedge B \in \tau$.

(3) $\forall$ (Aj)j€J $\in \tau \Rightarrow \vee_{j\in J} A_j \in \tau$.

The pair (X, τ) is called a fuzzy topological space or fts, for short. The elements of τ are called fuzzy open sets. A fuzzy set K is called fuzzy closed if $K^c \in \tau$. We denote by τ^c the collection of all fuzzy closed sets in this fuzzy topological space. Obviously, we have:

(a) $\alpha^C \in \tau^C$

(b) if K , M $\in \tau^C$ then $K \vee M \in \tau^c$ and

(c) if {Kj : j $\in$ J} $\in \tau^c$, then $\wedge$ {Kj : j $\in$ J} $\in \tau^c$.

2.2 Base and Subbase for FTS

In this section, we shall establish several definitions and thermos Fuzzy topological space.

Definition 2.2.1 . A base for a fuzzy topological space (X, τ) is a sub collection B of τ such that each member A of τ can be written as A = Vj€∧Aj , where each Aj $\in$ B.

Definition 2.2.2 . A subbase for a fuzzy topological space(X, τ) is a subcollection S of τ such that the collection of infimum of finite subfamilies of S forms a base for (X, τ).

Definition 2.2.3. Let (X, τ) be an fts. Suppose A is any subset of X. Then (A, τ_A) is called a fuzzy subspace of (X, τ), where $\tau_A = \{BA : B \in \tau\}$, $B = \{(x, \mu_B(x)) : x \in X\}$ and $BA = \{(x, \mu_{B/A}(x)) : x \in A\}$.

Definition 2.2.4. A fuzzy point P in X is a special fuzzy set with membership function defined by

$$P(x) = \begin{cases} \lambda & \text{if } x = y, \\ 0 & \text{if } x \neq y; \end{cases}$$

where $0 < \lambda \leq 1$. P is said to have support y, value λ and is denoted by P_y^λ or $P(y, \lambda)$.

Let A be a fuzzy set in X, then P_y^α c $A \Leftrightarrow \alpha \leq A(y)$. In particular, P_y^α C $P_z^\beta \Leftrightarrow y = Z, \alpha \leq \beta$. A fuzzy point P_y^α is said to be in A, denoted by $P_y^\alpha \in A$, $\Leftrightarrow$ $a \leq A(y)$.

The complement of the fuzzy point P_y^λ is denoted either by $P_x^{1-\lambda}$ or by $(P_x^\lambda)^c$.

Definition 2.2.5. The fuzzy point $P^\lambda x$ is said to be contained in a fuzzy set A, or to belong to A, denoted by $P_x^\lambda \in A$ if and only if $\lambda < A(x)$.

Every fuzzy set A can be expressed as the union of all the fuzzy points which belong to A. That is, if $A(x)$ is not zero for $x \in X$, then $A(x) = \sup\{\lambda : P_x^\lambda ; 0 < \lambda \leq A(x)\}$.

Definition 2.2.6. Two fuzzy sets A, B in X are said to be intersecting if and only if there exists a point $x \in X$ such that $(A \wedge B)(x) \neq 0$. For such a case, we say that A and B intersect at x.

Let $A, B \in I^X$. Then $A = B$ if and only if $P \in A \Leftrightarrow P \in B$ for every fuzzy point P in X.

Proposition 2.2.7 Let $\{Aj : j \in J\}$ be a family of fuzzy sets in X, P_x^a and P_y^b be fuzzy points in X and f be a map of X into Y . Then we have the following:

1. $P_x^a \in \vee\{Aj : j \in J\}$ if and only if there exists $j \in J$ such that $P_x^a \in Aj$.
2. If $P_x^a \in \wedge\{Aj : j \in J\}$, then for every $j \in J$ we have $P_x^a \in Aj$.
3. $P_x^a \in P_y^b$ if and only if $x = y$ and $a < b$.
4. If $P_x^a \in P_y^b$ and for every $j \in J$; $P_y^b \in Aj$, then $P_x^a \in \wedge\{Aj : j \in J\}$.
5. If $P_x^a \in A$, where A is a fuzzy set in X, then there exists $a < b$ such that $P_y^b \in A$.
6. $f(P_x^a) = P_{f(x)}^a$
7. $f((P_x^a)^c) = (f(P_x^a))^c$
8. If $P_x^a \in A$, then $f(P_x^a) \in f(A)$

9. If $P_x^a \in f^{-1}$(B), then $P_{f(x)}^a \in$ B, where B is a fuzzy set in Y .

10. If $P_y^b \in f$ (A), then there exists $x \in$ X such that $f(x) = y$ and $P_x^a \in$ A.

11. If $P_y^b \in$ B and $y \in f$ (X), then for every $x \in f^{-1}(y)$ we have $P_x^b \in f^{-1}$(B).

Proof. (1) $P_x^a \in \vee\{Aj : j \in J\}$ if and only if there exists j ∈ J such that $P_x^a \in$ Aj .

Let $P_x^a \in$ Aj $\Rightarrow a \le Aj(x) \Rightarrow a \le \max\{Aj(x) : j \in J\} \Rightarrow a \le (\vee Aj)(x)$.

Again $P_x^a \in \vee\{Aj : j \in J\} \Rightarrow a \le (\vee j \in J\ Aj)(x) \Rightarrow a \le Aj(x) \Rightarrow P_x^a \in Aj$; j ∈ J.

(4) If $P_x^a \in P_y^b$ and for every j ∈ J, $P_y^b \in$ Aj , then $P_x^a \in \wedge\{Aj : j \in J\}$.

$P_x^a \in P_y^b \in Aj \Rightarrow P_x^a \in Aj \Rightarrow a \le Aj(x) \Rightarrow a \le \min\{Aj(x) : j \in J\} \Rightarrow a \le (\wedge Aj)(x) \Rightarrow P_x^a \in \wedge\{Aj : j \in J\}$.

(5) $f(P_x^a) = P_{f(x)}^a$

$$f(P_x^a)(y) = \begin{cases} \sup\{P_x^a(z) : z \in f^{-1}(y) & \text{if } f^{-1}(y) \ne \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

$$= \begin{cases} a & \text{if } x \in f^{-1}(y) \\ 0 & \text{otherwise;} \end{cases}$$

$$= \begin{cases} a & \text{if } f(x) = y \\ 0 & \text{otherwise;} \end{cases}$$

$= P_{f(x)}^a(y)\ \forall y \in Y \Rightarrow f(P_x^a) = P_{f(x)}^a$

(6) $f((P_x^a)^c) = (f(P_x^a))^c$

$f((P_x^a)^c)(y) =$

$$(P^{a}_{f(x)})^{c}(y) = \begin{cases} 1-a & \text{if } y = f(x), \\ 0 & \text{otherwise; ……………(i)} \end{cases}$$

Now

$$(P^{a}_{x})^{c}(z) = \begin{cases} 1-a & \text{if } z = x, \\ 0 & \text{if } z \neq x; \end{cases}$$

So

$$f((P^{a}_{x})^{c})(y) = \begin{cases} \text{Sup}\{(P^{a}_{x})^{c}(z) : z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

$$= \begin{cases} 1-a & \text{if } x \in f^{-1}(y), \\ 0 & \text{otherwise;} \end{cases}$$

$$= \begin{cases} 1-a & \text{if } f(x) = y, \\ 0 & \text{otherwise; ….. (ii)} \end{cases}$$

Thus $f((P^{a}_{x})^{c}) = (f(P^{a}_{x}))^{c}$.

Theorem 2.2.8 . B is a base for an fts (X, τ) iff ∀A∈ τ and for every fuzzy point P in A, ∃B ∈ B such that P ∈ B ⊂ A.

Proof. Assume that B is a base for τ , that is, every A$\in \tau$ is a union of members of B.

Let A $\in \tau$ and P_x^α $\in$A. So A $\in \tau$ $\Rightarrow$ A = $\cup_{i\in I}$ {Bi : Bi $\in$ B} $\Rightarrow P_x^\alpha \in$A= $\cup_{i\in I}${ Bi : Bi $\in$ B} Conversely, assume that for each A $\in \tau$ and for each $P_x^\alpha \in$ A, $\exists B_x$ such that $P_x^\alpha \in B_x \subset$ A.

Let A $\in \tau$ To prove that A can be written as a union of members of B consider any arbitrary $P_x^\alpha \in$ A. So by hypothesis $\exists B_x \in$ B such that $P_x^\alpha \in B_x \subset A \Rightarrow A \subset \cup_{P_x^\alpha \in A} B_x$. Since $B_x \subset$ A, for each $P_x^\alpha \in$ A, therefore A = $\cup_{P_x^\alpha \in A}$ B.

2.3 Closure and Interior of Fuzzy Sets

Definition 2.3.1 [6]. The closure $\bar{A}$ and the interior A^o of a fuzzy set A of X are defined as

$$\bar{A} = \inf\{K : A \le K, K^c \in \tau\}$$

$$A^o = \sup\{O : O \le A, O \in \tau\} \quad \text{respectively.}$$

2.4 Neighborhood

In this section, we shall establish theorems of neighborhood.

Definition 2.4.1 [6]. A fuzzy point P_x^λ is said to be quasi-coincident with A, denoted by $P_x^\lambda{}_q$ A, if and only if $\lambda > A^c(x)$, or $\lambda + A(x) > 1$.

Definition 2.4.3 . A fuzzy set A in (X, τ) is called a neighborhood of fuzzy point P_x^λ if and only if there exists a B $\in \tau$ such that $P_x^\lambda \in$ B$\le$ A; a neighborhood A is said to be open if and only if A is open.

The family consisting of all the neighborhoods of P_x^λ x is called the system of neighborhoods of P_x^λ .

Definition 2.4.4. A fuzzy set A in (X, τ) is called a Q-neighborhood of fuzzy point P_x^λ if and only if there exists a B $\in \tau$ such that $P_x^\lambda{}_q$B $\le$ A. The family consisting of all the Q-neighborhoods of P_x^λ x is called the system of Q-neighborhoods of P_x^λ .

Theorem-2.4.5. A fuzzy point e $\in A^o$ if and only if e has a neighborhood contained in A.

Theorem 2.4.6. A fuzzy point e = P_x^λ $\in$ A if and only if each Q-neighborhood of e is quasi- coincident with A.

Definition 2.4.7. A fuzzy point e is called an adherence point of a fuzzy set A, if and only if, every Q-neighborhood of e is quasi-coincident with A.

Definition 2.4.8 . A fuzzy point e is called a boundary point of a fuzzy set A if and only if e $\in \bar{A} \wedge \bar{A}^c$. The union of all the boundary points of A is called a boundary of A, denoted by b(A). It is clear that b(A) = $\bar{A} \wedge \bar{A}^c$.

Definition 2.4.9. A fuzzy point e is called an accumulation point of a fuzzy set A if and only if e is an adherence point of A and every Q-neighborhood of e and A are quasi-coincident at some point different from supp(e), whenever e $\in$ A. The union of all the accumulation points of A is called the derived set of A, denoted by A^d. It is evident that $A^d \subset \bar{A}$.

2.5 Fuzzy Continuous Map

In this section, we generalize the notion of continuity to what we call F-continuous functions. As a preliminary, we shall establish several properties of fuzzy sets induced by mappings.

Definition 2.5.1 . Given fuzzy topological space (X, τ) and (Y, γ), a function f : X → Y is fuzzy continuous if the inverse image under f of any open fuzzy set in Y is an open fuzzy set in X; that is if $f^{-1}(\upsilon) \in \tau$ whenever $\upsilon \in \gamma$.

Definition 2.5.2. Let f be a function from X to Y. Let B be a fuzzy set in Y with membership function $\mu_B(y)$. Then the inverse of B, written as $f^{-1}[B]$, is a fuzzy set in X whose membership function is defined by

$$\mu_{f^{-1}[B]}(x) = P_B(f(x)) \text{ for all x in X.}$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(X)$. The image of A, written as f [A], is a fuzzy set in Y whose membership function is given by

$\mu_{f[A]}(x) = \sup\{\mu_A(x)$ if f^{-1} (y)is not empty,z$\in f^{-1}(y)$ = 0 otherwise, for all y in Y,

where $f^{-1}[y] = \{x|$ If $(x) = y\}$

Theorem 2.5.1, Let f be a function from X to Y. Then ;

(a) $f^{-1}[B'] = \{ f^{-1}[B]\}'$ for any fuzzy set B in Y.

(b) $f[A'] \supset \{ f[A]\}'$ for any fuzzy set A in X.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}[B_1] \subset f^{-1}[B_2]$, where B_1, B_2 are fuzzy sets in Y.

(d) $A_1 \subset A_2 \Rightarrow f[A_1] \subset f[A_2]$, where A_1 and A_2 are fizzy sets in X.

(e) $B \supset f[f^{-1}[B]]$ for any fuzzy set B in Y.

(f) $A \subset f^{-1}[f[A]]$ for any fuzzy set A in X.

(g) Let f be a function from X to Y and g be a function from Y to Z. Then
$(g \circ f)^{-1}[C] = f^{-1}[g^{-1}[C]]$ for any fuzzy set C in Z, where g of is the composition of g and f .

Definition 2.5.3. A function f from a fts (X, T) to a fts (Y, U) is F-continuous iff the inverse of each U-open fuzzy set is T-open.

Clearly, if f is an F-continuous function on X to Y and g is an F-continuous function on Y to Z, then the composition g of is an F-continuous function on X to Z, for $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$ for each fuzzy set V in Z, and using the F-continuity of g and f it follows that if V is open so is $(g \text{ of })^{-1}[V]$.

Theorem 2.5.2. If X and Y are tfs's, and f is a function on X to Y, then the conditions below are related as follows: (a) and (b) are equivalent; (c) and(d) are equivalent; (a) implies (c), and (d) implies (e).

(a) The function f is F-continuous.

(b) The inverse of every closed fuzzy set is closed,

(c) For each fuzzy set A in X, the inverse of every neighbourhood of $f[A]$ is a neighbourhood of A.

(d) For each fuzzy set A in X and each neighbourhood V of $f[A]$, there is a neighbourhood W of A such that $f[W] \subset V$.

(e) For each sequence of fuzzy sets $\{A_n, n = 1,2,...\}$ in X which converges to a fuzzy set A in X, the sequence $\{f[A_n], n = 1, 2,...\}$ converges to f [A].

2.6 Sequences of Fuzzy Sets

Definition 2.6.1. A sequence of fuzzy sets, say $\{A_n \ n = 1,2,...\}$, is eventually contained in a fuzzy set A iff there is an integer m such that, if $n \geq m$, then $A_n \subset A$. The sequence is frequently contained in A iff for each integer m there is an integer n such that $n \geq m$ and $A_n \subset A$. If the sequence is in a fts (X, T), then we say that the sequence convergence to a fuzzy set A iff it is eventually contained in each neighbourhood of A.

Definition 2.6.2. Let N be a map from the set of non-negative integers to the set of non-negative integers. Then the sequence $\{B_i, i = 1,2,...\}$ is a subsequence of a sequence $\{A_n, n = 1, 2,...\}$ iff there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$.

Definition 2.6.3 A fuzzy set A in a fts (X, τ) is a cluster fuzzy set of a sequence of fuzzy sets iff the sequence is frequently contained in every neighbourhood of A.

Theorem 2.6.1. If the neighbourhood system of each fuzzy set in a fts (X, τ) is countable, then ;

(a) A fuzzy set A is open iff each sequence of fuzzy sets,{ A_n, n = 1,2,...}, which converges to a fuzzy set B contained in A is eventually contained in A.

(b) If A is a cluster fuzzy set of a sequence{A_n, , n = 1, 2,...} of fuzzy sets, then there is a subsequence of the sequence convergent to A.

CHAPTER - 3

COMPACTNESS AND SEPARATION AXIOMS

In this chapter, we study some theorems, propositions, axioms of Fuzzy topological spaces

3.1. Compact Fuzzy Topological Space

We now consider a fuzzy compact space constructed around a fuzzy topology.

Definition 3.1.1. A family A of fuzzy sets is a cover of a fuzzy set B iff $B \subset \cup\{A \mid A \in A\}$. It is an open cover iff each member of A is an open fuzzy set. A subcover of A is a subfamily of A which is also a cover.

Definition 3.1.2. A fuzzy topological space (X, τ) is compact if every cover of X by members of τ contains a finite subcover, i.e. if $A_i \in \tau$, for every $i \in I$, and $\vee_{i\in I}A_i = \bar{1}$, then there are finitely many indices $i_1, i_2, \ldots\ldots i_0 \in I$ such that $V_{j-1}^{n} A_{ij} = \bar{1}$.

Theorem 3.1.3. Let (X, τ) and (Y, γ) be fuzzy topological spaces with (X, τ) compact, and let $f : X \to Y$ be a fuzzy continuous surjection. Then (Y, γ) is also compact.

Proof. Let $B_i \in \gamma$ for each $i \in I$, and assume that $\vee_{i\in I}B_i = \bar{1}_{\gamma}$. For each $x \in X$, $\vee_{i\in I}f^{-1}(B_i)(x) = \vee_{i\in I}B_i(f(x)) = \bar{1}_X$. So the τ-open fuzzy sets $f^{-1}(B_i)(i \in I)$ cover X. Thus, for finitely many indices $i_1, i_2, \ldots i_n \in I$, $V_{j-1}^{n} f^{-1}(B_{ij}) = \bar{1}_x$. If B is any fuzzy set in Y , the fact that f is a surjection mapping onto Y implies that, for any $y \in Y$; $f(f^{-1}(B))(y) = \sup\{f^{-1}(B)(z) : z \in f^{-1}(y)\} = \sup\{(B)\, f(z) : f(z) = y\} = B(y) \Rightarrow f(f^{-1}(B)) = B$. Thus, $\bar{1}_Y = f(\bar{1}_X) = f(V_{j-1}^{n} (f^{-1}(B_{ij}))) = V_{j-1}^{n} f(f^{-1}(B_{ij})) = V_{j-1}^{n} B_{ij}$.

Therefore, (Y, γ) is also compact.

Lemma 3.1.4 (Alexander Subbase Lemma) [1]. If S is a subbase for a fuzzy topo-logical space (X, τ), then (X, τ) is compact iff every cover of X by members of S has a finite sub cover (i.e. if $A_\alpha \in S$ for each $\alpha \in \wedge$ and $\vee_{\alpha\in\wedge} A_\alpha = \bar{1}$, then there are finitely many indices α_i, $(i = 1, 2, \ldots .n)$ such that $V_{j-1}^{n} A_{\alpha i} = \bar{1}$).

Definition 3.1.5. Let (X_i, τ_i) be a fuzzy topological space, for each index $i \in I$. The product fuzzy topology $\tau = \prod_{i\in I} \tau_i$ on the set $X = \prod_{i\in I} X_i$ is the coarsest fuzzy topology on X making all the projection mappings $\Pi_i : X \to X_i$ fuzzy continuous.

Theorem 3.1.6 (Fuzzy Tychonoff Theorem) [1].

Let n be a positive integer and for each i = 1, 2, ... n, let (X_i, τ_i) be a compact fuzzy topological space. Then (X, τ) = ($\prod_{i-1}^{n} x_i, \prod_{i-1}^{n} \tau_i$) is compact.

Proof. We will say that a collection of open fuzzy sets of a fuzzy topological space has the finite union property (FUP) if none of its finite sub collections cover the space (i.e. none of its finite sub collections have supremum identically equal to $\bar{1}$).

Since S = {Π_i^{-1} (Ai) : $A_i \in \tau_i$, i = 1, 2, _ _ _ , n} is a subbase for (X, τ), by the Lemma it suffices to show that no sub collection of S with FUP covers X. Let C be a sub collection of S with FUP.
For each i = 1, 2, ... , n let C_i = {A ∈ τ_i : Π_i^{-1} (A) ∈ C}.

Then C_i is a collection of open fuzzy sets in (X_i, τ_i) with FUP. Indeed, if $A_{i,1}$, $A_{i,2}$, ... , $A_{i,k}$ ∈ C_i satisfy $V_{j-1}^{k} A_{i,j} = \bar{1}X_i$, then $V_{j-1}^{k}\ \Pi_i^{-1}(A_{i,j}) = \Pi_i^{-1}(V_{j-1}^{k}\ A_{i,j}) = \Pi_i^{-1}(\bar{1}X_i) = \bar{1}_X$, and this would contradict the fact that C has FUP.

Therefore, by the compactness of (X_i, τ_i), the collection C_i cannot cover X_i, and we can select a point $x_i \in X_i$ such that (∨C_i)(x_i) = ai < 1.

Now if we consider the point = (x 1, x 2, ... x n) X and the collection = { (A) : A τi} ∩C, then it follows that (∨)( ) =∨{ (A)(x) : A τi and (A) C} =∨{A(x i) : A τi and (A) C } = (∨Ci)(x i) = ai.

Further noting that C = , we obtain (∨C)(x) = (∨)(x) = (∨)(x i) = ai which is strictly less than 1 since each of the finitely many real numbers ai is strictly less than 1. Thus ∨ C ≠ , as desired.

Definition .3.1.7 A family A of fuzzy sets has the finite intersection property if the intersection of the members of each finite subfamily of A is nonempty.

Theorem 3.1.8. Afts is compact if and only if each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection. proof. If A is a family of fuzzy sets in a fts (X, T), then A is a cover of X iff ∪ {A | A ∊ A} = X, or iff (∪ [A | A ∊ A]}′ = X′ = ϕ, or iff ∩ {A′ | A ∊ A} = ϕ by the De Morgan's laws. Hence, the fuzzy space X is compact iff each family of open fuzzy sets in X such that no finite subfamily covers

X, fails to be a cover, and this is true iff each family of closed fuzzy sets which possesses the finite intersection property has a nonempty intersection.

Theorem 3.1.9. Let f be an F-continuous function carrying the compact fts X onto the fts Y. Then Y is compact.

Proof. Let B be an open cover of Y. Then, since for all $x \in$ X, the family of all fuzzy sets of the form f -l [B], for B in B, is an open cover of X which has a finite sub cover. However, if f is onto, then it is easily seen that f[f-1 [B]] = B f or any fuzzy set B in Y. Thus, the family of images of members of the sub cover is a finite subfamily of B which covers Y and consequently Y is compact.

3.2. Fuzzy Regular Space

Definition 3.2.1. An fts (X, τ) will be called regular if for each fuzzy point P and each fuzzy closed set C such that P ∧ C = $\bar{0}$ there exist fuzzy open sets U and V such that P∈ U and C ⊆ V .

Proposition 3.2.2. Every subspace of regular space is also regular.

Proof. Let X be a fuzzy regular space and A be a subspace of X. We have to prove that A is regular. Recall that τA = {GA : G τ}, where G = {(x, μG(x)) : x X} and GA ={(x, μG/A(x)) : x A}. Let be fuzzy point in A and FA is closed set of A such that FA. Since A is a subspace of X, therefore X and there is a closed set F in X, which generated the closed subset FA of A. Since X is regular space and ∧ F = there exist open sets U and V such that ⊆ U = (x, μU) and F⊆ V = (, μV). Thus UA = (x, μUj\A), VA = (x, μVj\A) are open sets in A such that ⊆ UA and FA ⊆ VA. Hence A is a regular subspace of X.

Proposition 3.2.3. If a space X is a regular space, then for any open set U and a fuzzy point P ∈ X such that P ∩ U′ = $\bar{0}$, there exists an open set V such that P ∈ V ⊆ $\bar{V}$ ⊆ U.

Proof. Suppose that X is a fuzzy regular space. Let U = {(x , μU) : x X} be a fuzzy open set of X such that U′ = 0. Then U′ = (x, 1 – μU) is fuzzy closed set of X such that U′ = (x, 1 – μU) and hence, U. Since X is regular, therefore there exist two disjoint fuzzy open set V and W such that V and U′ ⊆ W. Now W′ is a closed set of X such that V ⊆ W′ ⊆ U. Thus, V ⊆ V and V⊆ W′ ⊆ U and hence, V ⊆ U.

This proves that V ⊆ ⊆ U.

3.3. Fuzzy Normal Space

Definition 3.3.1. A fuzzy topological space (X, T) will be called normal if for each pair of fuzzy closed sets C_1 and C_2 such that $C_1 \wedge C_2 = 0$ there exist fuzzy open sets M_1 and M_2 such that $C_i \subseteq Mi(i = 1, 2)$ and $M_1 \wedge M_2 = 0$.

Proposition-3.3.2. If a space X is a normal space, then for each closed set F of X and any open set G of X such that $F \wedge G' = 0$ there exists an open set G_F such that $F \subseteq G_F \subseteq \overline{G_F} \subseteq G$.

Proof. Let X be a normal space. Let F be a fuzzy closed set in X and G be an fuzzy open set in X such that $F \wedge G' = 0$, then $F \subseteq G$. Let $G = (x, \mu G)$ and $F = (x, \mu F)$, then F and G′ are two disjoint fuzzy closed sets of X. Since X is fuzzy normal, so two disjoint fuzzy open sets GF and GG′ such that $F \subseteq GF$ and $G' \subseteq GG'$ and $GF \wedge GG' = 0$. Thus, $GF \subseteq G'G'$, but G′G′ is a fuzzy closed set and hence $GF \subseteq G'G'$. Thus from the above we have $F \subseteq GF \subseteq \subseteq G$.

3.4. Other Separation Axioms

Definition 3.4.1 [9]. An fts (X, τ) is said to be fuzzy T_0 iff $\forall x, y \in X, x \neq y$, $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$ or $U(y) = 1$ and $U(x) = 0$.

Definition 3.4.2 (a) An fts (X, τ) is said to be fuzzy T1- topological space iff x , y X, $x \neq y$, U, V τ such that U() = 1, U(y) = 0 and V (y) = 1, V (x) = 0.

Definition 3.4.2(b) . An fts (X, τ) is F - T_1 iff singletons are closed.

Definition 3.4.3(a). An fts (X, τ) is said to be Hausdorff or fuzzy $T_2\in$ iff the following conditions hold:

If p, q are any two disjoint fuzzy points in X then

(i) if $x_p \neq x_q$, $\exists$ open sets V_p and V_q, such that $p \in V_{p,\, q} \notin V_{p;\, q} \in V_{q,\, p} \notin V_{q;}$

(ii) if $x_p = x_q$, and $\mu_p(x_p) < \mu_q(x_p)$, then $\exists$ an open set Vp such that $p \in V_p$, but q V p.

Definition 3.4.3(b) A fts (X, τ) is said to be fuzzy Hausdorff iff for any two distinct fuzzy points $p, q \in X$, there exist disjoint U, $V \in \tau$ with $p \in U$ and $q \in V$.

Definition 3.4.4. An fts (X, τ) is F - T_3 iff it is T_1, or F - T_1 and regular.

Definition 3.4.5 . An fts (X, τ) is F - T_4 iff it is T_1, or F - T_1 and normal.

Proposition 3.4.6. Every subspace of T_1-space is T_1.

Proof. Let X be a T1 fuzzy topological space and A be a subspace of X. So τA = {GA |GA = (x , μG|A), G τ}. Let x , y A such that x ≠ y. Then x , y X are two distinct points and as X is T1, there exist U, V τ such that U(x) = 1, U(y) = 0 and V (y) = 1, V (x) = 0. Then, UA and VA are fuzzy open sets of A such that UA(x) = 1, UA(y) = 0 and VA(y) = 1, VA(x) = 0. This shows that A is also T1.

Theorem 3.4.7 . A fuzzy subspace of a fuzzy Hausdorff topological space is fuzzy Hausdorff.

Proof. Let X be a fuzzy Hausdorff topological space and A be a subspace of X. Let P_x^α , P_y^β be any two arbitrary points in A with $P_x^\alpha \neq P_y^\beta$. Then, we have P_x^α , P_y^β ∈ X, with$P_x^\alpha \neq P_y^\beta$. Since, X is a Hausdorff space therefore ∃U, V ∈ τ such that P_x^α ∈ U, P_y^β ∈ Vand U ∩ V = 0.

Since U, V are fuzzy open subsets of X and μU(z) ∧ μV (z) = 0, for every z X, therefore UA = (x, μUjA) and VA = (x , μVjA) are fuzzy open subsets of A such that UA, VA and UA \ VA = 0, . Thus (A, τA) is also a fuzzy Hausdorff topological space.

Proposition 3.4.8. No subset of a Hausdorff fts can be compact.

Corollary 3.4.9 . Singletons in an Hausdorff fts are not compact.

Theorem 3.4.10. If {(X_i, τ_i) : i ∈ I} is a family of fuzzy Hausdorff topological spaces, their product(X, τ) is also fuzzy Hausdorff.

Proof. Let {X_i : I ∈ I} be a family of fuzzy Haudorff spaces and X =$\prod_{i \in I} X_i$. We haveto show that X is fuzzy Hausdorff.

Let P_x^α, P_y^β ∈ X with $P_x^\alpha \neq P_y^\beta$

We know that the projection $P_i : X \to X_i$, I ∈ I is fuzzy continuous.

$P_x^\alpha \neq P_y^\beta \Rightarrow$ there exists some $i_0 \in I$ such that say P_x^α = m and P_y^β = n; $P_{i0}(m) = P_{i0}(n) \Rightarrow m_{i0} \neq n_{i0}$ and we have $P_{i0} : X \to X_{i0}$ and here m_{i0}, $n_{i0} \in X_{i0}$ with $m_{i0} \neq n_{i0}$. X_{i0} is fuzzy Hausdorff ⇒ there exists open sets U and V in X_{i0} such that m_{i0} ∈ U and n_{i0} ∈ V and U ∩ V = ∅ $P_{i0}^{-1}(U)$ open ⊂ X and $P_{i0}^{-1}(V)$ open ⊂ X.

Since

P_{i0} is continuous $m_{i0} \in U \Rightarrow P_{i0}(m) \in U \Rightarrow m \in P_{i0}^{-1}(U)$ again $n_{i0} \in V \Rightarrow P_{i0}(n) \in V \Rightarrow n \in P_{i0}^{-1}(V)$.

Claim. $P_{i0}^{-1}(U) \cap P_{i0}^{-1}(V) = \emptyset$. Suppose to the contrary $P_{i0}^{-1}(U) \cap P_{i0}^{-1}(V)$ $\emptyset$. This$\Rightarrow$ some $q \in P_{i0}^{-1}(U) \cap P_{i0}^{-1}(V)$ $\Rightarrow q \in P_{i0}^{-1}(U)$ and $q \in P_{i0}^{-1}(V) \Rightarrow P_{i0}(q) \in U$ and $P_{i0}(q) \in V \Rightarrow q_{i0} \in U$ and $q_{i0} \in V \Rightarrow U \cap V = \emptyset$ which is a contradiction. Therefore (X, τ) is also fuzzy Hausdorff.

Proposition 3.4.11. Every subspace of T_3-space is T_3.

Proof. We know that $T_3 = T_1$+Regular. The proof follows by noting that every subspace of T_1-space is T_1 and every subspace of regular space is regular.

Proposition 3.4.12. Every subspace of T_4-space is T_4

Proof. We know that $T_4 = T_1$+Normal. Since every subspace of T_1-space is T_1 and every subspace of normal space is normal, therefore every subspace of a T_4-space is T_4.

Theorem 3.4.13. An $F - T_2$ -space is an $F - T_1$-space.

Proof. Let p be a fuzzy point in X. Then any point q {p}´ belongs to an open set Vq such that μ{p}´(xp) ≥μVq (xp). So Vq {p}´. If, on the other hand, p is crisp, let xq X – {xp} be arbitrary. If {qn, n N} be a sequence of fuzzy points, where xqn = xq, n N and the sequence {μqn(x q); n N} is decreasing and converges to zero, then there exists a sequence of open sets {Vpqn, n N}, such that p Vpqn and qn Vpqn, n N, as (X, τ) is Hausdorff. Therefore, if P =∩n N V pqn, then P is a closed set, where μp(x q) = 0 and μp(x p) = 1. So P´ is an open set contained in {p}´ and containing the crisp point q.

Theorem 3.4.14 . An $F - T_3$ -space is an $F - T_2$-space.

Proof. Let p, q be two fuzzy points, where x p ≠ x q and let w be a third fuzzy point, where x w = x p and μw(p) > 1 - μp(x p). Then {w}´ is open and

$$\mu_{\{w\}1}(x) = \begin{cases} 1-\mu_w(x_p) < \mu_p(x) & \text{for } x = x_p; \\ 1 & \text{otherwise:} \end{cases}$$

Therefore $q \in \{w\}^1$, but p/∈{w}´. Now since (X, T) is regular, there exists $V_q \in \tau$ such that $q \in V_q \subset \bar{V}_q \subset \{w\}'$. Obviously then, $p \in \bar{V}_q$. Similarly, an open set V_p can be determined such that $p \in V_p$ and $q \in \bar{V}_p$.

Theorem 3.4.15. An $F - T_4$-space is an $F - T_3$-space.

Proof. Let (X, τ) be a regular space. Let $p \in X$ and $V \in \tau$. Since X is $F - T_4$ it is $F - T_1$ and normal. Since X is $F - T_1$. {p} is a closed set in X. Since X is normal. There exists $G \in \tau$ such that $\{p\} \subset G \subset \bar{G} \subset V \Rightarrow p \in G \subset \bar{G} \subset V$.

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